Chapter 4
Correlation and Dependence

Correlation and dependence play a capital role in risk management, in particular when assessing or preventing any potential "domino effect" arising from interactions between different entities exposed to uncertainties.* This chapter presents several standard models for the statistical interactions that arise in the modeling of correlated risk. For this, we use the concept of copulas, which can model the uncertainty and dependence properties observed between random variables or data samples.

4.1 Joint Bernoulli Distribution

Our study of dependence structures starts with the simplest case of two correlated random variables X and Y, each of them taking only two possible values. For this, let X and Y by two Bernoulli random variables, with

$$p_X = \mathbb{P}(X = 1) = \mathbb{E}[\mathbb{1}_{\{X=1\}}] \quad \text{and} \quad p_Y = \mathbb{P}(Y = 1) = \mathbb{E}[\mathbb{1}_{\{Y=1\}}]$$

and correlation coefficient

$$\rho := \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}}$$

* Correlation does not imply causation. Try "Spurious Correlations".

$$= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}}$$
$$= \frac{\mathbb{P}(X = 1 \text{ and } Y = 1) - p_X p_Y}{\sqrt{p_X(1-p_X)p_Y(1-p_Y)}},$$

with $\rho \in [-1,1]$ from the Cauchy-Schwarz inequality. We note that in this case, the joint distribution $\mathbb{P}(X = i \text{ and } Y = j)$, $i,j = 0,1$, is fully determined by the data of $p_X = \mathbb{P}(X = 1)$, $p_Y = \mathbb{P}(Y = 1)$ and the correlation coefficient $\rho \in [-1,1]$, as

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = \mathbb{E}[XY] \\ \qquad = p_X p_Y + \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = \mathbb{E}[(1-X)Y] = \mathbb{P}(Y = 1) - \mathbb{P}(X = 1 \text{ and } Y = 1) \\ \qquad = (1-p_X)p_Y - \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = \mathbb{E}[X(1-Y)] = \mathbb{P}(X = 1) - \mathbb{P}(X = 1 \text{ and } Y = 1) \\ \qquad = p_X(1-p_Y) - \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = \mathbb{E}[(1-X)(1-Y)] \\ \qquad = (1-p_X)(1-p_Y) + \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \end{cases}$$

see Exercise 4.2.

4.2 Joint Gaussian Distribution

Consider now two *centered* Gaussian random variables $X \simeq \mathcal{N}(0,\sigma^2)$ and $Y \simeq \mathcal{N}(0,\eta^2)$ with probability density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-x^2/(2\sigma^2)} \quad \text{and} \quad f_Y(x) = \frac{1}{\sqrt{2\pi\eta^2}}\mathrm{e}^{-x^2/(2\eta^2)}, \qquad x \in \mathbb{R}.$$

Let

$$\rho = \mathrm{corr}\,(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}}.$$

When the covariance matrix

$$\Sigma := \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho\sigma\eta \\ \rho\sigma\eta & \eta^2 \end{bmatrix} \tag{4.1}$$

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with determinant

$$\begin{aligned}\det \Sigma &= \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[XY])^2 \\ &= \mathbb{E}[X^2]\mathbb{E}[Y^2](1 - (\operatorname{corr}(X,Y))^2) \\ &\geqslant 0,\end{aligned}$$

is invertible, there exists a probability density function

$$\begin{aligned} f_\Sigma(x,y) &= \frac{1}{\sqrt{2\pi \det \Sigma}} \exp\left(-\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^\top \Sigma^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right) \qquad (4.2) \\ &= \frac{1}{\sqrt{2\pi \det \Sigma}} \exp\left(-\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right), \end{aligned}$$

with respective marginals $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(0,\eta^2)$.

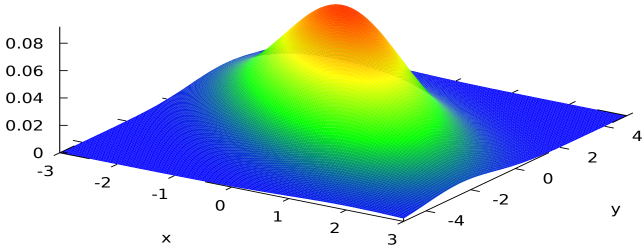

Fig. 4.1: Joint Gaussian probability density.

The probability density function (4.2) is called the centered joint (bivariate) Gaussian probability density with covariance matrix Σ.

Note that when $\rho = \operatorname{corr}(X,Y) = \pm 1$ we have $\det \Sigma = 0$ and the joint probability density function $f_\Sigma(x,y)$ is *not defined*.

Definition 4.1. *A random vector $(X_1,\ldots,X_n)$ is said to have a multivariate centered Gaussian distribution if every linear combination*

$$a_1X_1 + \cdots + a_nX_n, \quad a_1,\ldots,a_n \in \mathbb{R}, \; n \geqslant 1,$$

has a centered Gaussian distribution.

