Chapter 9
Credit Risk - Structural Approach

Credit risk can be defined as the risk of default on the payment of a debt. In this chapter, credit risk is modeled using the value of a firm's assets, *i.e.* the default event is said to occur when the value of assets drops below a certain pre-defined level. This is in contrast to the reduced form approach to credit risk of Chapter 10, in which stochastic processes are used to model default probabilities. We also consider the modeling of correlation and dependence between multiple default times.

9.1 Merton Model

The Merton (1974) credit risk model reframes corporate debt as an option on a firm's underlying value. Precisely the value S_t of a firm's asset is modeled by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

under the historical (or physical) measure $\mathbb{P}$. Recall that, using the standard Brownian motion

$$\widehat{B}_t = \frac{\mu - r}{\sigma} t + B_t, \qquad t \geqslant 0,$$

under the risk-neutral probability measure $\mathbb{P}^*$, the process $(S_t)_{t\in\mathbb{R}_+}$ is modeled as

$$dS_t = rS_t dt + \sigma S_t d\widehat{B}_t.$$

Assumption 9.1. *The company's debt is represented by an amount $K > 0$ in bonds to be paid at maturity T, see* e.g. *§ 4.1 of Grasselli and Hurd (2010).*

In this setting, default occurs if $S_T < K$ with probability $\mathbb{P}(S_T < K)$, the bond holder will receive the recovery value S_T. Otherwise, if $S_T \geqslant K$ the bond holder receives K and the equity holder is entitled to receive $S_T - K$, which can be represented as $(S_T - K)^+$ in general.

The discounted expected cash flow (or dividend) $\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t]$ received by the equity holder can be estimated at time $t \in [0, T]$ as the price of a European call option, from the Black-Scholes formula

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] &= S_t\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right)\\ &-K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t \leqslant T,\end{aligned}$$

see Proposition 5.4.

Proposition 9.2. *The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed from the lognormal distribution of S_T as*

$$\mathbb{P}(S_T < K \mid \mathcal{F}_t) = \Phi\big(-d_-^\mu\big), \tag{9.1}$$

where Φ is the cumulative distribution function of the standard normal distribution, and

$$d_-^\mu := \frac{(\mu-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

Proof. The default probability $\mathbb{P}(S_T < K \mid \mathcal{F}_t)$ can be computed from the lognormal distribution of S_T as

$$\begin{aligned}\mathbb{P}(S_T < K \mid \mathcal{F}_t) &= \mathbb{P}\big(S_0\mathrm{e}^{\sigma B_T+(\mu-\sigma^2/2)T} < K \mid \mathcal{F}_t\big)\\ &= \mathbb{P}\left(B_T < \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)T+\log\frac{K}{S_0}\right)\ \Big|\ \mathcal{F}_t\right)\\ &= \mathbb{P}\left(B_T - B_t + y < \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)T+\log\frac{K}{S_0}\right)\right)_{|y=B_t}\\ &= \mathbb{P}\left(B_T - B_t + \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)t+\log\frac{K}{x}\right)\right.\\ &\qquad\left.< \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)T+\log\frac{K}{S_0}\right)\right)_{|x=S_t}\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{\sqrt{2(T-t)\pi}} \int_{-\infty}^{(-(\mu-\sigma^2/2)(T-t)+\log(K/S_t))/\sigma} \mathrm{e}^{-x^2/(2(T-t))} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(-(\mu-\sigma^2/2)(T-t)+\log(K/S_t))/(\sigma\sqrt{T-t})} \mathrm{e}^{-x^2/2} dx \\
&= 1 - \Phi\left(\frac{(\mu-\sigma^2/2))(T-t)+\log(S_t/K))}{\sigma\sqrt{T-t}} \right) \\
&= 1 - \Phi(d_-^\mu) \\
&= \Phi(-d_-^\mu).
\end{aligned}
$$

