Chapter 9
Volatility Estimation

Volatility estimation methods include historical, implied and local volatility, and the VIX® volatility index. This chapter presents such estimation methods, together with examples of how the Black-Scholes formula can be fitted to market data. While the market parameters r, t, S_t, T, and K used in Black-Scholes option pricing can be easily obtained from market terms and data, the estimation of volatility parameters can be a more complex task.

9.1 Historical Volatility

We consider the problem of estimating the parameters μ and σ from market data in the stock price model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{9.1}$$

Historical trend estimation

By discretization of (9.1) along a family $t_0, t_1, \ldots, t_N$ of observation times as

$$\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} = (t_{k+1} - t_k)\mu + (B_{t_{k+1}} - B_{t_k})\sigma, \quad k = 0, 1, \ldots, N-1, \tag{9.2}$$

a natural estimator for the trend parameter μ can be constructed as

$$\widehat{\mu}_N := \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S^M_{t_{k+1}}-S^M_{t_k}}{S^M_{t_k}}\right), \tag{9.3}$$

where $(S^M_{t_{k+1}} - S^M_{t_k})/S^M_{t_k}$, $k = 0, 1, \ldots, N-1$ denotes market returns observed at discrete times $t_0, t_1, \ldots, t_N$.

Historical log-return estimation

Alternatively, observe that, replacing* (9.3) by the log-returns

$$\begin{aligned}\log\frac{S_{t_{k+1}}}{S_{t_k}} &= \log S_{t_{k+1}} - \log S_{t_k}\\ &= \log\left(1+\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}\right)\\ &\simeq \frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}},\end{aligned}$$

with $t_{k+1} - t_k = T/N$, $k = 0, 1, \ldots, N-1$, one can replace (9.3) with the simpler telescoping estimate

$$\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\log S_{t_{k+1}} - \log S_{t_k}\right) = \frac{1}{T}\log\frac{S_T}{S_0}.$$

Historical volatility estimation

The volatility parameter σ can be estimated by writing, from (9.2),

$$\sigma^2\sum_{k=0}^{N-1}\frac{(B_{t_{k+1}}-B_{t_k})^2}{t_{k+1}-t_k} = \sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}} - (t_{k+1}-t_k)\mu\right)^2,$$

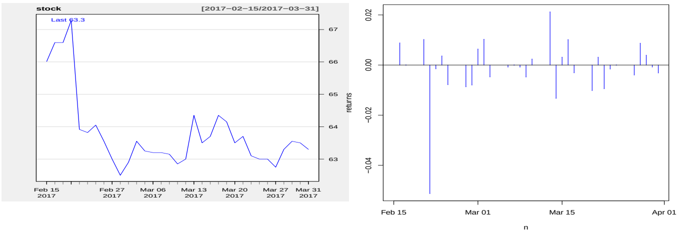
which yields the (unbiased) realized variance estimator

$$\widehat{\sigma}^2_N := \frac{1}{N-1}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}} - (t_{k+1}-t_k)\widehat{\mu}_N\right)^2.$$

```
library(quantmod)
getSymbols("0005.HK",from="2017-02-15",to=Sys.Date(),src="yahoo")
stock=Ad(`0005.HK`)
chartSeries(stock,up.col="blue",theme="white")
```

* This approximation does not include the correction term $(dS_t)^2/(2S_t^2)$ in the Itô formula $d\log S_t = dS_t/S_t - (dS_t)^2/(2S_t^2)$.

```
stock=Ad(`0005.HK`);logreturns=diff(log(stock));returns=(stock-lag(stock))/lag(stock)
times=index(returns);returns <- as.vector(returns)
n = sum(is.na(returns))+sum(!is.na(returns))
plot(times,returns,pch=19,cex=0.05,col="blue", ylab="returns", xlab="n", main = '')
segments(x0 = times, x1 = times, cex=0.05,y0 = 0, y1 = returns,col="blue")
abline(seq(1,n),0,FALSE);dt=1.0/365;mu=mean(returns,na.rm=TRUE)/dt
sigma=sd(returns,na.rm=TRUE)/sqrt(dt);mu;sigma
```



(a) Underlying asset price. (b) Log-returns.

Fig. 9.1: Graph of underlying asset price *vs.* log-returns.

```
library(PerformanceAnalytics);
returns <- exp(CalculateReturns(stock,method="compound")) - 1; returns[1,] <- 0
histvol <- rollapply(returns, width = 30, FUN=sd.annualized)
myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme(); myTheme$col$line.col <- "blue"
dev.new(width=16,height=7)
chart_Series(stock,name="0005.HK",pars=myPars,theme=myTheme)
add_TA(histvol, name="Historical Volatility")
```

Figure 9.2 presents a historical volatility graph with a 30 days rolling window.

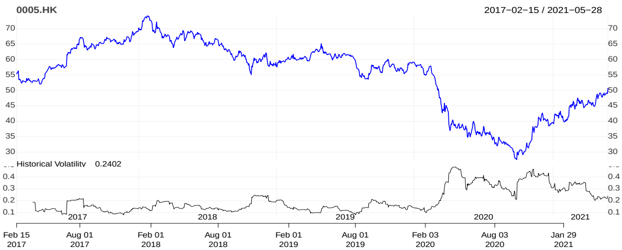


Fig. 9.2: Historical volatility graph.

Parameter estimation based on historical data usually requires a lot of samples and it can only be valid on a given time interval, or as a moving average. Moreover, it can only rely on past data, which may not reflect future data.

Fig. 9.3: "The fugazi: it's a wazy, it's a woozie. It's fairy dust."*

9.2 Implied Volatility

Recall that when $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE is given by

$$\mathrm{Bl}(t,x,K,\sigma,r,T) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

and

$$\begin{cases} d_+(T-t) = \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) = \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \end{cases}$$

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data. Equating the Black-Scholes formula

$$\mathrm{Bl}(t,S_t,K,\sigma,r,T) = M \tag{9.4}$$

to the observed value M of a given market price allows one to infer a value of σ when t, S_t, r, T are known, as in *e.g.* Figure 9.4.

* Scorsese (2013) Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

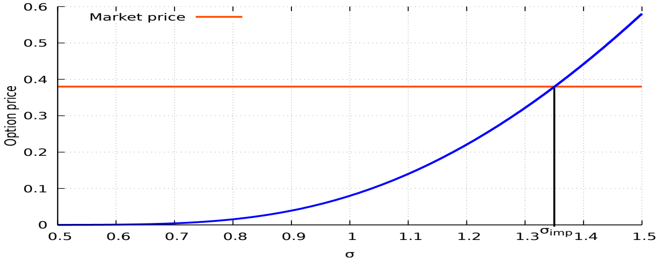


Fig. 9.4: Option price as a function of the volatility σ.

This value of σ is called the implied volatility, and it is denoted here by $\sigma_{\rm imp}(K,T)$, cf. *e.g.* Exercise 6.6. Various algorithms can be implemented to solve (9.4) numerically for $\sigma_{\rm imp}(K,T)$, such as the bisection method and the Newton-Raphson method.*

```
BS <- function(S, K, T, r, sig){d1 <- (log(S/K) + (r + sig^2/2)*T) / (sig*sqrt(T))
d2 <- d1 - sig*sqrt(T);return(S*pnorm(d1) - K*exp(-r*T)*pnorm(d2))}
implied.vol <- function(S, K, T, r, market){
sig <- 0.20;sig.up <- 10;sig.down <- 0.0001;count <- 0;err <- BS(S, K, T, r, sig) - market
while(abs(err) > 0.00001 && count<1000){
if(err < 0){sig.down <- sig;sig <- (sig.up + sig)/2} else{sig.up <- sig;sig <- (sig.down + sig)/2}
err <- BS(S, K, T, r, sig) - market;count <- count +
      1};if(count==1000){return(NA)}else{return(sig)}}
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02; implied.vol(S, K, T, r, market)
BS(S, K, T, r, implied.vol(S, K, T, r, market))
```

The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, market option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula, see, *e.g.*, Figure S.24.

```
library(devtools); install_github("https://github.com/cran/fOptions")
library(fOptions)
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02
sig=GBSVolatility(market,"c",S,K,T,r,r,1e-4,maxiter = 10000)
BS(S, K, T, r, sig)
```

* Download the corresponding R **code** or the **IPython notebook** that can be run **here** or **here**.