Recall that if $(X_1,\ldots,X_n)$ has a multivariate centered Gaussian distribution, then its probability density function takes the form

$$f_\Sigma(x_1,\ldots,x_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(x_1,\ldots,x_n)^T\Sigma^{-1}(x_1,\ldots,x_n)\right),$$

$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where Σ is the covariance matrix

$$\Sigma = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}(X_1,X_2) & \cdots & \mathrm{Cov}(X_1,X_{n-1}) & \mathrm{Cov}(X_1,X_n) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Var}[X_2] & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \mathrm{Var}[X_{n-1}] & \mathrm{Cov}(X_{n-1},X_n) \\ \mathrm{Cov}(X_1,X_n) & \mathrm{Cov}(X_2,X_n) & \cdots & \mathrm{Cov}(X_{n-1},X_n) & \mathrm{Var}[X_n] \end{bmatrix}.$$

The next remark plays an important role in the modeling of joint default probabilities, see **here** for a detailed discussion.

Remark 4.2. *There exist couples (X, Y) of random variables with Gaussian marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$, such that*

i) *(X, Y) does not have the bivariate Gaussian distribution with probability density function $f_\Sigma(x, y)$, where Σ is the covariance matrix (4.1) of (X, Y).*
ii) *the random variable $X + Y$ is not even Gaussian.*

Proof. See Exercise 4.5. □

4.3 Copulas and Dependence Structures

The word copula derives from the Latin noun for a “link” or “tie” that connects two different objects or concepts.

Definition 4.3. *A two-dimensional copula is any joint cumulative distribution function*

$$\begin{aligned} C : [0,1] \times [0,1] &\longrightarrow [0,1] \\ (u,v) &\longmapsto C(u,v) \end{aligned}$$

with uniform $[0,1]$-valued marginals.

In other words, any copula function $C(u, v)$ can be written as

$$C(u,v) = \mathbb{P}(U \leqslant u \text{ and } V \leqslant v), \qquad 0 \leqslant u, v \leqslant 1,$$

where U and V are uniform $[0,1]$-valued random variables.

Examples.

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i) The copula corresponding to independent uniform random variables (U, V) is given by

$$\begin{aligned} C(u,v) &= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}(U \leqslant u)\mathbb{P}(V \leqslant v) \\ &= uv, \qquad 0 \leqslant u, v \leqslant 1. \end{aligned}$$

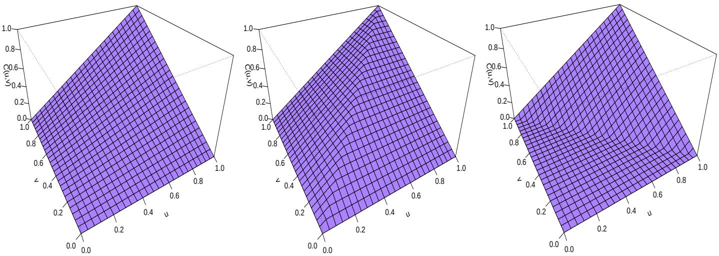
ii) The copula corresponding to the fully correlated case $U = V$ is given by

$$\begin{aligned} C(u,v) &= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}(U \leqslant \min(u,v)) \\ &= \min(u,v), \qquad 0 \leqslant u, v \leqslant 1. \end{aligned}$$

iii) The copula corresponding to the fully anticorrelated case $U = 1 - V$ is given by

$$\begin{aligned} C(u,v) &:= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}(U \leqslant u \text{ and } 1 - U \leqslant v) \\ &= \mathbb{P}(1 - v \leqslant U \leqslant u) \\ &= (u + v - 1)^+, \qquad 0 \leqslant u, v \leqslant 1. \end{aligned}$$

The above copulas are plotted in Figure 4.3a.



(a) uv. (b) $\min(u,v)$. (c) $(u+v-1)^+$.

Fig. 4.2: Copula graphs $C(u,v) = uv$, $C(u,v) = \min(u,v)$, $C(u,v) = (u+v-1)^+$.

In what follows, F_X^{-1} denotes the inverse of the *Cumulative Distribution Function* F_X of X.

Lemma 4.4. *Assume that the random variable X has a* continuous and strictly increasing *cumulative distribution function* $F_X(x) := \mathbb{P}(X \leqslant x)$, $x \in \mathbb{R}$. *Then, $U := F_X(X)$ is uniformly distributed on* $[0,1]$.

Proof. We have

$$\begin{aligned} F_U(u) &= \mathbb{P}(U \leqslant u) \\ &= \mathbb{P}(F_X(X) \leqslant u) \\ &= \mathbb{P}(X \leqslant F_X^{-1}(u)) \\ &= F_X(F_X^{-1}(u)) \\ &= u, \qquad 0 \leqslant u \leqslant 1. \end{aligned}$$

□

As in Lemma 4.4, given (X, Y) a couple of random variables with joint cumulative distribution function

$$F_{(X,Y)}(x, y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y), \qquad x, y \in \mathbb{R},$$

and continuous strictly increasing marginal cumulative distribution functions

$$F_X(x) = F_{(X,Y)}(x, \infty) = \mathbb{P}(X \leqslant x) \text{ and } F_Y(y) = F_{(X,Y)}(\infty, y) = \mathbb{P}(Y \leqslant y),$$

we note the following points.

i) The random variables

$$U := F_X(X) \quad \text{and} \quad V := F_Y(Y)$$

are uniformly distributed on $[0, 1]$.

