Chapter 10
Credit Risk - Reduced-Form Approach

In this chapter, credit risk is estimated by modeling default probabilities using stochastic failure rate processes. In addition, information on default events is incorporated to the model by the use of exogeneous random variables and enlargement of filtrations. This is in contrast to the structural approach to credit risk of Chapter 9, in which bankruptcy is modeled from a firm's asset value. Applications are given to the pricing of default bonds.

10.1 Survival Probabilities

The reduced-form approach to credit risk relies on the concept of survival probability, defined as the probability $\mathbb{P}(\tau > t)$ that a random system with lifetime τ survives at least over t years, $t > 0$. Assuming that survival probabilities $\mathbb{P}(\tau > t)$ are strictly positive for all $t > 0$, we can compute the conditional probability for that system to survive up to time T, given that it was still functioning at time $t \in [0, T]$, as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T,$$

with

$$\mathbb{P}(\tau \leqslant T \mid \tau > t) = 1 - \mathbb{P}(\tau > T \mid \tau > t)$$

$$= \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}$$
$$= \frac{\mathbb{P}(\tau \leqslant T) - \mathbb{P}(\tau \leqslant t)}{\mathbb{P}(\tau > t)}$$
$$= \frac{\mathbb{P}(t < \tau \leqslant T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T. \tag{10.1}$$

Such survival probabilities are typically found in life (or mortality) tables:

Age t	$\mathbb{P}(\tau \leqslant t+1 \mid \tau > t)$
20	0.0894%
30	0.1008%
40	0.2038%
50	0.4458%
60	0.9827%

Table 10.1: Mortality table.

The corresponding conditional survival probability distribution can be computed as follows:

$$\begin{aligned}\mathbb{P}(\tau \in dx \mid \tau > t) &= \mathbb{P}(x < \tau \leqslant x + dx \mid \tau > t)\\ &= \mathbb{P}(\tau \leqslant x + dx \mid \tau > t) - \mathbb{P}(\tau \leqslant x \mid \tau > t)\\ &= \frac{\mathbb{P}(\tau \leqslant x + dx) - \mathbb{P}(\tau \leqslant x)}{\mathbb{P}(\tau > t)}\\ &= \frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau \leqslant x)\\ &= -\frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau > x), \qquad x > t.\end{aligned}$$

Proposition 10.1. *The* failure rate *function, defined as*

$$\lambda(t) := \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt},$$

satisfies

$$\mathbb{P}(\tau > t) = \exp\left(-\int_0^t \lambda(u) du\right), \qquad t \geqslant 0. \tag{10.2}$$

Proof. By (10.1), we have

$$\lambda(t) := \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt}$$

$$
\begin{aligned}
&= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(t < \tau \leqslant t + dt)}{dt} \\
&= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > t + dt)}{dt} \\
&= -\frac{d}{dt} \log \mathbb{P}(\tau > t) \\
&= -\frac{1}{\mathbb{P}(\tau > t)} \frac{d}{dt} \mathbb{P}(\tau > t), \qquad t > 0,
\end{aligned}
$$

and the differential equation

$$
\frac{d}{dt}\mathbb{P}(\tau > t) = -\lambda(t)\mathbb{P}(\tau > t),
$$

which can be solved as in (10.2) under the initial condition $\mathbb{P}(\tau > 0) = 1$.
□

Proposition 10.1 allows us to rewrite the (conditional) survival probability as

$$
\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp\left(-\int_t^T \lambda(u) du \right), \qquad 0 \leqslant t \leqslant T,
$$

with

$$
\mathbb{P}(\tau > t + h \mid \tau > t) = \mathrm{e}^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \qquad [h \searrow 0],
$$

and

$$
\mathbb{P}(\tau \leqslant t + h \mid \tau > t) = 1 - \mathrm{e}^{-\lambda(t)h} \simeq \lambda(t)h, \qquad [h \searrow 0],
$$

as h tends to 0. When the failure rate $\lambda(t) = \lambda > 0$ is a constant function of time, Relation (10.2) shows that

$$
\mathbb{P}(\tau > T) = \mathrm{e}^{-\lambda T}, \qquad T \geqslant 0,
$$

i.e. τ has the exponential distribution with parameter λ. Note that given $(\tau_n)_{n \geqslant 1}$ a sequence of *i.i.d.* exponentially distributed random variables, letting

$$
T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \qquad n \geqslant 1,
$$

defines the sequence of jump times of a standard Poisson process with intensity $\lambda > 0$, see Proposition 3.3.