Option chain data in

```
install.packages("quantmod");install.packages("jsonlite");
library(devtools); remotes::install_github("joshuaulrich/quantmod")
library(quantmod);library(jsonlite)
getSymbols("^GSPC",src="yahoo",from=as.Date("2018-01-01"), to = as.Date("2018-03-01"))
head(GSPC)
# Only the front-month expiry
SPX.OPT <- getOptionChain("^SPX")
AAPL.OPT <- getOptionChain("AAPL")
# All expiries
SPX.OPTS <- getOptionChain("^SPX", NULL)
AAPL.OPTS <- getOptionChain("AAPL", NULL)
# All 2021 to 2023 expiries
SPX.OPTS <- getOptionChain("^SPX", "2021/2023")
AAPL.OPTS <- getOptionChain("AAPL", "2021/2023")
```

Exporting option price data

```
write.table(AAPL.OPT$puts, file = "AAPLputs")
write.csv(AAPL.OPT$puts, file = "AAPLputs.csv")
install.packages("xlsx")
library(xlsx)
write.xlsx(AAPL.OPTS$Jun.19.2020$puts, file = "AAPL.OPTS$Jun.19.2020$puts.xlsx")
```

Volatility smiles

Given two European call options with strike prices K_1, resp. K_2, maturities T_1, resp. T_2, and prices C_1, resp. C_2, on the same stock S, this procedure should yield two estimates $\sigma_{\text{imp}}(K_1, T_1)$ and $\sigma_{\text{imp}}(K_2, T_2)$ of implied volatilities according to the following equations.

$$\begin{cases} \text{Bl}(t, S_t, K_1, \sigma_{\text{imp}}(K_1, T_1), r, T_1) = M_1, & (9.5a) \\ \\ \text{Bl}(t, S_t, K_2, \sigma_{\text{imp}}(K_2, T_2), r, T_2) = M_2, & (9.5b) \end{cases}$$

Clearly, there is no reason a priori for the implied volatilities $\sigma_{\text{imp}}(K_1, T_1)$, $\sigma_{\text{imp}}(K_2, T_2)$ solutions of (9.5a)-(9.5b) to coincide across different strike prices and different maturities. However, in the standard Black-Scholes model the value of the parameter σ should be unique for a given stock S. This contradiction between a model and market data is motivating the development of more sophisticated stochastic volatility models.

Figure 9.5 presents an estimation of implied volatility surface for Asian options on light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange (CME).

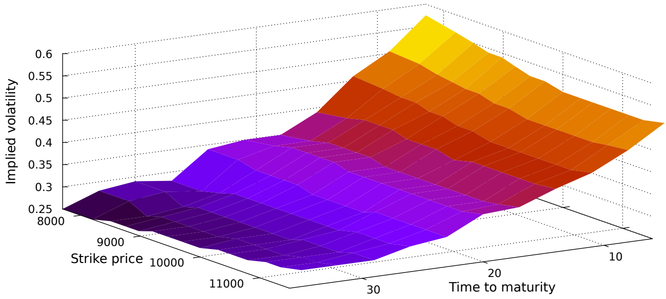


Fig. 9.5: Implied volatility surface of Asian options on light sweet crude oil futures.*

As observed in Figure 9.5, the volatility surface can exhibit a *smile* phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike price values.

```
remotes::install_github("joshuaulrich/quantmod")
install.packages("jsonlite");install.packages("lubridate")
library(jsonlite);library(lubridate);library(quantmod)
# Maturity to be updated as needed
maturity <- as.Date("2021-08-20", format="%Y-%m-%d")
CHAIN <- getOptionChain("GOOG",maturity)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
getSymbols("GOOG", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(GOOG))))
T <- as.numeric(difftime(maturity, lastBusDay, units = "days")/365);r = 0.02;ImpVol<-1:1;
        S=as.vector(tail(Ad(GOOG),1))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S,CHAIN$calls$Strike[i],T,r,
        CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab =
        "Implied volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4,
        data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=2)
```

* © Tan Yu Jia.

```
currentyear<-format(Sys.Date(), "%Y")
# Maturity to be updated as needed
maturity <- as.Date("2021-12-17", format="%Y-%m-%d")
CHAIN <- getOptionChain("^SPX",maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
getSymbols("^SPX", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(SPX))))
T <- as.numeric(difftime(maturity, lastBusDay, units = "days")/365);r = 0.02;ImpVol<-1:1;
      S=as.vector(tail(Ad(SPX),1))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S, CHAIN$calls$Strike[i], T, r,
      CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab =
      "Implied volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4,
      data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=3)
```

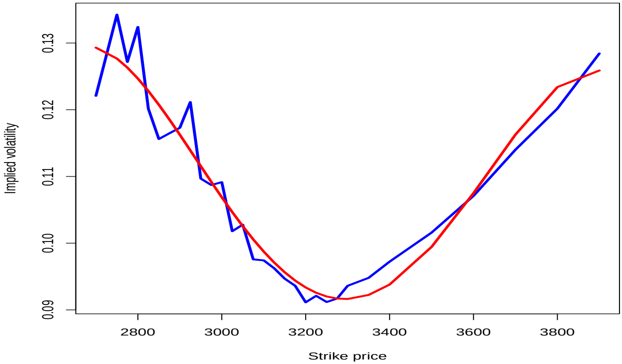


Fig. 9.6: S&P500 option prices plotted against strike prices.

When reading option prices on the volatility scale, the smile phenomenon shows that the Black-Scholes formula tends to underprice extreme events for which the underlying asset price S_T is far away from the strike price K. In that sense, the Black-Scholes formula, which is modeling asset returns using Gaussian distribution tails, tends to underestimate the probability of extreme events.

Plotting the different values of the implied volatility σ as a function of K and T will yield a three-dimensional plot called the volatility surface.*

* Download the corresponding **IPython notebook** that can be run **here** or **here** (© Qu Mengyuan).

Black-Scholes formula *vs.* market data - Call option example

On July 28, 2009 a call option has been issued by Merrill Lynch on the stock price S of Cheung Kong Holdings (0001.HK) with strike price K=$109.99, Maturity T = December 13, 2010, and entitlement ratio 100, cf. page 10.