ii) The copula function

$$C_{(X,Y)}(u, v) := \mathbb{P}(U \leqslant u \text{ and } V \leqslant v), \qquad 0 \leqslant u, v \leqslant 1,$$

satisfies

$$\begin{aligned} C_{(X,Y)}(u, v) &:= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}\big(F_X(X) \leqslant u \text{ and } F_Y(Y) \leqslant v\big) \\ &= \mathbb{P}\big(X \leqslant F_X^{-1}(u) \text{ and } Y \leqslant F_Y^{-1}(v)\big) \\ &= F_{(X,Y)}\big(F_X^{-1}(u), F_Y^{-1}(v)\big), \qquad 0 \leqslant u, v \leqslant 1. \end{aligned}$$

iii) The joint cumulative distribution function of (X, Y) can be recovered from the copula $C_{(X,Y)}$ and the marginal cumulative distribution functions F_X, F_Y as

$$\begin{aligned} F_{(X,Y)}(x, y) &= \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) \\ &= \mathbb{P}\big(F_X(X) \leqslant F_X(x) \text{ and } F_Y(Y) \leqslant F_Y(v)\big) \end{aligned}$$

$$= \mathbb{P}\big(U \leqslant F_X(x) \text{ and } V \leqslant F_Y(v)\big)$$
$$= C_{(X,Y)}\big(F_X(x), F_Y(y)\big), \qquad x, y \in \mathbb{R}.$$

Higher dimensional copulas

Definition 4.5. *An n-dimensional copula is any joint cumulative distribution function*

$$\begin{array}{rcl} C : [0,1] \times \cdots \times [0,1] & \longrightarrow & [0,1] \\ (u_1, \ldots, u_n) & \longmapsto & C(u_1, \ldots, u_n) \end{array}$$

of n uniform $[0,1]$-valued random variables.

Consider the joint cumulative distribution function

$$F_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n) := \mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n)$$

of a family $(X_1, \ldots, X_n)$ of random variables with marginal cumulative distribution functions

$$F_{X_i}(x) = F_{(X_1,\ldots,X_n)}(\infty, \ldots, +infty, x, \infty, \ldots, \infty), \quad x \in \mathbb{R},$$

$i = 1, 2, \ldots, n$. The copula defined in the next Sklar's Theorem 4.6 encodes the dependence structure of the vector $(X_1, \ldots, X_n)$.

Theorem 4.6. *[Sklar's theorem* (Sklar (1959; 2010))] Given a joint cumulative distribution function $F_{(X_1,\ldots,X_n)}$, there exists an n-dimensional copula $C(u_1, \ldots, u_n)$ such that*

$$F_{(X_1,\ldots,X_n)}(x_1, x_2, \ldots, x_n) = C\big(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\big),$$

$x_1, x_2, \ldots, x_n \in \mathbb{R}$.

The following corollary is a consequence of Sklar's Theorem 4.6.

Corollary 4.7. *Assume that the marginal distribution functions F_{X_i} are continuous and strictly increasing. Then the joint cumulative distribution function $F_{(X_1,\ldots,X_n)}$ defines a n-dimensional copula*

$$C(u_1, \ldots, u_n) := F_{(X_1,\ldots,X_n)}\big(F_{X_1}^{-1}(u_1), \ldots, F_{X_n}^{-1}(u_n)\big), \tag{4.3}$$

* "The author considers continuous non-decreasing functions C_n on the n-dimensional cube $[0,1]^n$ with $C_n(0, \ldots, 0) = 0$, $C_n(1, \ldots, 1, \alpha, 1, \ldots, 1) = \alpha$. Several theorems are stated relating n-dimensional distribution functions and their marginals in terms of functions C_n. No proofs are given." M. Loève, Math. Reviews MR0125600.

$u_1, u_2, \ldots, u_n \in [0,1]$, *which encodes the dependence structure of the vector* $(X_1, \ldots, X_n)$.

It can be checked as in Lemma 4.4 that $C(u_1, \ldots, u_n)$ defined in (4.3) has uniform marginal distributions on $[0,1]$, as

$$\begin{aligned} & C(1, \ldots, 1, u, 1, \ldots, 1) \\ & \quad = F_{(X_1,\ldots,X_n)}\big(F_{X_1}^{-1}(1), \ldots, F_{X_{i-1}}^{-1}(1), F_{X_i}^{-1}(u), F_{X_{i+1}}^{-1}(1), \ldots, F_{X_n}^{-1}(1)\big) \\ & \quad = F_{(X_1,\ldots,X_n)}\big(\infty, \ldots, \infty, F_{X_i}^{-1}(u), \infty, \ldots, \infty\big) \\ & \quad = F_{X_i}\big(F_{X_i}^{-1}(u)\big) \\ & \quad = u, \qquad 0 \leqslant u \leqslant 1. \end{aligned}$$

In the following proposition, we construct a vector of random variables from the data of a copula and a family of marginal distributions.

Proposition 4.8. *Given a family of continuous strictly increasing cumulative distribution functions* $F_1, \ldots, F_n$ *and a multidimensional copula* $C(u_1, \ldots, u_n)$*, the function*

$$F^C_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n) := C(F_1(x_1), \ldots, F_n(x_n)), \quad x_1, x_2, \ldots, x_n \in \mathbb{R}, \tag{4.4}$$

defines a joint cumulative distribution function with marginals $F_1, \ldots, F_n$.