□

The formula (9.2) can be implemented as follows.

```
P <- function(S, K, mu, T, sigma)
{d1 <- (log(S/K)+(mu-sigma^2/2)*T)/(sigma*sqrt(T))
P = pnorm(d1);P}
```

Note that under the risk-neutral probability measure $\mathbb{P}^*$ we have, replacing μ with r,

$$\mathbb{P}^*(S_T < K \mid \mathcal{F}_t) = \Phi(-d_-^r),$$

with

$$d_-^r = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}},$$

which implies the relation

$$d_-^r = d_-^\mu - \frac{\mu - r}{\sigma}\sqrt{T-t},$$

or, denoting by Φ^{-1} the inverse function of Φ,

$$\Phi^{-1}(\mathbb{P}(S_T < K \mid \mathcal{F}_t)) = -\frac{\mu-r}{\sigma}\sqrt{T-t} + \Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathcal{F}_t)).$$

If the level of the firm's assets falls below the level K at time T, default may have occurred at a random time τ such that

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) = \mathbb{P}(S_T < K \mid \mathcal{F}_t).$$

In this case, the result of Proposition 9.2 can be reinterpreted in the next corollary.

Corollary 9.3. *The conditional distribution of the default time τ is given by*

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) = \mathbb{P}(S_T < K \mid \mathcal{F}_t) = \Phi\left(-\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right), \tag{9.2}$$

$0 \leqslant t \leqslant T$.

We also have

$$\begin{aligned}
\mathbb{P}(\tau < T \mid \mathcal{F}_t) &= \mathbb{P}(S_T < K \mid \mathcal{F}_t) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathcal{F}_t)) - \frac{\mu - r}{\sigma}\sqrt{T-t}\right) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}^*(\tau < T \mid \mathcal{F}_t)) - \frac{\mu - r}{\sigma}\sqrt{T-t}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}^*(\tau < T \mid \mathcal{F}_t) &= \mathbb{P}^*(S_T < K \mid \mathcal{F}_t) \\
&= \Phi\left(-\frac{(r - \sigma^2/2))(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}(S_T < K \mid \mathcal{F}_t)) + \frac{\mu - r}{\sigma}\sqrt{T-t}\right) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}(\tau < T \mid \mathcal{F}_t)) + \frac{\mu - r}{\sigma}\sqrt{T-t}\right).
\end{aligned} \tag{9.3}$$

Note that when $\mu < r$, we have

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) > \mathbb{P}^*(\tau < T \mid \mathcal{F}_t),$$

whereas when $\mu > r$ we get

$$\mathbb{P}(\tau < T \mid \mathcal{F}_t) < \mathbb{P}^*(\tau < T \mid \mathcal{F}_t),$$

as illustrated in the next Figure 9.1.

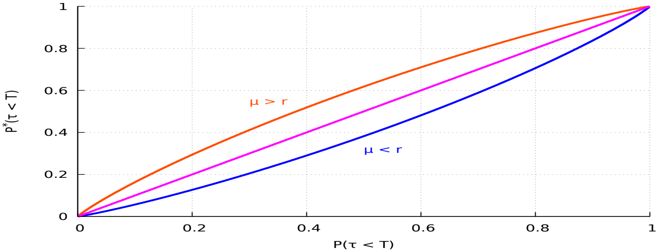

Fig. 9.1: Graph of the function $x \longmapsto \Phi\big(\Phi^{-1}(x) - (\mu - r)\sqrt{T}/\sigma\big)$ for $\mu > r$, $\mu = r$, and $\mu < r$.

9.2 Default Bonds

In the following proposition we price at time $t \in [0, T]$ the amount $\min(S_T, K)$ received by the bond holder (or junior creditor) at maturity, based on the recovery value S_T when $S_T < K$. This price can interpreted at the price $P(t, T)$ at time $t \in [0, T]$ of a default bond with face value \$1, maturity T and recovery value $\min(S_T/K, 1)$.