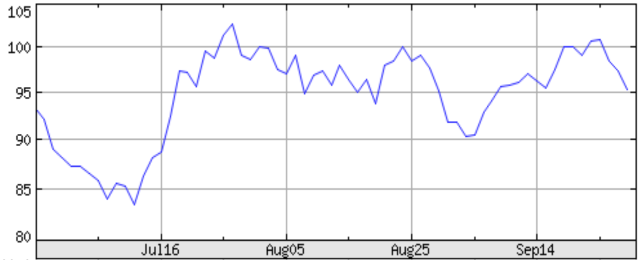
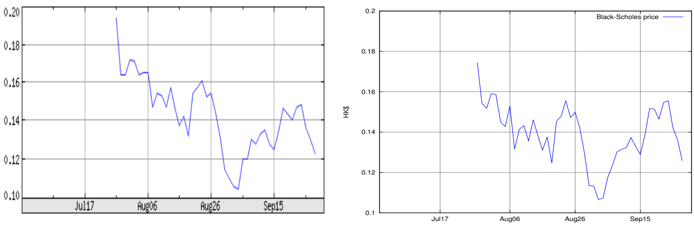


Fig. 9.7: Graph of the (market) stock price of Cheung Kong Holdings.

The market price of the option (17838.HK) on September 28 was $12.30, as obtained from https://www.hkex.com.hk/eng/dwrc/search/listsearch.asp.

The next graph in Figure 9.8a shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying asset price.



(a) Graph of (market) option prices. (b) Graph of Black-Scholes prices.

Fig. 9.8: Comparison of market call option prices *vs.* calibrated Black-Scholes prices.

In Figure 9.8b we have fitted the time evolution $t \mapsto g_c(t, S_t)$ of Black-Scholes prices to the data of Figure 9.8a using the market stock price data of Figure 9.7, by varying the values of the volatility σ.

Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

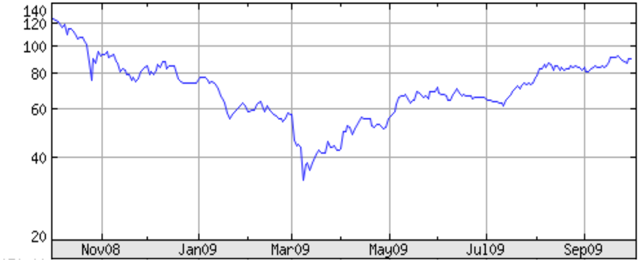
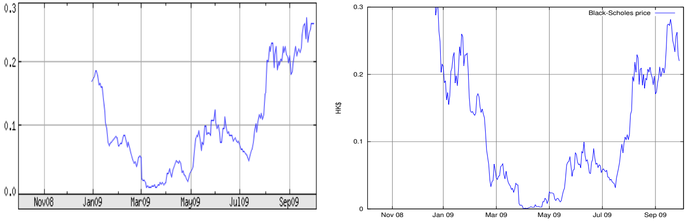


Fig. 9.9: Graph of the (market) stock price of HSBC Holdings.

Next, we consider the graph of the price of the call option issued by Societe Generale on 31 December 2008 with strike price K=$63.704, maturity T = October 05, 2009, and entitlement ratio 100, cf. page 10.



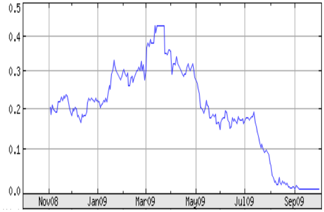
(a) Graph of (market) option prices. (b) Graph of Black-Scholes prices.

Fig. 9.10: Comparison of market call option prices *vs.* calibrated Black-Scholes prices.

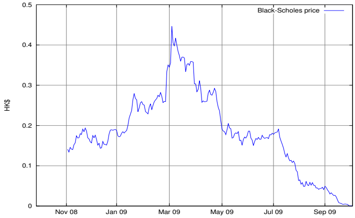
As above, in Figure 9.10b we have fitted the path $t \longmapsto g_c(t, S_t)$ of the Black-Scholes option price to the data of Figure 9.10a using the stock price data of Figure 9.9.
In this case the option is *in the money* at maturity. We can also check that the option is worth $100 \times 0.2650 =$ $26.650 at that time, which, according to absence of arbitrage, is quite close to the actual value $90 - $63.703 = $26.296 of its payoff.

Put option example

In Figure 9.11 we consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 on the underlying asset HSBC, with strike price K=$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593, cf. page 10.



(a) Graph of (market) option prices.



(b) Graph of Black-Scholes prices.

Fig. 9.11: Comparison of market put option prices *vs.* calibrated Black-Scholes prices.

One checks easily that at maturity, the price of the put option is worth \$0.01 (a market price cannot be lower), which almost equals the option payoff \$0, by absence of arbitrage opportunities. Figure 9.11b is a fit of the Black-Scholes put price graph

$$t \longmapsto g_{\mathrm{p}}(t, S_t)$$

to Figure 9.11a as a function of the stock price data of Figure 9.10b. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.

The normalized market data graph in Figure 9.12 shows how the option price can track the values of the underlying asset price. Note that the range of values $[26.55, 26.90]$ for the underlying asset price corresponds to $[0.675, 0.715]$ for the option price, meaning 1.36% *vs.* 5.9% in percentage. This is a European call option on the ALSTOM underlying asset with strike price $K = €20$, maturity March 20, 2015, and entitlement ratio 10, cf. page 10.

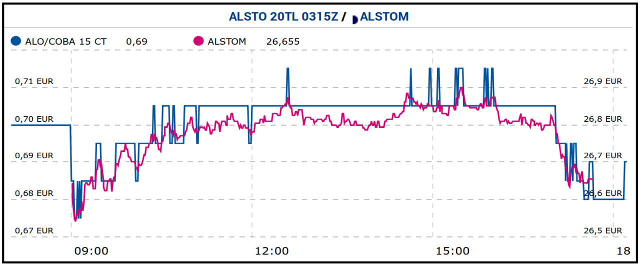


Fig. 9.12: Call option price *vs.* underlying asset price.

9.3 Local Volatility

As the constant volatility assumption in the Black-Scholes model does not appear to be satisfactory due to the existence of volatility smiles, it can make more sense to consider models of the form

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t$$

where σ_t is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dB_t \tag{9.6}$$

where $\sigma(t,x) \geqslant 0$ is a deterministic function of time t and of the underlying asset price x. Such models are called local volatility models.

As an example, consider the stochastic differential equation with local volatility

$$dY_t = rY_t dt + \sigma Y_t^2 dB_t, \tag{9.7}$$

where $\sigma > 0$, see also Problem 7.28.

```
dev.new(width=16,height=7)
N=10000; t <- 0:(N-1); dt <- 1.0/N;r=0.5;sigma=1.2;
Z <- rnorm(N,mean=0,sd=sqrt(dt));Y <- rep(0,N);Y[1]=1
for (j in 2:N){ Y[j]=max(0,Y[j-1]+r*Y[j-1]*dt+sigma*Y[j-1]**2*Z[j])}
plot(t*dt, Y, xlab = "t", ylab = "", type = "l", col = "blue", xaxs='i', yaxs='i', cex.lab=2,
     cex.axis=1.6,las=1)
abline(h=0)
```

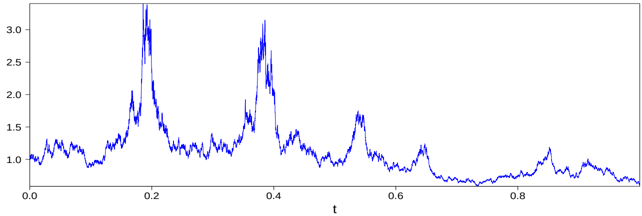


Fig. 9.13: Simulated path of (9.7) with $r = 0.5$ and $\sigma = 1.2$.