Proof. Given $(U_1, \ldots, U_n)$ a vector of n uniform random variables having the copula $C(u_1, \ldots, u_n)$ for cumulative distribution function, we let

$$X_1 := F_1^{-1}(U_1), \ldots, X_n := F_n^{-1}(U_n).$$

Then, we have

$$\begin{aligned} \mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) &= \mathbb{P}\big(F_1^{-1}(U_1) \leqslant x_1, \ldots, F_n(U_n) \leqslant x_n\big) \\ &= \mathbb{P}(U_1 \leqslant F_1(x_1), \ldots, U_n \leqslant F_n(x_n)) \\ &= C(F_1(x_1), \ldots, F_n(x_n)) \\ &= F^C_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n), \quad x_1, x_2, \ldots, x_n \in \mathbb{R}. \end{aligned}$$

We can also check that the marginal distributions generated by $F^C_{(X_1,\ldots,X_n)}$ coincide with the respective marginals of $(X_1, \ldots, X_n)$, as we have

$$\begin{aligned} & F^C_{(X_1,\ldots,X_n)}(\infty, \ldots, \infty, u, \infty, \ldots, \infty) \\ & \quad = C\big(F_1(\infty), \ldots, F_{i-1}(\infty), F_i(u), F_{i+1}(\infty), \ldots, F_n(\infty)\big) \\ & \quad = C\big(1, \ldots, 1, F_i(u), 1, \ldots, 1\big) \\ & \quad = F_i(u), \qquad 0 \leqslant u \leqslant 1. \end{aligned}$$

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□

4.4 Examples of Copulas

Gaussian copulas

The choice of (4.2) above as joint probability density function, see Figure 4.1, actually induces a particular dependence structure between the Gaussian random variables X and Y, and corresponding to the joint cumulative distribution function

$$\begin{aligned}\Phi_\Sigma(x,y) &:= \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y)\\ &= \frac{1}{\sqrt{2\pi \det \Sigma}} \int_{-\infty}^{x}\int_{-\infty}^{y} \exp\left(-\frac{1}{2}\left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \Sigma^{-1}\begin{bmatrix} u \\ v \end{bmatrix}\right\rangle\right) dudv,\end{aligned}$$

$x, y \in \mathbb{R}$. In case the random variables X, Y are normalized centered Gaussian random variables with unit variance, Σ is given by

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

with correlation parameter $\rho \in (-1, 1)$. Letting

$$F_X(x) := \mathbb{P}(X \leqslant x) \quad \text{and} \quad F_Y(y) := \mathbb{P}(Y \leqslant y),$$

denote the cumulative distribution functions of X and Y, the random variables $F_X(X)$ and $F_Y(Y)$ are known to be uniformly distributed on $[0,1]$, and $(F_X(X), F_Y(Y))$ is a $[0,1]\times[0,1]$-valued random variable with joint cumulative distribution function

$$\begin{aligned}C_\Sigma(u,v) &:= \mathbb{P}\big(F_X(X) \leqslant u \text{ and } F_Y(Y) \leqslant v\big)\\ &= \mathbb{P}\big(X \leqslant F_X^{-1}(u) \text{ and } Y \leqslant F_Y^{-1}(v)\big)\\ &= \Phi_\Sigma\big(F_X^{-1}(u), F_Y^{-1}(v)\big), \qquad 0 \leqslant u, v \leqslant 1. \qquad (4.5)\end{aligned}$$

The function $C_\Sigma(u,v)$, which is the joint cumulative distribution function of a couple of uniformly distributed $[0,1]$-valued random variables, is called the *Gaussian copula* generated by the jointly Gaussian distribution of (X,Y) with covariance matrix Σ.

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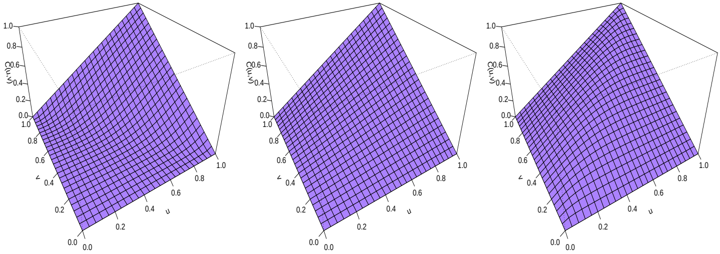
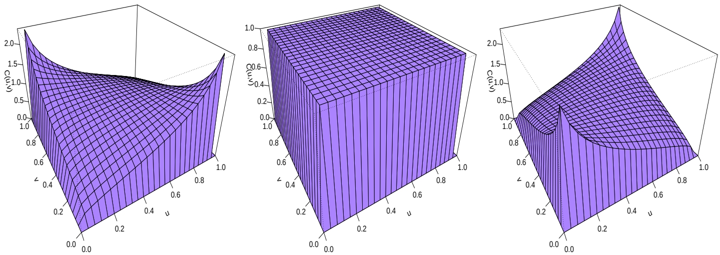


(a) $\rho = -0.9$. (b) $\rho = 0$. (c) $\rho = 0.9$.

Fig. 4.3: Gaussian copula graphs for $\rho = -0.9$, $\rho = 0$, and $\rho = 0.9$.

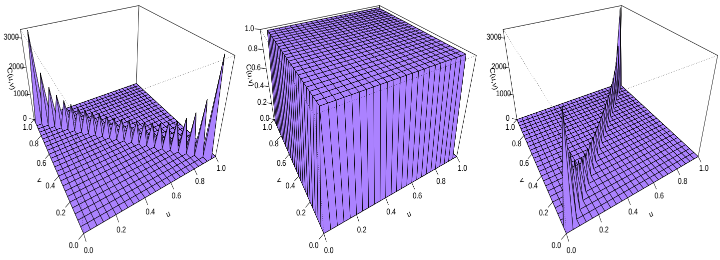
The graphs of Figures 4.3-(a) and 4.3-(c) correspond to intermediate dependence levels given by Gaussian copulas, cf. (4.5).