Proposition 9.4. *The amount received by the bond holder (or junior creditor) at maturity is priced at time $t \in [0, T]$ as*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\min(S_T, K) \mid \mathcal{F}_t\big] = K\mathrm{e}^{-(T-t)r}\Phi(d_-^r) - S_t\Phi(-d_+^r), \quad 0 \leqslant t \leqslant T.$$

Proof. Using the Black-Scholes put option pricing formula and the identity

$$\min(x, K) = K - (K - x)^+, \qquad x \in \mathbb{R},$$

we have

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\min(S_T, K) \mid \mathcal{F}_t\big] &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[K - (K - S_T)^+ \mid \mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}K - \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(K - S_T)^+ \mid \mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}K + S_t\Phi(-d_+^r) - K\mathrm{e}^{-(T-t)r}\Phi(-d_-^r) \\
&= K\mathrm{e}^{-(T-t)r}\Phi(d_-^r) - S_t\Phi(-d_+^r).
\end{aligned}$$

□

Writing

$$\begin{aligned}P(t,T) &= \mathrm{e}^{-(T-t)y_{t,T}}\\ &= \frac{1}{K}\mathrm{e}^{-(T-t)r}\mathbb{E}^*[\min(S_T,K) \mid \mathcal{F}_t]\\ &= \mathrm{e}^{-(T-t)r}\Phi(d_-^r) - \frac{S_t}{K}\Phi(-d_+^r),\end{aligned}$$

gives the default bond yield

$$\begin{aligned}y_{t,T} &= -\frac{1}{T-t}\log(P(t,T))\\ &= -\frac{1}{T-t}\log\left(\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\min\left(1,\frac{S_T}{K}\right)\Big|\mathcal{F}_t\right]\right)\\ &= r-\frac{1}{T-t}\log\left(\mathbb{E}^*\left[\min\left(1,\frac{S_T}{K}\right)\Big|\mathcal{F}_t\right]\right)\\ &= r-\frac{1}{T-t}\log\left(\frac{1}{K}\mathbb{E}^*\left[\min\left(K,S_T\right)\Big|\mathcal{F}_t\right]\right)\\ &= r-\frac{1}{T-t}\log\left(\Phi(d_-^r)-\frac{S_t}{K}\mathrm{e}^{(T-t)r}\Phi(-d_+^r)\right),\end{aligned}$$

which is usually higher than the risk-free yield r.

9.3 Black-Cox Model

In the Black and Cox (1976) model the firm has to maintain an account balance above the level K throughout time, therefore default occurs at the first time the process S_t hits the level K, cf. § 4.2 of Grasselli and Hurd (2010). The default time τ_K is therefore the first hitting time

$$\tau_K := \inf\left\{t \geqslant 0 \ : \ S_t := S_0\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t} \leqslant K\right\},$$

of the level K by

$$(S_t)_{t\in\mathbb{R}_+} = \left(S_0\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t}\right)_{t\in\mathbb{R}_+},$$

after starting from $S_0 > K$.

Proposition 9.5. *The probability distribution function of the default time τ_K is given by*

$$\mathbb{P}\big(\tau_K \leqslant T\big) = \mathbb{P}(S_T \leqslant K) + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right), \tag{9.4}$$

with $S_0 \geqslant K$.

Proof. By *e.g.* Corollary 7.2.2 and pages 297-299 of Shreve (2004), or from Relation (10.13) in Privault (2022), we have

$$\begin{aligned}
\mathbb{P}(\tau_K \leqslant T) &= \mathbb{P}\left(\min_{t\in[0,T]} S_t \leqslant K\right) \\
&= \mathbb{P}\left(\min_{t\in[0,T]} \mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t} \leqslant \frac{K}{S_0}\right) \\
&= \mathbb{P}\left(\min_{t\in[0,T]} \left(B_t + \frac{(\mu-\sigma^2/2)t}{\sigma}\right) \leqslant \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\right) \\
&= \Phi\left(\frac{\log(K/S_0)-(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2} \Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \qquad (9.5)\\
&= \mathbb{P}(S_T \leqslant K) + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2} \Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

with $S_0 \geqslant K$. □

We check that when $S_0 = K$, using (9.1), Relation (9.4) reads

$$\begin{aligned}
\mathbb{P}(\tau_K \leqslant T) &= \Phi\left(-\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&\quad + \Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&= 1.
\end{aligned}$$