In the general case, the corresponding Black-Scholes PDE for the option prices

$$g(t,x,K) := \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T - K)^+ \,\big|\, S_t = x\big], \tag{9.8}$$

where $(S_t)_{t\in\mathbb{R}_+}$ is defined by (9.6), can be written as

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https://personal.ntu.edu.sg/nprivault/indext.html

$$\begin{cases} rg(t,x,K) = \dfrac{\partial g}{\partial t}(t,x,K) + rx\dfrac{\partial g}{\partial x}(t,x,K) + \dfrac{1}{2}x^2\sigma^2(t,x)\dfrac{\partial^2 g}{\partial x^2}(t,x,K), \\ \\ g(T,x,K) = (x-K)^+, \end{cases} \tag{9.9}$$

with terminal condition $g(T,x,K) = (x-K)^+$, *i.e.* we consider European call options.

Lemma 9.1. *(Relation (1) in Breeden and Litzenberger (1978)). Consider a family* $(C^M(T,K))_{T,K>0}$ *of market call option prices with maturities* T *and strike prices* K *given at time* 0*. Then, the probability density function* $\varphi_T(y)$ *of* S_T *is given by*

$$\varphi_T(K) = \mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial K^2}(T,K), \qquad K,T>0. \tag{9.10}$$

Proof. Assume that the market option prices $C^M(T,K)$ match the Black-Scholes prices $\mathrm{e}^{-rT}\mathbb{E}^*[(S_T-K)^+]$, $K>0$. Letting $\varphi_T(y)$ denote the probability density function of S_T, Condition (9.13) can be written at time $t=0$ as

$$\begin{aligned} C^M(T,K) &= \mathrm{e}^{-rT}\mathbb{E}^*[(S_T-K)^+] \\ &= \mathrm{e}^{-rT}\int_0^\infty (y-K)^+\varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty (y-K)\varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty y\varphi_T(y)dy - K\,\mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy \qquad (9.11) \\ &= \mathrm{e}^{-rT}\int_K^\infty y\varphi_T(y)dy - K\,\mathrm{e}^{-rT}\mathbb{P}(S_T \geqslant K). \end{aligned}$$

By differentiation of (9.11) with respect to K, one gets

$$\begin{aligned} \frac{\partial C^M}{\partial K}(T,K) &= -\,\mathrm{e}^{-rT}K\varphi_T(K) - \mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy + \mathrm{e}^{-rT}K\varphi_T(K) \\ &= -\,\mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy, \end{aligned}$$

which yields (9.10) by twice differentiation of $C^M(T,K)$ with respect to K. □

In order to implement a stochastic volatility model such as (9.6), it is important to first calibrate the local volatility function $\sigma(t,x)$ to market data.

In principle, the Black-Scholes PDE (9.9) could allow one to recover the value of $\sigma(t,x)$ as a function of the option price $g(t,x,K)$, as

$$\sigma(t,x)=\sqrt{\frac{2rg(t,x,K)-2\frac{\partial g}{\partial t}(t,x,K)-2rx\frac{\partial g}{\partial x}(t,x,K)}{x^2\frac{\partial^2 g}{\partial x^2}(t,x,K)}}, \qquad x,t>0,$$

however, this formula requires the knowledge of the option price for different values of the underlying asset price x, in addition to the knowledge of the strike price K.

The Dupire (1994) formula brings a solution to the local volatility calibration problem by providing an estimator of $\sigma(t,x)$ as a function $\sigma(t,K)$ based on the values of the strike price K. See Exercises 9.2-9.3 for conditions ensuring the nonnegativity of the square root argument in (9.12) via absence of arbitrage.

Proposition 9.2. *(Dupire (1994), Derman and Kani (1994)) Consider a family $(C^M(T,K))_{T,K>0}$ of market call option prices at time 0 with maturity T and strike price K, and define the volatility function $\sigma(t,y)$ by*

$$\sigma(t,y):=\sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y)+2ry\frac{\partial C^M}{\partial y}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}}=\frac{\sqrt{\frac{\partial C^M}{\partial t}(t,y)+ry\frac{\partial C^M}{\partial y}(t,y)}}{y\,\mathrm{e}^{-rT/2}\sqrt{\varphi_t(y)/2}}, \tag{9.12}$$

where $\varphi_t(y)$ denotes the probability density function of S_t, $t\in[0,T]$. Then, the prices generated from the Black-Scholes PDE (9.9) will be compatible with the market option prices $C^M(T,K)$ in the sense that

$$C^M(T,K)=\mathrm{e}^{-rT}\mathbb{E}^*[(S_T-K)^+], \qquad K>0. \tag{9.13}$$

Proof. For any sufficiently smooth function $f\in\mathcal{C}_0^\infty(\mathbb{R})$, with $\lim_{x\to-\infty}f(x)=\lim_{x\to+\infty}f(x)=0$, using the Itô formula, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}f(y)\varphi_T(y)dy=\mathbb{E}^*[f(S_T)]\\
&=\mathbb{E}^*\left[f(S_0)+r\int_0^T S_tf'(S_t)dt+\int_0^T S_tf'(S_t)\sigma(t,S_t)dB_t\right.\\
&\qquad\left.+\frac{1}{2}\int_0^T S_t^2f''(S_t)\sigma^2(t,S_t)dt\right]\\
&=f(S_0)+\mathbb{E}^*\left[r\int_0^T S_tf'(S_t)dt+\frac{1}{2}\int_0^T S_t^2f''(S_t)\sigma^2(t,S_t)dt\right]\\
&=f(S_0)+r\int_0^T\mathbb{E}^*[S_tf'(S_t)]dt+\frac{1}{2}\int_0^T\mathbb{E}^*[S_t^2f''(S_t)\sigma^2(t,S_t)]dt
\end{aligned}$$

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$$= f(S_0) + r\int_{-\infty}^{\infty} y f'(y)\int_0^T \varphi_t(y)dtdy$$
$$+\frac{1}{2}\int_{-\infty}^{\infty} y^2 f''(y)\int_0^T \sigma^2(t,y)\varphi_t(y)dtdy,$$

hence, after differentiating both sides of the equality with respect to T,

$$\int_{-\infty}^{\infty} f(y)\frac{\partial\varphi_T}{\partial T}(y)dy = r\int_{-\infty}^{\infty} y f'(y)\varphi_T(y)dy + \frac{1}{2}\int_{-\infty}^{\infty} y^2 f''(y)\sigma^2(T,y)\varphi_T(y)dy.$$

Integrating by parts in the above relation yields

$$\int_{-\infty}^{\infty}\frac{\partial\varphi_T}{\partial T}(y)f(y)dy$$
$$= -r\int_{-\infty}^{\infty} f(y)\frac{\partial}{\partial y}(y\varphi_T(y))dy + \frac{1}{2}\int_{-\infty}^{\infty} f(y)\frac{\partial^2}{\partial y^2}\big(y^2\sigma^2(T,y)\varphi_T(y)\big)dy,$$

for all smooth functions $f(y)$ with compact support in $\mathbb{R}$, hence

$$\frac{\partial\varphi_T}{\partial T}(y) = -r\frac{\partial}{\partial y}(y\varphi_T(y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(y^2\sigma^2(T,y)\varphi_T(y)\big), \qquad y\in\mathbb{R}.$$

From Relation (9.10) in Lemma 9.1, we have

$$\frac{\partial\varphi_T}{\partial T}(K) = r\,\mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial K^2}(T,K) + \mathrm{e}^{rT}\frac{\partial^3 C^M}{\partial T\partial K^2}(T,K),$$

hence we get

$$-r\frac{\partial^2 C^M}{\partial y^2}(T,y) - \frac{\partial^3 C^M}{\partial T\partial y^2}(T,y)$$
$$= r\frac{\partial}{\partial y}\left(y\frac{\partial^2 C^M}{\partial y^2}(T,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right), \qquad y\in\mathbb{R}.$$