(a) $\rho = -0.35$. (b) $\rho = 0$. (c) $\rho = 0.35$.

Fig. 4.4: Different Gaussian copula *density* graphs for $\rho = -0.35$, $\rho = 0$ and $\rho = 0.35$.

Figures 4.4 and 4.5 present the corresponding copula density graphs.



(a) $\rho = -0.999$.

(b) $\rho = 0$.

(c) $\rho = 0.999$.

Fig. 4.5: Different Gaussian copula *density* graphs for $\rho = -0.999$, $\rho = 0$ and $\rho = 0.999$.

Figure 4.4-(a) represents a uniform (product) probability density function on the square $[0,1] \times [0,1]$, which corresponds to two independent uniformly distributed $[0,1]$-valued random variables U, V. Figure 4.4-(c) shows the probability distribution of the fully correlated couple (U,U), which does not admit a probability density on the square $[0,1] \times [0,1]$.

The Gaussian copula $C_\Sigma(u,u)$ admits a probability density function on $[0,1] \times [0,1]$ given by

$$
\begin{aligned}
c_\Sigma(u,v) &= \frac{\partial^2 C_\Sigma}{\partial u \partial v}(u,v) \\
&= \frac{\partial^2}{\partial u \partial v} \Phi_\Sigma\big(F_X^{-1}(u), F_Y^{-1}(v)\big) \\
&= \frac{\partial}{\partial u}\left(\frac{1}{F_Y'(F_Y^{-1}(v))} \frac{\partial \Phi_\Sigma}{\partial y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right)\right) \\
&= \frac{\partial}{\partial u}\left(\frac{1}{f_Y(F_Y^{-1}(v))} \frac{\partial \Phi_\Sigma}{\partial y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right)\right) \\
&= \frac{1}{f_X(F_X^{-1}(u)) f_Y(F_Y^{-1}(v))} \frac{\partial^2 \Phi_\Sigma}{\partial x \partial y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right) \\
&= \frac{f_\Sigma\big(F_X^{-1}(u), F_Y^{-1}(v)\big)}{f_X(F_X^{-1}(u)) f_Y(F_Y^{-1}(v))},
\end{aligned}
$$

hence the Gaussian copula $C_\Sigma(u,v)$ can be computed as

$$
C_\Sigma(u,v) = \int_0^u \int_0^v c_\Sigma(a,b)\, da\, db
$$

$$= \int_0^u \int_0^v \frac{f_\Sigma\big(F_X^{-1}(a), F_Y^{-1}(b)\big)}{f_X(F_X^{-1}(a)) f_Y(F_Y^{-1}(b))} dadb, \qquad 0 \leqslant u, v \leqslant 1.$$

The joint cumulative distribution function $F_{(X,Y)}(x, y)$ of (X, Y) can be recovered from Corollary 4.7 as

$$F_{(X,Y)}(x, y) = C_\Sigma(F_X(x), F_Y(y)), \qquad x, y \in \mathbb{R}. \tag{4.6}$$

from the Gaussian copula $C_\Sigma(x, y)$ and the respective cumulative distribution functions $F_X(x)$, $F_Y(y)$ of X and Y.

In that sense, the Gaussian copula $C_\Sigma(x, y)$ encodes the Gaussian dependence structure of the covariance matrix Σ. Moreover, the Gaussian copula $C_\Sigma(x, y)$ can be used to generate a joint distribution function $F^C_{(X,Y)}(x, y)$ by letting

$$F^C_{(X,Y)}(x, y) := C_\Sigma(F_X(x), F_Y(y)), \qquad x, y \in \mathbb{R}, \tag{4.7}$$

based on other, *possibly non-Gaussian* cumulative distribution functions $F_X(x)$, $F_Y(y)$ of two random variables X and Y. In this case we note that the marginals of the joint cumulative distribution function $F^C_{(X,Y)}(x, y)$ are $F_X(x)$ and $F_Y(y)$ because $C_\Sigma(x, y)$ has uniform marginals on $[0, 1]$.

Gumbel copula

The Gumbel copula is given by

$$C(u, v) = \exp\left(- \left((-\log u)^\theta + (-\log v)^\theta \right)^{1/\theta} \right), \qquad 0 \leqslant u, v \leqslant 1,$$

with $\theta \geqslant 1$, and $C(u, v) = uv$ when $\theta = 1$.