10.2 Stochastic Default

When the random time τ is a *stopping time* with respect to $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ we have

$$\{\tau > t\} \in \mathcal{F}_t, \qquad t \geqslant 0,$$

i.e. the knowledge of whether default or bankruptcy has already occurred at time t is contained in $\mathcal{F}_t$, $t \in \mathbb{R}_+$, cf. *e.g.* Section 14.3 of Privault (2022). As a consequence, we can write

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{E}\left[\mathbb{1}_{\{\tau > t\}} \mid \mathcal{F}_t\right] = \mathbb{1}_{\{\tau > t\}}, \qquad t \geqslant 0.$$

In what follows we will not assume that τ is an $\mathcal{F}_t$-stopping time, and by analogy with (10.2) we will write $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$ as

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t \geqslant 0, \tag{10.3}$$

where the failure rate function $(\lambda_t)_{t\in\mathbb{R}_+}$ is modeled as a random process adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Remark 10.2. *The process $(\lambda_t)_{t\in\mathbb{R}_+}$ can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In Lando (1998), the process $(\lambda_t)_{t\in\mathbb{R}_+}$ is constructed as $\lambda_t := h(X_t)$, $t \in \mathbb{R}_+$, where h is a nonnegative function and $(X_t)_{t\in\mathbb{R}_+}$ is a stochastic process generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. The default time τ is then* defined *as*

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \int_0^t h(X_u)du \geqslant L\right\},$$

where L is an exponentially distributed random variable with parameter $\mu > 0$ and distribution function $\mathbb{P}(L > x) = \mathrm{e}^{-\mu x}$, $x \geqslant 0$, independent of $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

In Remark 10.2, as τ is not an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time, we have

$$\begin{aligned}\mathbb{P}(\tau > t \mid \mathcal{F}_t) &= \mathbb{P}\left(\int_0^t h(X_u)du < L \;\middle|\; \mathcal{F}_t\right)\\ &= \exp\left(-\mu\int_0^t h(X_u)du\right)\\ &= \exp\left(-\mu\int_0^t \lambda_u du\right), \qquad t \geqslant 0.\end{aligned}$$

Definition 10.3. *Let $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be the filtration defined by $\mathcal{G}_\infty := \mathcal{F}_\infty \vee \sigma(\tau)$ and*

$$\mathcal{G}_t := \big\{B \in \mathcal{G}_\infty \ : \ \exists A \in \mathcal{F}_t \text{ such that } A \cap \{\tau > t\} = B \cap \{\tau > t\}\big\}, \tag{10.4}$$

with $\mathcal{F}_t \subset \mathcal{G}_t$, $t \geqslant 0$.

In other words, $\mathcal{G}_t$ contains insider information on whether default at time τ has occurred or not before time t, and τ is a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-stopping time. Note that this information on τ may not be available to a generic user who has only access to the smaller filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. The next key Lemma 10.4, see Lando (1998), Guo et al. (2007), allows us to price a contingent claim given the information in the larger filtration $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$, by only using information in $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and factoring in the default rate factor $\exp\left(-\int_t^T \lambda_u du\right)$.