After a first integration with respect to y under the boundary condition $\lim_{y\to+\infty} C^M(T,y) = 0$, we obtain

$$-r\frac{\partial C^M}{\partial y}(T,y) - \frac{\partial}{\partial T}\frac{\partial C^M}{\partial y}(T,y)$$
$$= ry\frac{\partial^2 C^M}{\partial y^2}(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right),$$

i.e.

$$-r\frac{\partial C^M}{\partial y}(T,y) - \frac{\partial}{\partial T}\frac{\partial C^M}{\partial y}(T,y)$$

$$= r\frac{\partial}{\partial y}\left(y\frac{\partial C^M}{\partial y}(T,y)\right) - r\frac{\partial C^M}{\partial y}(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right),$$

or

$$-\frac{\partial}{\partial y}\frac{\partial C^M}{\partial T}(T,y) = r\frac{\partial}{\partial y}\left(y\frac{\partial C^M}{\partial y}(T,y)\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right).$$

Integrating one more time with respect to y yields

$$-\frac{\partial C^M}{\partial T}(T,y) = ry\frac{\partial C^M}{\partial y}(T,y) - \frac{1}{2}y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y), \qquad y \in \mathbb{R}, \tag{9.14}$$

which conducts to (9.12) and is called the Dupire (1994) PDE. □

Partial derivatives in time can be approximated using *forward* finite difference approximations as

$$\frac{\partial C}{\partial t}(t_i, y_j) \simeq \frac{C(t_{i+1}, y_j) - C(t_i, y_j)}{\Delta t}, \tag{9.15}$$

or, using *backward* finite difference approximations, as

$$\frac{\partial C}{\partial t}(t_i, y) \simeq \frac{C(t_i, y_j) - C(t_{i-1}, y_j)}{\Delta t}. \tag{9.16}$$

First order spatial derivatives can be approximated as

$$\frac{\partial C}{\partial y}(t, y_j) \simeq \frac{C(t_i, y_j) - C(t_i, y_{j-1})}{\Delta y}, \quad \frac{\partial C}{\partial y}(t, y_{j+1}) \simeq \frac{C(t_i, y_{j+1}) - C(t_i, y_j)}{\Delta y}. \tag{9.17}$$

Reusing (9.17), second order spatial derivatives can be similarly approximated as

$$\begin{aligned}\frac{\partial^2 C}{\partial y^2}(t_i, y_j) &\simeq \frac{1}{\Delta y}\left(\frac{\partial C}{\partial y}(t_i, y_{j+1}) - \frac{\partial C}{\partial y}(t_i, y_j)\right) \\ &\simeq \frac{C(t_i, y_{j+1}) - 2C(t_i, y_j) + C(t_i, y_{j-1})}{(\Delta y)^2}.\end{aligned} \tag{9.18}$$

Figure 9.14* presents an estimation of local volatility by the finite differences (9.15)-(9.18), based on Boeing (NYSE:BA) option price data.

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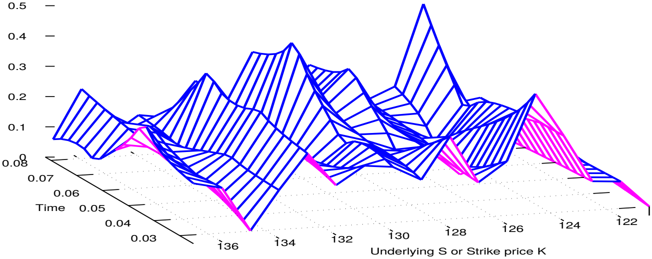

Fig. 9.14: Local volatility estimated from Boeing Co. option price data.

See Achdou and Pironneau (2005) and in particular Figure 8.1 therein for numerical methods applied to local volatility estimation using spline functions instead of the discretization (9.15)-(9.18). See also Ackerer et al. (2020), Chataigner et al. (2021) for deep learning approaches to the estimation of local volatility.

The attached **R code*** plots a local volatility estimate for a given stock, see also this **version** for the use of Black-Scholes derivatives as in Exercise 9.5-(9).

Based on (9.12), the local volatility $\sigma(t, y)$ can also be estimated by computing $C^M(T, y)$ from the Black-Scholes formula, from a value of the implied volatility σ.

Local volatility from put option prices

Note that by the call-put parity relation

$$C^M(T, y) = P^M(T, y) + x - y\,\mathrm{e}^{-rT}, \qquad y, T > 0,$$

where $S_0 = y$, cf. (6.24), we have

$$\begin{cases} \dfrac{\partial C^M}{\partial T}(T, y) = ry\,\mathrm{e}^{-rT} + \dfrac{\partial P^M}{\partial T}(T, y), \\[2ex] \dfrac{\partial P^M}{\partial y}(T, y) = \mathrm{e}^{-rT} + \dfrac{\partial C^M}{\partial y}(T, y), \end{cases}$$

and

$$\frac{\partial C^M}{\partial T}(T, y) + ry\frac{\partial C^M}{\partial y}(T, y) = \frac{\partial P^M}{\partial T}(T, y) + ry\frac{\partial P^M}{\partial y}(T, y).$$

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Consequently, the local volatility in Proposition 9.2 can be rewritten in terms of market put option prices as

$$\sigma(t,y) := \sqrt{\frac{2\dfrac{\partial P^M}{\partial t}(t,y) + 2ry\dfrac{\partial P^M}{\partial y}(t,y)}{y^2\dfrac{\partial^2 P^M}{\partial y^2}(t,y)}} = \frac{\sqrt{\dfrac{\partial P^M}{\partial t}(t,y) + ry\dfrac{\partial P^M}{\partial y}(t,y)}}{y\,\mathrm{e}^{-rT/2}\sqrt{\varphi_t(y)/2}},$$

which is formally identical to (9.12) after replacing market call option prices $C^M(T,K)$ with market put option prices $P^M(T,K)$. In addition, we have the relation

$$\varphi_T(K) = \mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial y^2}(T,y) = \mathrm{e}^{rT}\frac{\partial^2 P^M}{\partial y^2}(T,y) \tag{9.19}$$

between the probability density function φ_T of S_T and the call/put option pricing functions $C^M(T,y)$, $P^M(T,y)$.

9.4 The VIX® Index

Other ways to estimate market volatility include the CBOE Volatility Index® (VIX) for the S&P 500 Index (SPX). Let the asset price process $(S_t)_{t\in\mathbb{R}_+}$ satisfy

$$dS_t = rS_t dt + \sigma_t S_t dB_t,$$

i.e.

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dB_s + rt - \frac{1}{2}\int_0^t \sigma_s^2 ds\right), \qquad t \geqslant 0,$$

where, as in Section 8.2, $(\sigma_t)_{t\in\mathbb{R}_+}$ denotes a stochastic volatility process.

Lemma 9.3. *Let $\phi \in \mathcal{C}^2((0,\infty))$. For all $y > 0$, we have the following version of the Taylor formula:*

$$\phi(x) = \phi(y) + (x-y)\phi'(y) + \int_0^y (z-x)^+\phi''(z)dz + \int_y^\infty (x-z)^+\phi''(z)dz,$$

$x > 0$.