Uniform marginals with given copulas

The following R code generates random samples according to the Gaussian, Student, and Gumbel copulas with uniform marginals, as illustrated in Figure 4.6.

```
install.packages("copula"); install.packages("gumbel")
library(copula);library(gumbel)
norm.cop <- normalCopula(0.35); norm.cop
persp(norm.cop, pCopula, n.grid = 51, xlab="u", ylab="v", zlab="C(u,v)", main="", sub="",
      col='lightblue')
persp(norm.cop, dCopula, n.grid = 51, xlab="u", ylab="v", zlab="c(u,v)", main="", sub="",
      col='lightblue')
norm <- rCopula(4000,normalCopula(0.7))
plot(norm[,1],norm[,2],cex=3,pch='.',col='blue')
```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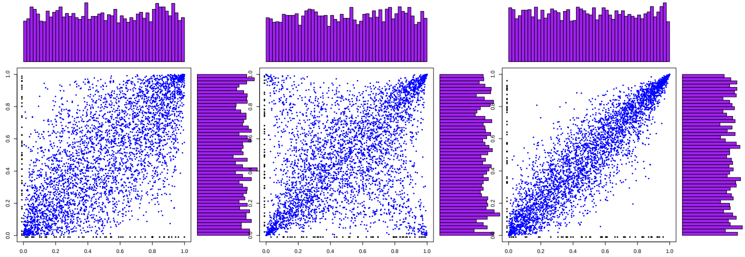
```
stud <- rCopula(4000,tCopula(0.5,dim=2,df=1))
points(stud[,1],stud[,2],cex=3,pch='.',col='red')
gumb <- rgumbel(4000,4)
points(gumb[,1],gumb[,2],cex=3,pch='.',col='green')
```



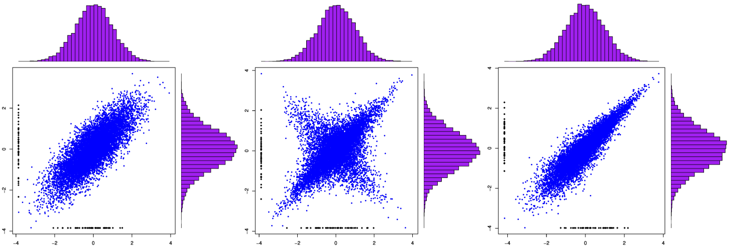
(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Fig. 4.6: Samples with uniform marginals and given copulas.

The following R code is plotting the histograms of Figure 4.6.

```
joint_hist <- function(u) {x <- u[,1]; y <- u[,2]
xhist <- hist(x, breaks=40,plot=FALSE) ; yhist <- hist(y, breaks=40,plot=FALSE)
top <- max(c(xhist$counts, yhist$counts))
nf <- layout(matrix(c(2,0,1,3),2,2,byrow=TRUE), c(3,1), c(1,3), TRUE)
par(mar=c(3,3,1,1))
plot(x, y, xlab="", ylab="",col="blue",pch=19,cex=0.4)
points(x, -0.01+rep(min(y),length(x)), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(x),length(y)), y, xlab="", ylab="",col="black",pch=18,cex=0.8)
par(mar=c(0,3,1,1))
barplot(xhist$counts, axes=FALSE, ylim=c(0, top), space=0,col="purple")
par(mar=c(3,0,1,1))
barplot(yhist$counts, axes=FALSE, xlim=c(0, top), space=0, horiz=TRUE,col="purple")}
joint_hist(norm);joint_hist(stud);joint_hist(gumb)
```

Gaussian marginals with given copulas

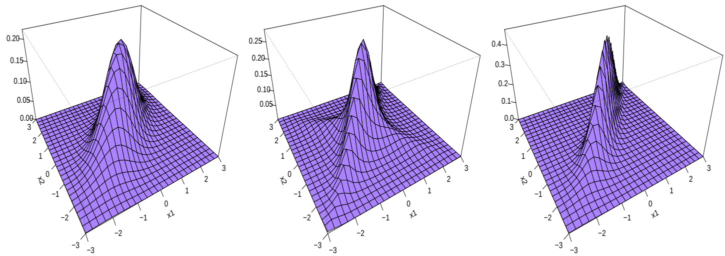


(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Fig. 4.7: Samples with Gaussian marginals and given copulas.

The next R code generates random samples according to the Gaussian, Student, and Gumbel copulas with Gaussian marginals, as illustrated in Figure 4.7.

```
set.seed(100);N=10000
gaussMVD<-mvdc(normalCopula(0.8), margins=c("norm","norm"),
      paramMargins=list(list(mean=0,sd=1),list(mean=0,sd=1)))
norm <- rMvdc(N,gaussMVD)
studentMVD<-mvdc(tCopula(0.5,dim=2,df=1), margins=c("norm","norm"),
      paramMargins=list(list(mean=0,sd=1),list(mean=0,sd=1)))
stud <- rMvdc(N,studentMVD)
gumbelMVD<-mvdc(gumbelCopula(param=4, dim=2), margins=c("norm","norm"),
      paramMargins=list(list(mean=0,sd=1),list(mean=0,sd=1)))
gumb <- rMvdc(N,gumbelMVD)
plot(norm[,1],norm[,2],cex=3,pch='.',col='blue')
points(norm[,1], -0.01+rep(min(norm[,2]),N), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(norm[,1]),N), norm[,2], xlab="", ylab="",col="black",pch=18,cex=0.8)
plot(stud[,1],stud[,2],cex=3,pch='.',col='blue')
points(stud[,1], -0.01+rep(min(stud[,2]),N), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(stud[,1]),N), stud[,2], xlab="", ylab="",col="black",pch=18,cex=0.8)
plot(gumb[,1],gumb[,2],cex=3,pch='.',col='blue')
points(gumb[,1], -0.01+rep(min(gumb[,2]),N), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(gumb[,1]),N), gumb[,2], xlab="", ylab="",col="black",pch=18,cex=0.8)
joint_hist(norm);joint_hist(stud);joint_hist(gumb)
```

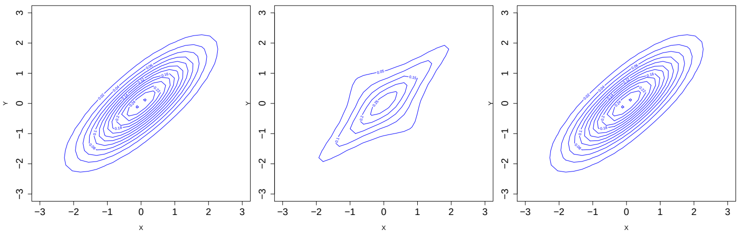


(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Fig. 4.8: Joint densities with Gaussian marginals and given copulas.