The cash flow

$$(S_T - K)^+ \mathbb{1}_{\{\tau_K > T\}} = (S_T - K)\mathbb{1}_{\left\{\min_{t\in[0,T]} S_t > K\right\}}$$

received at maturity T by the equity holder can be priced at time $t \in [0, T]$ as a down-and-out barrier call option with strike price K and barrier level K is priced in the next proposition, in which BS_c denotes the Black-Scholes call pricing formula.

Proposition 9.6. *We have*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\,\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T}S_t>K\right\}}\,\middle|\,\mathcal{F}_t\right]=\mathbb{1}_{\left\{\min\limits_{t\in[0,T]}S_t>B\right\}}g(t,S_t),$$

$t\in[0,T]$, *where*

$$g(t,S_t)=\mathrm{BS}_c(S_t,K,r,T-t,\sigma)-S_t\left(\frac{K}{S_t}\right)^{2r/\sigma^2}\mathrm{BS}_c(K/S_t,1,r,T-t,\sigma),$$

$0\leqslant t\leqslant T$.

Proof. By *e.g.* Relation (11.10) and Exercise 11.1 in Privault (2022), we have

$$\mathbb{E}^*\left[(S_T-K)^+\,\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T}S_t>K\right\}}\,\middle|\,\mathcal{F}_t\right]=\mathbb{1}_{\left\{\min\limits_{t\in[0,T]}S_t>B\right\}}g(t,S_t),$$

$t\in[0,T]$, where

$$\begin{aligned}
&g(t,S_t)\\
&=S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-\mathrm{e}^{-(T-t)r}K\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)\\
&\quad-K\left(\frac{K}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{K}{S_t}\right)\right)+\mathrm{e}^{-(T-t)r}K\left(\frac{S_t}{K}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{K}{S_t}\right)\right)\\
&=\mathrm{BS}_c(S_t,K,r,T-t,\sigma)\\
&\quad-K\left(\frac{S_t}{K}\right)^{-2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{K}{S_t}\right)\right)+\mathrm{e}^{-(T-t)r}S_t\left(\frac{S_t}{K}\right)^{-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{K}{S_t}\right)\right)\\
&=\mathrm{BS}_c(S_t,K,r,T-t,\sigma)-S_t\left(\frac{K}{S_t}\right)^{2r/\sigma^2}\mathrm{BS}_c(K/S_t,1,r,T-t,\sigma),
\end{aligned}$$

$0\leqslant t\leqslant T$. □

For $t\geqslant 0$, taking now

$$\tau_K:=\inf\left\{u\in[t,\infty)\ :\ S_u:=S_0\mathrm{e}^{\sigma B_u+(\mu-\sigma^2/2)u}\leqslant K\right\},$$

the recovery value received by the bond holder at time $\min\left(\tau_K,T\right)$ is K, and it can be priced as in the next proposition.

Proposition 9.7. *After discounting from time* $\min(\tau_K, T)$ *to time* $t \in [0, T]$*, we have*

$$\begin{aligned}&\mathbb{E}^*\left[K\mathrm{e}^{-(\min(\tau_K,T)-t)r} \mid \mathcal{F}_t\right]\\&= K\mathbb{1}_{\{\tau_K \geqslant t\}}\int_t^T \mathrm{e}^{-(u-t)r}\mathrm{d}\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big) + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big).\end{aligned}$$