Proof. We use the Taylor formula with integral remainder:

$$\phi(x) = \phi(y) + (x-y)\phi'(y) + |x-y|^2\int_0^1 (1-\tau)\phi''(\tau x + (1-\tau)y)d\tau, \quad x,y \in \mathbb{R}.$$

Letting $z = \tau x + (1-\tau)y = y + \tau(x-y)$, if $x \leqslant y$ we have

$$|x-y|^2\int_0^1 (1-\tau)\phi''(\tau x + (1-\tau)y)d\tau = |x-y|\int_y^x \left(1 - \frac{z-y}{x-y}\right)\phi''(z)dz$$

$$= \int_y^x (x-z)\phi''(z)dz$$
$$= \int_y^\infty (x-z)^+\phi''(z)dz.$$

If $x \geqslant y$, we have

$$|x-y|^2 \int_0^1 (1-\tau)\phi''(\tau x + (1-\tau)y)d\tau = |y-x| \int_x^y \left(1 - \frac{y-z}{y-x}\right)\phi''(z)dz$$
$$= \int_x^y (z-x)\phi''(z)dz$$
$$= \int_0^y (z-x)^+\phi''(z)dz.$$

□

The next Proposition 9.4, cf. Remark 5 in Friz and Gatheral (2005) and page 4 of the CBOE white paper, shows that the VIX® Volatility Index defined as

$$\mathrm{VIX}_t := \sqrt{\frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\int_0^{F_{t,t+\tau}} \frac{P(t,t+\tau,K)}{K^2}dK + \int_{F_{t,t+\tau}}^\infty \frac{C(t,t+\tau,K)}{K^2}dK\right)} \tag{9.20}$$

at time $t > 0$ can be interpreted as an average of future volatility values, see also § 3.1.1 of Papanicolaou and Sircar (2014). Here, τ =30 days,

$$F_{t,t+\tau} := \mathbb{E}^*[S_{t+\tau} \mid \mathcal{F}_t] = \mathrm{e}^{r\tau}S_t$$

represents the future price on $S_{t+\tau}$, and $P(t,t+\tau,K)$, $C(t,t+\tau,K)$ are OTM (Out-Of-the-Money) call and put option prices with respect to $F_{t,t+\tau}$, with strike price K and maturity $t+\tau$.

Proposition 9.4. *The value of the VIX® Volatility Index at time $t \geqslant 0$ is given from the averaged realized variance swap price as*

$$\mathrm{VIX}_t := \sqrt{\frac{1}{\tau}\mathbb{E}^*\left[\int_t^{t+\tau} \sigma_u^2 du \,\middle|\, \mathcal{F}_t\right]}.$$

Proof. We take $t = 0$ for simplicity. Applying Lemma 9.3 to the function

$$\phi(x) = \frac{x}{y} - 1 - \log\frac{x}{y}$$

with $\phi'(x) = 1/y - 1/x$ and $\phi''(x) = 1/x^2$ shows that

$$\frac{x}{y} - 1 - \log\frac{x}{y} = \int_0^y (z-x)^+\frac{1}{z^2}dz + \int_y^\infty (x-z)^+\frac{1}{z^2}dz, \quad x, y > 0.$$

Alternatively, we can use the following relationships which are obtained by integration by parts:

$$\begin{aligned}\int_0^y (z-x)^+ \frac{dz}{z^2} &= \mathbb{1}_{\{x\leqslant y\}} \int_x^y (z-x) \frac{dz}{z^2} \\ &= \mathbb{1}_{\{x\leqslant y\}} \left(\int_x^y \frac{dz}{z} - x \int_x^y \frac{dz}{z^2} \right) \\ &= \mathbb{1}_{\{x\leqslant y\}} \left(\frac{x}{y} - 1 + \log \frac{y}{x} \right),\end{aligned}$$

and

$$\begin{aligned}\int_y^\infty (x-z)^+ \frac{dz}{z^2} &= \mathbb{1}_{\{x\geqslant y\}} \int_y^x (x-z) \frac{dz}{z^2} \\ &= \mathbb{1}_{\{x\geqslant y\}} \left(x \int_y^x \frac{dz}{z^2} - \int_y^x \frac{dz}{z} \right) \\ &= \mathbb{1}_{\{x\geqslant y\}} \left(\frac{x}{y} - 1 + \log \frac{y}{x} \right).\end{aligned}$$

Hence, taking $y := F_{0,\tau} = \mathrm{e}^{r\tau} S_0$ and $x := S_\tau$, we have

$$\frac{S_\tau}{F_{0,\tau}} - 1 + \log \frac{F_{0,\tau}}{S_\tau} = \int_0^{F_{0,\tau}} (K - S_\tau)^+ \frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty (S_\tau - K)^+ \frac{dK}{K^2}. \quad (9.21)$$

Next, taking expectations under $\mathbb{P}^*$ on both sides of (9.21) and using (9.20), we find

$$\begin{aligned}\mathrm{VIX}_0^2 &= \frac{2\,\mathrm{e}^{r\tau}}{\tau} \left(\int_0^{F_{0,\tau}} \frac{P(0,\tau,K)}{K^2} dK + \int_{F_{0,\tau}}^\infty \frac{C(0,\tau,K)}{K^2} dK \right) \\ &= \frac{2}{\tau} \int_0^{F_{0,\tau}} \mathbb{E}^*[(K-S_\tau)^+] \frac{dK}{K^2} + \frac{2}{\tau} \int_{F_{0,\tau}}^\infty \mathbb{E}^*[(S_\tau - K)^+] \frac{dK}{K^2} \\ &= \frac{2}{\tau} \mathbb{E}^* \left[\int_0^{F_{0,\tau}} (K-S_\tau)^+ \frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty (S_\tau - K)^+ \frac{dK}{K^2} \right] \\ &= \frac{2}{\tau} \mathbb{E}^* \left[\frac{S_\tau}{F_{0,\tau}} - 1 + \log \frac{F_{0,\tau}}{S_\tau} \right] \\ &= \frac{2}{\tau} \left(\frac{\mathbb{E}^*[S_\tau]}{F_{0,\tau}} - 1 \right) + \frac{2}{\tau} \mathbb{E}^* \left[\log \frac{F_{0,\tau}}{S_\tau} \right] \\ &= \frac{2}{\tau} \mathbb{E}^* \left[\log \frac{F_{0,\tau}}{S_\tau} \right] \\ &= -\frac{2}{\tau} \mathbb{E}^* \left[\log \frac{\mathrm{e}^{-r\tau} S_\tau}{S_0} \right] \\ &= -\frac{2}{\tau} \mathbb{E}^* \left[\int_0^\tau \sigma_t dB_t - \frac{1}{2} \int_0^\tau \sigma_t^2 dt \right] \\ &= \frac{1}{\tau} \mathbb{E}^* \left[\int_0^\tau \sigma_t^2 dt \right],\end{aligned}$$

as in the proof of Proposition 8.1. □

The following R code allows us to estimate the VIX® index based on the discretization of (9.20) and market option prices on the S&P 500 Index (SPX). Here, the OTM put strike prices and call strike prices are listed as

$$K_1^{(p)} < \cdots < K_{n_p-1}^{(p)} < K_{n_p}^{(p)} := F_{t,t+\tau} =: K_0^{(c)} < K_1^{(c)} < \cdots < K_{n_c}^{(c)},$$