The following R code is plotting joint densities with Gaussian marginals and given copulas, as illustrated in Figure 4.8.

```
persp(gaussMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
persp(studentMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
persp(gumbelMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
```



(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Fig. 4.9: Joint density contour plots with Gaussian marginals and given copulas.

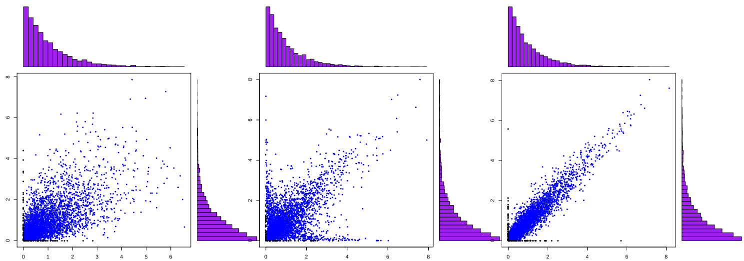
The following R code generates countour plots with Gaussian marginals and given copulas, as illustrated in Figure 4.9.

```
contour(gaussMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(studentMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(gaussMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
```

Exponential marginals with given copulas

The following R code generates random samples with exponential marginals according to the Gaussian, Student, and Gumbel copulas as illustrated in Figure 4.10.

```
library(copula);set.seed(100);N=4000
gaussMVD<-mvdc(normalCopula(0.7), margins=c("exp","exp"),
      paramMargins=list(list(rate=1),list(rate=1)))
norm <- rMvdc(N,gaussMVD)
studentMVD<-mvdc(tCopula(0.5,dim=2,df=1), margins=c("exp","exp"),
      paramMargins=list(list(rate=1),list(rate=1)))
stud <- rMvdc(N,studentMVD)
gumbelMVD<-mvdc(gumbelCopula(param=4, dim=2), margins=c("exp","exp"),
      paramMargins=list(list(rate=1),list(rate=1)))
gumb <- rMvdc(N,gumbelMVD)
plot(norm[,1],norm[,2],cex=3,pch='.',col='blue'); plot(stud[,1],stud[,2],cex=3,pch='.',col='blue');
      plot(gumb[,1],gumb[,2],cex=3,pch='.',col='blue')
persp(gaussMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
persp(studentMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
persp(gumbelMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
contour(gaussMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(studentMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(gaussMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
joint_hist(norm); joint_hist(stud); joint_hist(gumb)
```



(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Fig. 4.10: Samples with exponential marginals and given copulas.

Exercises

Exercise 4.1 Copulas. In what follows, U denotes a uniformly distributed $[0,1]$-valued random variable.

a) To which couple (U,V) of uniformly distributed $[0,1]$-valued random variables does the copula function

$$C_M(u,v) = \min(u,v), \qquad 0 \leqslant u,v \leqslant 1,$$

correspond?

b) Show that the function

$$C_m(u,v) := (u+v-1)^+, \qquad 0 \leqslant u,v \leqslant 1,$$

is the copula on $[0,1]\times[0,1]$ corresponding to $(U,V)=(U,1-U)$.

c) Show that for any copula function $C(u,v)$ on $[0,1]\times[0,1]$ we have

$$C(u,v) \leqslant C_M(u,v), \qquad 0 \leqslant u,v \leqslant 1. \tag{4.8}$$

d) Show that for any copula function $C(u,v)$ on $[0,1]\times[0,1]$ we also have

$$C_m(u,v) \leqslant C(u,v), \qquad 0 \leqslant u,v \leqslant 1. \tag{4.9}$$

Hint: For fixed $v\in[0,1]$, let $h(u) := C(u,v)-(u+v-1)$ and show that $h(1)=0$ and $h'(u)\leqslant 0$.

Exercise 4.2 Consider two Bernoulli random variables X and Y, with $p_X = \mathbb{P}(X=1)$, $p_Y = \mathbb{P}(Y=1)$, correlation coefficient $\rho\in[-1,1]$, and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1) = p_X p_Y + \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X=0 \text{ and } Y=1) = (1-p_X)p_Y - \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X=1 \text{ and } Y=0) = p_X(1-p_Y) - \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X=0 \text{ and } Y=0) = (1-p_X)(1-p_Y) + \rho\sqrt{p_X p_Y(1-p_X)(1-p_Y)}. \end{cases}$$

a) Is it possible to have $\rho=1$ *without* having $p_X=p_Y$ and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1) = p_X = p_Y, \\ \\ \mathbb{P}(X=0 \text{ and } Y=1) = 0, \\ \\ \mathbb{P}(X=1 \text{ and } Y=0) = 0, \\ \\ \mathbb{P}(X=0 \text{ and } Y=0) = 1-p_X = 1-p_Y \ ? \end{cases}$$

b) Similarly, is it possible to have $\rho=1$ *without* having $p_X=1-p_Y$ and

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = 0, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = p_Y, \\ \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = 0 \ ? \end{cases}$$