Lemma 10.4. *(Guo et al. (2007), Theorem 1) For any $\mathcal{F}_T$-measurable integrable random variable F, we have*

$$\begin{aligned}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\mathbb{P}(\tau>T \mid \tau>t) \mid \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T.\end{aligned}$$

Proof. By (10.3) we have

$$\frac{\mathbb{P}(\tau>T \mid \mathcal{F}_T)}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} = \frac{\mathrm{e}^{-\int_0^T \lambda_u du}}{\mathrm{e}^{-\int_0^t \lambda_u du}} = \exp\left(-\int_t^T \lambda_u du\right),$$

hence, since F is $\mathcal{F}_T$-measurable,

$$\begin{aligned}&\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\frac{\mathbb{P}(\tau>T \mid \mathcal{F}_T)}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \,\Big|\, \mathcal{F}_t\right] \\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)}\mathbb{E}\left[F\mathbb{E}[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t\right] \\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)}\mathbb{E}\left[\mathbb{E}[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\big|\, \mathcal{F}_t\right]}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)} \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\big|\, \mathcal{G}_t\right] \\ &= \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\big|\, \mathcal{G}_t\right], \qquad 0 \leqslant t \leqslant T.\end{aligned}$$

In the last step of the above argument, we used the key relation

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\Big|\, \mathcal{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathcal{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right],$$

cf. Relation (75.2) in § XX-75 page 186 of Dellacherie et al. (1992), Theorem VI-3-14 page 371 of Protter (2004), and Lemma 3.1 of Elliott et al.

(2000), under the conditional probability measure $\mathbb{P}_{|\mathcal{F}_t}$, $0 \leqslant t \leqslant T$. Indeed, according to (10.4), for any $B \in \mathcal{G}_t$ we have, for some event $A \in \mathcal{F}_t$,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_B \mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] &= \mathbb{E}\left[\mathbb{1}_{B\cap\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{A\cap\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E}\big[F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\big]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E}\big[F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\big]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathcal{F}_t]}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E}\big[F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\big]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E}\big[F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\big]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_B \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E}\big[F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\big]\right],
\end{aligned}$$

hence by a standard characterization of conditional expectations, see *e.g.* Relation (A.26), we have

$$\mathbb{E}\big[\mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\big] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)} \mathbb{E}\big[F \mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\big]$$

□

Taking $F = 1$ in Lemma 10.4 allows one to write the survival probability up to time T, given the information known up to time t, as

$$\begin{aligned}
\mathbb{P}(\tau > T \mid \mathcal{G}_t) &= \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right) \;\middle|\; \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{10.5}$$

In particular, applying Lemma 10.4 for $t = T$ and $F = 1$ shows that

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \mid \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}},$$

which shows that $\{\tau > t\} \in \mathcal{G}_t$ for all $t > 0$, and recovers the fact that τ is a $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$-stopping time, while in general, τ is not $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time.

The computation of $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ according to (10.5) is then similar to that of a bond price, by considering the failure rate $\lambda(t)$ as a "virtual" short-term interest rate. In particular the failure rate $\lambda(t, T)$ can be modeled in the HJM framework, cf. *e.g.* Chapter 18.3 of Privault (2022), and

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda(t,u)du\right) \;\Big|\; \mathcal{F}_t\right]$$

can then be computed by applying HJM bond pricing techniques.

The computation of expectations given $\mathcal{G}_t$ as in Lemma 10.4 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration $\mathcal{G}_t$ while the ordinary trader has only access to $\mathcal{F}_t$, therefore generating two different prices $\mathbb{E}^*[F \mid \mathcal{F}_t]$ and $\mathbb{E}^*[F \mid \mathcal{G}_t]$ for the same claim payoff F under the same risk-neutral probability measure $\mathbb{P}^*$. This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a $(\mathcal{F}_t)_{t\geqslant 0}$-martingale *vs.* a $(\mathcal{G}_t)_{t\geqslant 0}$-martingale instead of using different forward measures as in *e.g.* § 19.1 of Privault (2022). This can be obtained by the technique of enlargement of filtration, cf. Jeulin (1980), Jacod (1985), Yor (1985), Elliott and Jeanblanc (1999).