and (9.20) may for example be discretized as

$$\begin{aligned}
\mathrm{VIX}_t^2 &= \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\int_0^{F_{t,t+\tau}} \frac{P(t,t+\tau,K)}{K^2}dK + \int_{F_{t,t+\tau}}^{\infty} \frac{C(t,t+\tau,K)}{K^2}dK\right)\\
&= \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\sum_{i=1}^{n_p-1}\int_{K_i^{(p)}}^{K_{i+1}^{(p)}} \frac{P(t,t+\tau,K)}{K^2}dK + \sum_{i=1}^{n_c}\int_{K_{i-1}^{(c)}}^{K_i^{(c)}} \frac{C(t,t+\tau,K)}{K^2}dK\right)\\
&\simeq \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\sum_{i=1}^{n_p-1}\int_{K_i^{(p)}}^{K_{i+1}^{(p)}} \frac{P\big(t,t+\tau,K_i^{(p)}\big)}{K^2}dK + \sum_{i=1}^{n_c}\int_{K_{i-1}^{(c)}}^{K_i^{(c)}} \frac{C\big(t,t+\tau,K_i^{(c)}\big)}{K^2}dK\right)\\
&= \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\sum_{i=1}^{n_p-1} P\big(t,t+\tau,K_i^{(p)}\big)\left(\frac{1}{K_i^{(p)}} - \frac{1}{K_{i+1}^{(p)}}\right)\right.\\
&\qquad \left. + \sum_{i=1}^{n_c}\int_{K_{i-1}^{(c)}}^{K_i^{(c)}} C\big(t,t+\tau,K_i^{(c)}\big)\left(\frac{1}{K_{i-1}^{(c)}} - \frac{1}{K_i^{(c)}}\right)\right),
\end{aligned}$$

see page 158 of Gatheral (2006) for the implementation of the discretization of the CBOE white paper.

```r
remotes::install_github("joshuaulrich/quantmod")
library(quantmod); getSymbols("^SPX", src = "yahoo")
lastBusDay=as.Date(last(row.names(as.data.frame(Ad(SPX)))))
S0 = as.vector(tail(Ad(SPX),1)); T = 30/365;r=0.02;F0 = S0*exp(r*T)
maturity <- as.Date(lastBusDay+31, format="%Y-%m-%d") # Choose a maturity in 30 days
SPX.OPTS <- getOptionChain("^SPX", maturity)
Call <- as.data.frame(SPX.OPTS$calls);Put <- as.data.frame(SPX.OPTS$puts)
Call_OTM <- Call[Call$Strike>F0,];Put_OTM <-Put[Put$Strike<F0,];
Call_OTM$dif = c(1/F0-1/min(Call_OTM$Strike),-diff(1/Call_OTM$Strike))
Put_OTM$dif = c(-diff(1/Put_OTM$Strike),1/max(Put_OTM$Strike)-1/F0)
VIX_imp = 100*sqrt((2*exp(r*T)/T)*(sum(Put_OTM$Last*Put_OTM$dif)
      +sum(Call_OTM$Last*Call_OTM$dif)))
getSymbols("^VIX", src = "yahoo", from = lastBusDay);VIX_market = as.vector(Ad(VIX)[1])
c("Estimated VIX"= VIX_imp, "CBOE VIX"=VIX_market)
VIX_imp - (F0/max(Put$Strike[Put$Strike<F0])-1)^2/T
VIX.OPTS <- getOptionChain("^VIX")
```

The following R code is fetching VIX® index data using the quantmod package.

```
library(quantmod)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo")
dev.new(width=16,height=7); myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(Ad(`GSPC`),name="S&P500",pars=myPars,theme=myTheme)
add_TA(Ad(`VIX`))
```

The impact of various world events can be identified on the VIX® index in Figure 9.15.

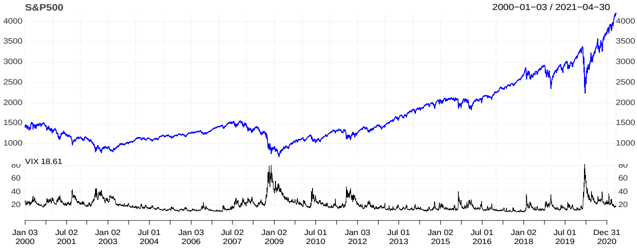


Fig. 9.15: VIX® Index *vs.* S&P 500.

```
library(quantmod);library(PerformanceAnalytics)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo");SP500=Ad(`GSPC`)
SP500.rtn <- exp(CalculateReturns(SP500,method="compound")) - 1;SP500.rtn[1,] <- 0
histvol <- rollapply(SP500.rtn, width = 30, FUN=sd.annualized)
dev.new(width=16,height=7)
myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(SP500,name="SP500",theme=myTheme,pars=myPars)
add_TA(histvol, name="Historical Volatility");add_TA(Ad(`VIX`), name="VIX")
```

Figure 9.16 compares the VIX® index estimate to the historical volatility of Section 9.1.

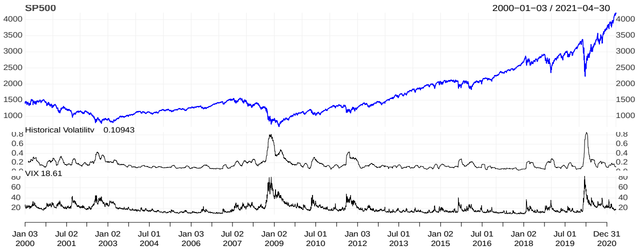
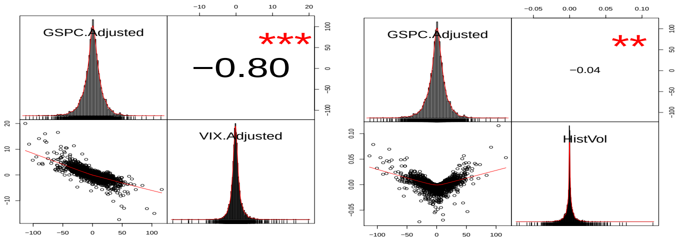


Fig. 9.16: VIX® Index *vs.* historical volatility for the year 2011.

We note that the variations of the stock index are negatively correlated to the variations of the VIX® index, however the same cannot be said of the correlation to the variations of historical volatility.



(a) GSPC returns *vs.* VIX®. (b) GSPC returns *vs.* hist. vol.

Fig. 9.17: Correlation estimates between GSPC and the VIX®.

```
chart.Correlation(cbind(Ad(`GSPC`)-lag(Ad(`GSPC`)),Ad(`VIX`)-lag(Ad(`VIX`))),
      histogram=TRUE, pch="+")
colnames(histvol) <- "HistVol"
chart.Correlation(cbind(Ad(`GSPC`)-lag(Ad(`GSPC`)),histvol-lag(histvol)), histogram=TRUE,
      pch="+")
```

Figure 9.18 shortens the time range to year 2011 and shows the increased reactivity of the VIX® index to volatility spikes, in comparison with the moving average of historical volatility.

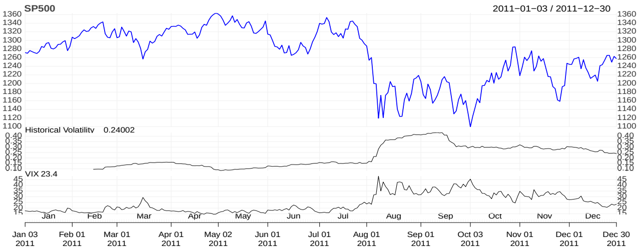


Fig. 9.18: VIX® Index *vs.* 30 day historical volatility for the S&P 500.