Proof. We have

$$\begin{aligned}&\mathbb{E}^*\left[K\mathrm{e}^{-(\min(\tau_K,T)-t)r} \mid \mathcal{F}_t\right]\\&= \mathbb{E}^*\left[K\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} + K\mathrm{e}^{-(T-t)r}\mathbb{1}_{\{\tau_K>T\}} \mid \mathcal{F}_t\right]\\&= K\mathbb{E}^*\left[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} \mid \mathcal{F}_t\right] + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big)\\&= K\mathbb{1}_{\{\tau_K\geqslant t\}}\mathbb{E}^*\left[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} \mid \mathcal{F}_t\right] + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big)\\&= K\mathbb{1}_{\{\tau_K\geqslant t\}}\int_t^T \mathrm{e}^{-(u-t)r}\mathrm{d}\mathbb{P}\big(\tau_K \leqslant u \mid \mathcal{F}_t\big) + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big),\end{aligned}$$

$0 \leqslant t \leqslant T$. □

The above probabilities $\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big)$ and $\mathbb{P}^*\big(\tau_K > T \mid \mathcal{F}_t\big) = 1 - \mathbb{P}^*\big(\tau_K \leqslant T \mid \mathcal{F}_t\big)$ can be computed from (9.5) as

$$\begin{aligned}\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big) &= \Phi\left(\frac{\log(K/S_t) - (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)\\&\quad + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2}\Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)\\&= \mathbb{P}(S_u \leqslant K \mid \mathcal{F}_t) + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2}\Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right),\end{aligned}$$

with $S_t \geqslant K$ and $u > t$, from which the probability density function of the hitting time τ_K can be estimated by differentiation with respect to $u > t$. Note also that we have

$$\begin{aligned}\mathbb{P}^*\big(\tau_K < \infty \mid \mathcal{F}_t\big) &= \lim_{u\to\infty}\mathbb{P}^*\big(\tau_K \leqslant u \mid \mathcal{F}_t\big)\\&= \begin{cases}\left(\dfrac{K}{S_t}\right)^{-1+2r/\sigma^2} & \text{if } r > \sigma^2/2\\[2ex] 1 & \text{if } r \leqslant \sigma^2/2.\end{cases}\end{aligned}$$

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9.4 Correlated Default Times

In order to model correlated default and possible "domino effects", one can regard two given default times τ_1 and τ_2 as correlated random variables, with correlation coefficient

$$\rho := \frac{\mathrm{Cov}(\tau_1, \tau_2)}{\sqrt{\mathrm{Var}[\tau_1]\,\mathrm{Var}[\tau_2]}} \in [-1, 1].$$

Given two default events $\{\tau_1 \leqslant T\}$ and $\{\tau_2 \leqslant T\}$ with probabilities

$$\mathbb{P}(\tau_1 \leqslant T) = 1 - \exp\left(-\int_0^T \lambda_1(s)ds\right) \text{ and } \mathbb{P}(\tau_2 \leqslant T) = 1 - \exp\left(-\int_0^T \lambda_2(s)ds\right)$$

we can define the default correlation $\rho^D \in [-1, 1]$ as

$$\rho^D = \frac{C_\Sigma(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}}. \quad (9.6)$$

When the default probabilities are specified in the Merton model of credit risk as

$$\begin{aligned}
\mathbb{P}(\tau_i \leqslant T) &= \mathbb{P}(S_T < K) \\
&= \mathbb{P}\left(\mathrm{e}^{\sigma_i B_T + (\mu_i - \sigma_i^2/2)T} < \frac{K}{S_0}\right) \\
&= \mathbb{P}\left(B_T \leqslant -\frac{(\mu_i - \sigma_i^2/2)T}{\sigma_i} + \frac{1}{\sigma_i}\log\frac{K}{S_0}\right) \\
&= \Phi\left(\frac{\log(K/S_0) - (\mu_i - \sigma_i^2/2)T}{\sigma_i\sqrt{T}}\right), \qquad i = 1, 2,
\end{aligned}$$