Exercise 4.3 Exponential copulas. Consider the random vector (X, Y) of nonnegative random variables, whose joint distribution is given by the survival function

$$\mathbb{P}(X \geqslant x \text{ and } Y \geqslant y) := \mathrm{e}^{-\lambda x - \mu y - \nu \operatorname{Max}(x,y)}, \qquad x, y \in \mathbb{R}_+,$$

where $\lambda, \mu, \nu > 0$.

a) Find the marginal distributions of X and Y.
b) Find the joint cumulative distribution function $F(x, y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y)$ of (X, Y).
c) Construct an "exponential copula" based on the joint cumulative distribution function of (X, Y).

Exercise 4.4 Gumbel bivariate logistic distribution. Consider the random vector (X, Y) of nonnegative random variables, whose joint distribution is given by the joint cumulative distribution function (CDF)

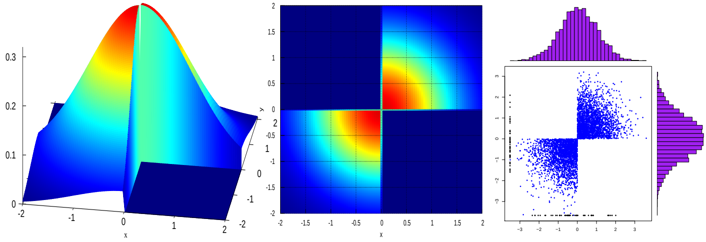
$$F_{(X,Y)}(x, y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) := \frac{1}{1 + \mathrm{e}^{-x} + \mathrm{e}^{-y}}, \qquad x, y \in \mathbb{R}.$$

a) Find the marginal distributions of X and Y.
b) Construct the copula based on the joint CDF of (X, Y).

Exercise 4.5 Consider the random vector (X, Y) with the joint probability density function

$$\widetilde{f}(x, y) := \frac{1}{\pi\sigma\eta} \mathbb{1}_{\mathbb{R}^2_-}(x, y) \mathrm{e}^{-x^2/(2\sigma^2) - y^2/(2\eta^2)} + \frac{1}{\pi\sigma\eta} \mathbb{1}_{\mathbb{R}^2_+}(x, y) \mathrm{e}^{-x^2/(2\sigma^2) - y^2/(2\eta^2)},$$

plotted as a heat map in Figure 4.11b.



(a) 3D view. (b) Heat map. (c) Samples.

Fig. 4.11: Truncated two-dimensional Gaussian density.

```
library(MASS)
Sigma <- matrix(c(1,0,0,1),2,2);N=10000
u<-mvrnorm(N,rep(0,2),Sigma);j=1
for (i in 1:N){
if (u[i,1]>0 && u[i,2]>0) {j<-j+1;}
if (u[i,1]<0 && u[i,2]<0) {j<-j+1;}}
v<-matrix(nrow=j-1, ncol=2);j=1
for (i in 1:N){
if (u[i,1]>0 && u[i,2]>0) {v[j,]=u[i,];j<-j+1;}
if (u[i,1]<0 && u[i,2]<0) {v[j,]=u[i,];j<-j+1;}}
joint_hist(v) # Function defined the previous section
```

a) Show that (X, Y) has the Gaussian marginals $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \eta^2)$.
b) Does the couple (X, Y) have the bivariate Gaussian distribution with probability density function $f_\Sigma(x, y)$, where Σ is the covariance matrix (4.1) of (X, Y)?
c) Show that the random variable $X + Y$ is not Gaussian (take $\sigma = \eta = 1$ for simplicity).
d) Show that under the rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

of angle $\theta \in [0, 2\pi]$ the random vector $(X\cos\theta - Y\sin\theta, X\sin\theta + Y\cos\theta)$ can have an arbitrary covariance depending on the value of $\theta \in [0, 2\pi]$.

Exercise 4.6 Let τ_1, τ_2 and τ denote three independent exponentially distributed random times with respective parameters $\lambda_1, \lambda_2, \lambda > 0$. Consider two firms with respective default times $\tau_1 \wedge \tau = \min(\tau_1, \tau)$ and $\tau_2 \wedge \tau = \min(\tau_2, \tau)$, where τ represents the time of a macro-economic shock.

a) Find the tail (or survival) distribution functions of $\tau_1 \wedge \tau$ and $\tau_2 \wedge \tau$.

b) Compute the joint survival probability

$$\mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t), \qquad s, t \geqslant 0.$$

Hint: Use the relation

$$\mathrm{Max}(s, t) = s + t - \min(s, t), \qquad s, t \geqslant 0.$$

c) Compute the joint cumulative distribution function

$$\mathbb{P}(\tau_1 \wedge \tau \leqslant s \text{ and } \tau_2 \wedge \tau \leqslant t), \qquad s, t \geqslant 0.$$

d) Compute the resulting copula

$$C(u, v) := F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)), \qquad 0 \leqslant u, v \leqslant 1.$$

e) Compute the resulting copula density function $\frac{\partial^2 C}{\partial u \partial v}(u, v)$, $u, v \in [0, 1]$.