Exercises

Exercise 9.1 We consider an entropy swap with discrete-time payoff

$$\frac{1}{T}\sum_{k=1}^{N} S_{t_k} \log \frac{S_{t_k}}{S_{t_{k-1}}} - \kappa_\sigma^2 = \frac{1}{T}\sum_{k=1}^{N} S_{t_k} (\log S_{t_k} - \log S_{t_{k-1}}) - \kappa_\sigma^2,$$

approximated in continous time as

$$\frac{1}{T}\int_0^T S_t d\log S_t - \kappa_\sigma^2,$$

where κ_σ is the volatility level.

a) Show that for any $K^* > 0$ we have

$$\int_0^T S_t d\log S_t = f_{K^*}(S_T) - f_{K^*}(S_0) + \int_0^T dS_t + \int_0^T \log\frac{S_t}{K^*} dS_t, \quad (9.22)$$

where $f_{K^*}(x) := x - K^* - x\log(x/K^*)$.
Hint: Use $d\log S_t$ as well.

b) Show that the payoff $f_{K^*}(S_T)$ can be hedged using a portfolio of call and put options.
Hint: Use Lemma 9.3 with $y := K^*$.

Exercise 9.2 Maturity (or calendar) arbitrage. We consider a market with vanishing risk-free interest rate $r = 0$.

a) Given a strike prices K and two maturities $T_1 < T_2$ with $K_2 - T_1 = \Delta T > 0$, write down a discretized expression of the first partial derivative

$$\frac{\partial C}{\partial T}(T, K)_{|T=(T_1+T_2)/2}.$$

b) Show that if $\frac{\partial C}{\partial T}(T, K)_{|T=(T_1+T_2)/2} < 0$, one can construct a portfolio leading to an arbitrage opportunity.

Exercise 9.3 Strike arbitrage.

a) Given a set of three strike prices $K_1 < K_2 < K_3$ with $K_3 - K_2 = K_2 - K_1 = \Delta K > 0$, write down a discretized expression of the second partial derivative

$$\frac{\partial^2 C}{\partial K^2}(T, K)_{|K=K_2}.$$

b) Show that if $\frac{\partial^2 C}{\partial K^2}(T, K)_{|K=K_2} < 0$, one can construct a portfolio leading to an arbitrage opportunity.

Hint: Choose your own values of K_1, K_2, K_3, use https://optioncreator.com/, and Save your portfolio design.

Exercise 9.4 Consider the Black-Scholes call pricing formula

$$C(T-t,x,K)=Kf\left(T-t,\frac{x}{K}\right)$$

written using the function

$$f(\tau,z):=z\Phi\left(\frac{(r+\sigma^2/2)\tau+\log z}{|\sigma|\sqrt{\tau}}\right)-\mathrm{e}^{-r\tau}\Phi\left(\frac{(r-\sigma^2/2)\tau+\log z}{|\sigma|\sqrt{\tau}}\right).$$

a) Compute $\frac{\partial C}{\partial x}$ and $\frac{\partial C}{\partial K}$ using the function f, and find the relation between $\frac{\partial C}{\partial K}(T-t,x,K)$ and $\frac{\partial C}{\partial x}(T-t,x,K)$.

b) Compute $\frac{\partial^2 C}{\partial x^2}$ and $\frac{\partial^2 C}{\partial K^2}$ using the function f, and find the relation between $\frac{\partial C^2}{\partial K^2}(T-t,x,K)$ and $\frac{\partial C^2}{\partial x^2}(T-t,x,K)$.

c) From the Black-Scholes PDE

$$rC(T-t,x,K)=\frac{\partial C}{\partial t}(T-t,x,K)+rx\frac{\partial C}{\partial x}(T-t,x,K)+\frac{\sigma^2x^2}{2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K),$$

recover the Dupire (1994) PDE (9.14) for the constant volatility σ.

Exercise 9.5

a) Using Lemma 9.1 and the Black-Scholes Greek Gamma expression

$$\begin{aligned}\frac{\partial^2 C}{\partial x^2}(T-t,x,K)&=\frac{1}{\sigma x\sqrt{T-t}}\Phi'(d_+(T-t))\\&=\frac{1}{\sigma x\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2},\end{aligned}$$

see Table 6.1, recover the lognormal probability density function $\varphi_T(y)$ of geometric Brownian motion S_T.

b) Using Proposition 9.12 and the expressions of the Black-Scholes Greeks Delta and Theta, see Table 6.1, recover the constant volatility value σ.

Exercise 9.6 The prices of call options in a certain local volatility model of the form $dS_t=S_t\sigma(t,S_t)dB_t$ with risk-free rate $r=0$ are given by

$$C(T,K)=\eta\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(K-S_0)^2/(2\eta^2T)}-(K-S_0)\Phi\left(-\frac{K-S_0}{\eta\sqrt{T}}\right),\quad K,T>0,$$

where $\eta > 0$. Recover the local volatility function $\sigma(t, x)$ of this model by applying the Dupire formula (9.12).

Exercise 9.7 Let $\sigma_{\rm imp}(K)$ denote the implied volatility of a call option with strike price K, defined from the relation

$$M_C(K, S, r, \tau) = C(K, S, \sigma_{\rm imp}(K), r, \tau),$$

where M_C is the market price of the call option, $C(K, S, \sigma_{\rm imp}(K), r, \tau)$ is the Black-Scholes call pricing function, S is the underlying asset price, τ is the time remaining until maturity, and r is the risk-free interest rate.

a) Compute the partial derivative

$$\frac{\partial M_C}{\partial K}(K, S, r, \tau).$$

using the functions C and $\sigma_{\rm imp}$.

b) Knowing that market call option prices $M_C(K, S, r, \tau)$ are *decreasing* in the strike prices K, find an upper bound for the slope $\sigma'_{\rm imp}(K)$ of the implied volatility curve.

c) Similarly, knowing that the market *put* option prices $M_P(K, S, r, \tau)$ are *increasing* in the strike prices K, find a lower bound for the slope $\sigma'_{\rm imp}(K)$ of the implied volatility curve.

Exercise 9.8 (Hagan et al. (2002)) Consider the European option priced as $\mathrm{e}^{-rT}\mathbb{E}^*[(S_T - K)^+]$ in a local volatility model $dS_t = \sigma_{\rm loc}(S_t)S_t dB_t$. The implied volatility $\sigma_{\rm imp}(K, S_0)$, computed from the equation

$$\mathrm{Bl}\big(S_0, K, T, \sigma_{\rm imp}(K, S_0), r\big) = \mathrm{e}^{-rT}\mathbb{E}^*[(S_T - K)^+],$$

is known to admit the approximation

$$\sigma_{\rm imp}(K, S_0) \simeq \sigma_{\rm loc}\left(\frac{K + S_0}{2}\right).$$

a) Taking a local volatility of the form $\sigma_{\rm loc}(x) := \sigma_0 + \beta(x - S_0)^2$, estimate the implied volatility $\sigma_{\rm imp}(K, S)$ when the underlying asset price is at the level S.

b) Express the Delta of the Black Scholes call option price given by

$$\mathrm{Bl}\big(S, K, T, \sigma_{\rm imp}(K, S), r\big),$$

using the standard Black-Scholes Delta and the Black-Scholes Vega.

Exercise 9.9 Show that the result of Proposition 9.4 can be recovered from Lemma 8.2 and Relation (9.19).

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Exercise 9.10 (Exercise 8.8 continued). Find an expression for $\mathbb{E}^*\left[R_{0,T}^4\right]$ using call and put pricing functions.

Exercise 9.11 (Henry-Labordère (2009), § 3.5).

a) Using the gamma probability density function and integration by parts or Laplace transform inversion, prove the formula

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}} dx = \frac{\mu^\rho - \nu^\rho}{\rho} \Gamma(1-\rho)$$

for all $\rho \in (0,1)$ and $\mu, \nu > 0$, see Relation 3.434.1 in Gradshteyn and Ryzhik (2007).

b) By the result of Question (a), generalize the volatility swap pricing expression (8.19).

c) By Lemma 8.2 and the result of Question (b), find an expression of the volatility swap price using call and put functions.