where

$$\left(A_t^i\right)_{t\in\mathbb{R}_+} := \left(S_0 \mathrm{e}^{\sigma_i B_t + (\mu - \sigma_i^2/2)t}\right)_{t\in\mathbb{R}_+}, \qquad i = 1, 2,$$

the default correlation ρ^D becomes

$$\begin{aligned}
\rho^D &= \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}} \\
&= \frac{\Phi_\Sigma\left(\frac{\log(S_0/K) + (\mu_1 - \sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\log(S_0/K) + (\mu_2 - \sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}}.
\end{aligned}$$

When trying to build a dependence structure for the default times τ_1 and τ_2, the idea of Li (2000) is to use the normalized Gaussian copula $C_\Sigma(x, y)$, with

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

with correlation parameter $\rho \in [-1, 1]$, and to model the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$ as

$$\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) := C_\Sigma \left(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T) \right),$$

where C_Σ is given by (4.5). Namely, it was suggested to use a single *average correlation* estimate, see (8.1) page 82 of the Credit Metrics™ Technical Document Gupton et al. (1997), and Appendix F therein.

It is worth noting that the outcomes of this methodology have been discussed in a number of magazine articles in recent years, to name a few:

"Recipe for disaster: the formula that killed Wall Street", *Wired Magazine*, by F. Salmon (2009);

"The formula that felled Wall Street", *Financial Times Magazine*, by S. Jones (2009);

"Formula from hell", *Forbes.com*, by S. Lee (2009),

see also **here**.

On the other hand, a more proper definition of the default correlation ρ^D should be

$$\rho^D := \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1 - \mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1 - \mathbb{P}(\tau_2 \leqslant T))}},$$

which requires the actual computation of the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$. An exact expression for this joint default probability in the first passage time Black-Cox model, and the associated correlation, have been recently obtained in Li and Krehbiel (2016).

Multiple default times

Consider now a sequence $(\tau_k)_{k=1,2,\ldots,n}$ of random default times and, for more flexibility, a standardized random variable M with probability density function $\phi(m)$ and variance $\text{Var}[M] = 1$.

As in the Merton (1974) model, cf. § 9.1, a common practice, see Vašíček (1987), Gibson (2004), Hull and White (2004) is to parametrize the default probability associated to each τ_k by a conditioning of the form

$$\mathbb{P}(\tau_k \leqslant T \mid M = m) = \Phi\left(\frac{\Phi^{-1}(F_{\tau_k}(T)) - a_k m}{\sqrt{1-a_k^2}}\right), \tag{9.7}$$

see (9.3), where

$$F_{\tau_k}(T) := \mathbb{P}(\tau_k \leqslant T)$$

is the cumulative distribution function of τ_k, $k = 1, 2, \ldots, n$, and $a_1, \ldots, a_n \in (-1, 1)$. Note that we have

$$\begin{aligned}\mathbb{P}(\tau_k \leqslant T) &= \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leqslant T \mid M = m)\phi(m)dm \\ &= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k m}{\sqrt{1-a_k^2}}\right)\phi(m)dm, \qquad (9.8)\end{aligned}$$

and $\phi(m)$ can be typically chosen as a standard normal Gaussian probability density function.

Next, we present a dependence structure which implements of the Gaussian copula correlation method of Li (2000) in the case of multiple default times.

Definition 9.8. *Given $X_1, X_2, \ldots, X_n$ Gaussian samples defined as*

$$X_k := a_k M + \sqrt{1-a_k^2} Z_k, \quad k = 1, 2, \ldots, n, \tag{9.9}$$

conditionally to M, where $Z_1, Z_2, \ldots, Z_n$ are normal random variables with same cumulative distribution function Φ, independent of M, we construct the correlated default times $(\tau_1, \ldots, \tau_n)$ as

$$\tau_k := F_{\tau_k}^{-1}(\Phi_{X_k}(X_k)), \tag{9.10}$$

where $F_{\tau_k}^{-1}$ denotes the inverse function of F_{τ_k} and Φ_{X_k} denotes the cumulative distribution function of X_k, $k = 1, 2, \ldots, n$.