10.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition $P(T,T) = \$1$ according to which the bond payoff at maturity is always equal to \$1, and default does not occurs. In this chapter we allow for the possibility of default at a random time τ, in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price $P_d(t,T)$ at time t of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned} P_d(t,T) = {} & \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}}\exp\left(-\int_t^T r_u du\right) \;\Big|\; \mathcal{G}_t\right] \\ & + \mathbb{E}^*\left[\xi\mathbb{1}_{\{\tau\leqslant T\}}\exp\left(-\int_t^T r_u du\right) \;\Big|\; \mathcal{G}_t\right], \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

Proposition 10.5. *The default bond with maturity T and default time τ can be priced at time $t \in [0,T]$ as*

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T(r_u+\lambda_u)du\right) \Bigm| \mathcal{F}_t\right]$$
$$+\mathbb{E}^*\left[\xi\mathbb{1}_{\{\tau\leqslant T\}}\exp\left(-\int_t^T r_u du\right) \Bigm| \mathcal{G}_t\right], \quad 0\leqslant t\leqslant T.$$

Proof. We take $F = \exp\left(-\int_t^T r_u du\right)$ in Lemma 10.4, which shows that

$$\mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}}\exp\left(-\int_t^T r_u du\right) \Bigm| \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T(r_u+\lambda_u)du\right) \Bigm| \mathcal{F}_t\right],$$

cf. *e.g.* Lando (1998), Duffie and Singleton (2003), Guo et al. (2007). □

In the case of complete default (zero-recovery), we have $\xi = 0$ and

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T(r_s+\lambda_s)ds\right) \Bigm| \mathcal{F}_t\right], \quad 0\leqslant t\leqslant T. \quad (10.6)$$

From the above expression (10.6) we note that the effect of the presence of a default time τ is to decrease the bond price, which can be viewed as an increase of the short rate by the amount λ_u. In a simple setting where the interest rate $r > 0$ and failure rate $\lambda > 0$ are constant, the default bond price becomes

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}}\mathrm{e}^{-(r+\lambda)(T-t)}, \qquad 0\leqslant t\leqslant T.$$

In this case, the failure rate λ can be estimated at time $t\in[0,T]$ from a default bond price $P_d(t,T)$ and a non-default bond price $P(t,T) = \mathrm{e}^{-(T-t)r}$ as

$$\lambda = \frac{1}{T-t}\log\frac{P(t,T)}{P_d(t,T)}.$$

Finally, as in *e.g.* Proposition 19.1 in Privault (2022) the bond price (10.6) can also be expressed under the forward measure $\widehat{\mathbb{P}}$ with maturity T, as

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T(r_s+\lambda_s)ds\right) \Bigm| \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Bigm| \mathcal{F}_t\right]\widehat{\mathbb{E}}\left[\exp\left(-\int_t^T \lambda_s ds\right) \Bigm| \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}N_t\widehat{\mathbb{P}}(\tau>T \mid \mathcal{G}_t), \end{aligned}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is the numéraire process

$$N_t := P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Bigm| \mathcal{F}_t\right], \qquad 0\leqslant t\leqslant T,$$

and by (10.5),

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$$\widehat{\mathbb{P}}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \widehat{\mathbb{E}} \left[\exp\left(- \int_t^T \lambda_s ds \right) \Bigm| \mathcal{F}_t \right]$$

is the survival probability under the forward measure $\widehat{\mathbb{P}}$ defined as

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} := \frac{N_T}{N_0} e^{-\int_0^T r_t dt},$$

see Chen and Huang (2001) and Chen et al. (2008).

Estimating the default rates

Recall that the price of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(- \int_t^T (r_u + \lambda_u) du \right) \Bigm| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}^* \left[\xi \mathbb{1}_{\{\tau \leqslant T\}} \exp\left(- \int_t^T r_u du \right) \Bigm| \mathcal{G}_t \right], \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

where ξ denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure

$$\{t = T_0 < T_1 < \cdots < T_n = T\},$$

where

$$r(t) = \sum_{l=0}^{n-1} r_l \mathbb{1}_{(T_l, T_{l+1}]}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1} \lambda_l \mathbb{1}_{(T_l, T_{l+1}]}(t), \quad t \geqslant 0. \tag{10.7}$$

i) Estimating the default rates from default bond prices.