In the next proposition we compute the joint distribution of the default times $(\tau_1, \ldots, \tau_n)$ according to the above dependence structure.

Proposition 9.9. *The default times $(\tau_k)_{k=1,2,\ldots,n}$ have the joint distribution*

$$\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = C\big(\mathbb{P}(\tau_1 \leqslant y_1), \ldots, \mathbb{P}(\tau_n \leqslant y_n)\big), \tag{9.11}$$

where

$$\begin{aligned} &C(x_1, \ldots, x_n) \\ &:= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_k}^{-1}(x_1) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi_{X_k}^{-1}(x_n) - a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm,\end{aligned}$$

$x_1, x_2, \ldots, x_n \in [0,1]$.

Proof. We start by showing that Definition 9.8 recovers the conditional distribution (9.7), as follows:

$$\begin{aligned}
\mathbb{P}(\tau_k \leqslant T \mid M = m) &= \mathbb{P}\big(F_{\tau_k}^{-1}(\Phi_{X_k}(X_k)) \leqslant T \mid M = m\big) \\
&= \mathbb{P}\big(\Phi_{X_k}(X_k) \leqslant F_{\tau_k}(T) \mid M = m\big) \\
&= \mathbb{P}\big(X_k \leqslant \Phi_{X_k}^{-1}\big(F_{\tau_k}(T)\big) \mid M = m\big) \\
&= \mathbb{P}\left(a_k m + Z_k\sqrt{1-a_k^2} \leqslant \Phi_{X_k}^{-1}\big(F_{\tau_k}(T)\big)\right) \\
&= \mathbb{P}\left(Z_k\sqrt{1-a_k^2} \leqslant \Phi_{X_k}^{-1}\big(F_{\tau_k}(T)\big) - a_k m\right) \\
&= \mathbb{P}\left(Z_k \leqslant \frac{\Phi_{X_k}^{-1}\big(F_{\tau_k}(T)\big) - a_k m}{\sqrt{1-a_k^2}}\right) \\
&= \Phi\left(\frac{\Phi_{X_k}^{-1}\big(\mathbb{P}(\tau_k \leqslant T)\big) - a_k m}{\sqrt{1-a_k^2}}\right), \qquad k = 1,2,\ldots,n.
\end{aligned}$$

Note that the above recovers the correct marginal distributions (9.8), *i.e.* we have

$$\begin{aligned}
\mathbb{P}(\tau_k \leqslant y_k) &= \mathbb{P}(\tau_1 \leqslant \infty, \ldots, \tau_{k-1} \leqslant \infty, \tau_k \leqslant y_k, \tau_{k+1} \leqslant \infty, \ldots, \tau_n \leqslant \infty) \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_k}^{-1}\big(\mathbb{P}(\tau_k \leqslant y_k)\big) - a_k m}{\sqrt{1-a_k^2}}\right)\phi(m)dm \\
&= \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leqslant T \mid M = m)\phi(m)dm, \quad k = 1,2,\ldots,n.
\end{aligned}$$

Knowing that, given the sample $M = m$, the default times τ_k, $k = 1,2,\ldots,n$, are independent random variables, we can compute the joint distribution

$$\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n \mid M = m) \\
&\qquad = \mathbb{P}(\tau_1 \leqslant y_1 \mid M = m) \times \cdots \times \mathbb{P}(\tau_n \leqslant y_n \mid M = m),
\end{aligned}$$

conditionally to $M = m$. This yields

$$\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = \int_{-\infty}^{\infty} \mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n \mid M = m)\phi(m)dm \\
&= \int_{-\infty}^{\infty} \mathbb{P}(\tau_1 \leqslant y_1 \mid M = m) \cdots \mathbb{P}(\tau_n \leqslant y_n \mid M = m)\phi(m)dm \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_1}^{-1}\big(\mathbb{P}(\tau_1 \leqslant y_1)\big) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi_{X_n}^{-1}\big(\mathbb{P}(\tau_n \leqslant y_n)\big) - a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm,
\end{aligned}$$

which is (9.11). □

The next corollary deals with the case where M is normally distributed.

Corollary 9.10. *Assume that M has the standard normal distribution with probability density function ϕ and is independent of $Z_1, \ldots, Z_n$. Then, the joint distribution of the default times $(\tau_k)_{k=1,2,\ldots,n}$ is given by*

$$\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = C\big(\mathbb{P}(\tau_1 \leqslant y_1), \ldots, \mathbb{P}(\tau_n \leqslant y_n)\big),$$

where

$$C(x_1, \ldots, x_n) \\ := \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(x_1) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi^{-1}(x_n) - a_n m}{\sqrt{1-a_n^2}}\right) \phi(m) dm,$$

$x_1, x_2, \ldots, x_n \in [0,1]$, *is the Gaussian copula with covariance matrix*

$$\Sigma = \begin{bmatrix} 1 & a_1a_2 & \cdots & a_1a_{n-1} & a_1a_n \\ a_2a_1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & a_{n-1}a_n \\ a_na_1 & a_na_2 & \cdots & a_na_{n-1} & 1 \end{bmatrix}. \tag{9.12}$$

Proof. When the random variable M is normally distributed and independent of $Z_1, \ldots, Z_n$, the random vector $(X_1, \ldots, X_n)$ has the covariance matrix (9.12), and the function

$$C(x_1, \ldots, x_n) \\ := \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(x_1) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi^{-1}(x_n) - a_n m}{\sqrt{1-a_n^2}}\right) \phi(m) dm,$$

$x_1, x_2, \ldots, x_n \in [0,1]$, is a Gaussian copula on $[0,1]^n$, built as

$$C(x_1, \ldots, x_n) = F\big(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_n)\big),$$

from the Gaussian cumulative distribution function

$$\begin{aligned} F(x_1, \ldots, x_n) &:= \int_{-\infty}^{\infty} \Phi\left(\frac{x_1 - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{x_n - a_n m}{\sqrt{1-a_n^2}}\right) \phi(m) dm \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left(Z_1 \leqslant \frac{x_1 - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \mathbb{P}\left(Z_n \leqslant \frac{x_n - a_n m}{\sqrt{1-a_n^2}}\right) \phi(m) dm \\ &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n \mid M = m) \phi(m) dm \end{aligned}$$

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$$= \mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n), \qquad 0 \leqslant x_1, x_2, \ldots, x_n \leqslant 1,$$

of the vector $(X_1, \ldots, X_n)$, with covariance matrix given by (9.12). We conclude by Proposition 9.9. □

Exercises

Exercise 9.1 Compute the conditional probability density function of the default time τ defined in (9.2).

Exercise 9.2 Credit Default Contract. The assets of a company are modeled using a geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$ with drift $r > 0$ under the risk-neutral probability measure $\mathbb{P}^*$. A Credit Default Contract pays \$1 as soon as the asset S_t hits a level $K > 0$. Price this contract at time $t > 0$ assuming that $S_t > K$.

Exercise 9.3

a) Check that the vector $(X_1, X_2, \ldots, X_n)$ defined in (9.9) has the covariance matrix given by (9.12).
b) Show that the vector $(X_1, X_2, \ldots, X_n)$, with covariance matrix (9.12) has standard Gaussian marginals.
c) By computing explicitly the probability density function of $(X_1, \ldots, X_n)$, recover the fact that it is a jointly Gaussian random vector with covariance matrix (9.12).

Exercise 9.4 Compute the inverse Σ^{-1} of the covariance matrix (9.12) in case $n = 2$.

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