From Proposition 10.5, we have

$$\begin{aligned} P_d(t,T_k) &= \mathbb{1}_{\{\tau>t\}} \exp\left(- \int_t^{T_k} (r(u) + \lambda(u)) du \right) \\ &= \mathbb{1}_{\{\tau>t\}} \exp\left(- \sum_{l=0}^{k-1} (r_l + \lambda_l)(T_{l+1} - T_l) \right), \end{aligned}$$

$k = 1, 2, \ldots, n$, from which we can infer

$$\lambda_k = -r_k + \frac{1}{T_{k+1} - T_k} \log \frac{P_d(t,T_k)}{P_d(t,T_{k+1})} > 0, \quad k = 0, 1, \ldots, n-1.$$

ii) Estimating (implied) default probabilities $\mathbb{P}^*(\tau < T \mid \mathcal{G}_t)$ from default rates.

Based on the expression

$$\mathbb{P}^*(\tau > T \mid \mathcal{G}_t) = \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \Big| \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T, \tag{10.8}$$

of the survival probability up to time T, see (10.3), and given the information known up to time t, in terms of the hazard rate process $(\lambda_u)_{u\in\mathbb{R}_+}$ adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we find

$$\mathbb{P}(\tau > T \mid \mathcal{G}_{T_k}) = \mathbb{1}_{\{\tau>T_k\}} \exp\left(-\int_{T_k}^T \lambda_u du\right) = \mathbb{1}_{\{\tau>t\}} \exp\left(-\sum_{l=k}^{n-1} \lambda_l (T_{l+1} - T_l)\right), \quad k = 0, 1, \ldots, n-1,$$

where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leqslant u\} : 0 \leqslant u \leqslant t), \quad t \geqslant 0,$$

i.e. $\mathcal{G}_t$ contains the additional information on whether default at time τ has occurred or not before time t.

In Table 10.2, bond ratings are determined according to hazard (or failure) rate thresholds.

Bond Credit	Moody's		S & P	
Ratings	Municipal	Corporate	Municipal	Corporate
Aaa/AAAs	0.00	0.52	0.00	0.60
Aa/AA	0.06	0.52	0.00	1.50
A/A	0.03	1.29	0.23	2.91
Baa/BBB	0.13	4.64	0.32	10.29
Ba/BB	2.65	19.12	1.74	29.93
B/B	11.86	43.34	8.48	53.72
Caa-C/CCC-C	16.58	69.18	44.81	69.19
Investment Grade	0.07	2.09	0.20	4.14
Non-Invest. Grade	4.29	31.37	7.37	42.35
All	0.10	9.70	0.29	12.98

Table 10.2: Cumulative historic default rates (in percentage).*

* Sources: Moody's, S&P.

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Exercises

Exercise 10.1 Consider a standard zero-coupon bond with constant yield $r > 0$ and a defaultable (risky) bond with constant yield r_d and default probability $\alpha \in (0,1)$. Find a relation between r, r_d, α and the bond maturity T.

Exercise 10.2 A standard zero-coupon bond with constant yield $r > 0$ and maturity T is priced $P(t,T) = \mathrm{e}^{-(T-t)r}$ at time $t \in [0,T]$. Assume that the company can get bankrupt at a random time $t+\tau$, and default on its final \$1 payment if $\tau < T-t$.

a) Explain why the defaultable bond price $P_d(t,T)$ can be expressed as

$$P_d(t,T) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\mathbb{1}_{\{\tau > T-t\}}\big]. \tag{10.9}$$

b) Assuming that the default time τ is exponentially distributed with parameter $\lambda > 0$, compute the default bond price $P_d(t,T)$ using (10.9).

c) Find a formula that can estimate the parameter λ from the risk-free rate r and the market data $P_M(t,T)$ of the defaultable bond price at time $t \in [0,T]$.

Exercise 10.3 Consider an interest rate process $(r_t)_{t\in\mathbb{R}_+}$ and a default rate process $(\lambda_t)_{t\in\mathbb{R}_+}$, modeled according to the Vasicek processes

$$\begin{cases} dr_t = -ar_t dt + \sigma dB_t^{(1)}, \\ d\lambda_t = -b\lambda_t dt + \eta dB_t^{(2)}, \end{cases}$$

where $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-Brownian motions with correlation $\rho \in [-1,1]$, and $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$.

a) Taking $r_0 := 0$, show that we have

$$\int_t^T r_s ds = C(a,t,T)r_t + \sigma \int_t^T C(a,s,T) dB_s^{(1)},$$

and

$$\int_t^T \lambda_s ds = C(b,t,T)\lambda_t + \eta \int_t^T C(b,s,T) dB_s^{(2)},$$

where

$$C(a,t,T) = -\frac{1}{a}(\mathrm{e}^{-(T-t)a} - 1).$$

b) Show that the random variable

$$\int_t^T r_s ds + \int_t^T \lambda_s ds$$

is has a Gaussian distribution, and compute its conditional mean

$$\mathbb{E}^*\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathcal{F}_t\right]$$

and variance

$$\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathcal{F}_t\right],$$

conditionally to $\mathcal{F}_t$.

Exercise 10.4 (Exercise 10.3 continued). Consider a (random) default time τ with cumulative distribution function

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t \geqslant 0,$$

where λ_t is a (random) default rate process which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Recall that the probability of survival up to time T, given the information known up to time t, is given by

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathcal{F}_t\right],$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\} : 0 \leqslant u \leqslant t)$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. In this framework, the price $P(t,T)$ of defaultable bond with maturity T, short-term interest rate r_t and (random) default time τ is given by

$$\begin{aligned} P(t,T) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}}\exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right] \qquad (10.10)\\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \,\Big|\, \mathcal{F}_t\right]. \end{aligned}$$

a) Give a justification for the fact that

$$\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \,\Big|\, \mathcal{F}_t\right]$$

can be written as a function $F(t, r_t, \lambda_t)$ of t, r_t and λ_t, $t \in [0,T]$.

b) Show that

$$t \longmapsto \exp\left(-\int_0^t (r_s+\lambda_s)ds\right)\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u)du\right) \Bigm| \mathcal{F}_t\right]$$

is an $(\mathcal{F}_t)_{t\geqslant 0}$-martingale under $\mathbb{P}$.

c) Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t,x,y)$.

d) Compute $P(t,T)$ from its expression (10.10) as a conditional expectation.

e) Show that the solution $F(t,x,y)$ to the 2-dimensional PDE of Question (c) is

$$\begin{aligned} F(t,x,y) &= \exp\left(-C(a,t,T)x - C(b,t,T)y\right) \\ &\quad\times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &\quad\times \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right). \end{aligned}$$

f) Show that the defaultable bond price $P(t,T)$ can also be written as

$$P(t,T) = \mathrm{e}^{U(t,T)}\mathbb{P}(\tau > T \mid \mathcal{G}_t)\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Bigm| \mathcal{F}_t\right],$$

where

$$U(t,T) = \rho\frac{\sigma\eta}{ab}\left(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T)\right).$$

g) By partial differentiation of $\log P(t,T)$ with respect to T, compute the corresponding instantaneous short rate

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

h) Show that $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ can be written using an HJM type default rate as

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\exp\left(-\int_t^T f_2(t,u)du\right),$$

where

$$f_2(t,u) = \lambda_t \mathrm{e}^{-(u-t)b} - \frac{\eta^2}{2}C^2(b,t,u).$$

i) Show how the result of Question (f) can be simplified when the processes $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are independent.