Chapter 8
Stochastic Volatility

Stochastic volatility refers to the modeling of volatility using time-dependent stochastic processes, in contrast to the constant volatility assumption made in the standard Black-Scholes model. In this setting, we consider the pricing of realized variance swaps and options using moment matching approximations. We also cover the pricing of vanilla options by PDE arguments in the Heston model, and by perturbation analysis approximations in more general stochastic volatility models.

8.1 Stochastic Volatility Models

Time-dependent stochastic volatility

The next Figure 8.1 refers to the EURO/SGD exchange rate, and shows some spikes that cannot be generated by Gaussian returns with constant variance.

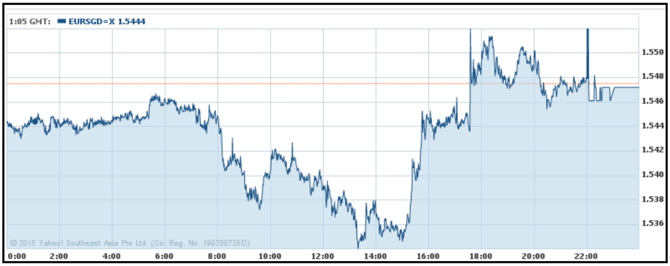

Fig. 8.1: Euro / SGD exchange rate.

This type data shows that, in addition to jump models that are commonly used to take into account the slow decrease of probability tails observed in market data, other tools should be implemented in order to model a possibly random and time-varying volatility.

We consider an asset price driven by the stochastic differential equation

$$dS_t = rS_t dt + S_t\sqrt{v_t}dB_t \tag{8.1}$$

under the risk-neutral probability measure $\mathbb{P}^*$, with solution

$$S_T = S_t \exp\left((T-t)r + \int_t^T \sqrt{v_s}dB_s - \frac{1}{2}\int_t^T v_s ds\right) \tag{8.2}$$

where $(v_t)_{t\in\mathbb{R}_+}$ is a (possibly random) squared volatility (or variance) process adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$.

Time-dependent deterministic volatility

When the variance process $(v(t))_{t\in\mathbb{R}_+}$ is a deterministic function of time, the solution (8.2) of (8.1) is a lognormal random variable at time T with conditional log-variance

$$\int_t^T v(s)ds$$

given $\mathcal{F}_t$. In particular, the European call option on S_T can be priced by the Black-Scholes formula as

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t] = \mathrm{Bl}\big(S_t, K, r, T-t, \sqrt{\widehat{v}(t)}\big),$$

with integrated squared volatility parameter

$$\widehat{v}(t) := \frac{\int_t^T v(s)ds}{T-t}, \qquad t\in[0,T).$$

Independent (stochastic) volatility

When the volatility $(v_t)_{t\in\mathbb{R}_+}$ is a random process generating a filtration $\big(\mathcal{F}_t^{(2)}\big)_{t\in\mathbb{R}_+}$, independent of the filtration $\big(\mathcal{F}_t^{(1)}\big)_{t\in\mathbb{R}_+}$ generated by the driving Brownian motion $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$, the equation (8.1) can still be solved as

$$S_T = S_t \exp\left((T-t)r + \int_t^T \sqrt{v_s}dB_s^{(1)} - \frac{1}{2}\int_t^T v_s ds\right),$$

and, given $\mathcal{F}_T^{(2)}$, the asset price S_T is a lognormal random variable with random variance

$$\int_t^T v_s ds.$$

In this case, taking

$$\mathcal{F}_t := \mathcal{F}_t^{(1)} \vee \mathcal{F}_t^{(2)}, \qquad 0 \leqslant t \leqslant T,$$

where $\big(\mathcal{F}_t^{(1)}\big)_{t\in\mathbb{R}_+}$ is the filtration generated by $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$, we can still price an option with payoff $\phi(S_T)$ on the underlying asset price S_T using the tower property

$$\mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t] = \mathbb{E}^*\big[\mathbb{E}^*\big[\phi(S_T) \,\big|\, \mathcal{F}_t^{(1)} \vee \mathcal{F}_T^{(2)}\big] \,\big|\, \mathcal{F}_t^{(1)} \vee \mathcal{F}_t^{(2)}\big].$$

As an example, the European call option on S_T can be priced by averaging the Black-Scholes formula as follows:

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t] \\
&\quad = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\mathbb{E}^*\big[(S_T-K)^+ \,\big|\, \mathcal{F}_t^{(1)} \vee \mathcal{F}_T^{(2)}\big] \,\big|\, \mathcal{F}_t^{(1)} \vee \mathcal{F}_t^{(2)}\big]. \\
&\quad = \mathbb{E}^*\left[\mathrm{Bl}\left(S_t, K, r, T-t, \sqrt{\frac{\int_t^T v_s ds}{T-t}}\right) \,\middle|\, \mathcal{F}_t\right] \\
&\quad = \mathbb{E}^*\big[\mathrm{Bl}\big(x, K, r, T-t, \sqrt{\widehat{v}(t,T)}\big) \,\big|\, \mathcal{F}_t^{(2)}\big]_{|x=S_t},
\end{aligned}$$

which represents an averaged version of Black-Scholes prices, with the random integrated volatility

$$\widehat{v}(t,T) := \frac{1}{T-t}\int_t^T v_s ds, \qquad 0 \leqslant t \leqslant T.$$

On the other hand, the probability distribution of the time integral $\int_t^T v_s ds$ given $\mathcal{F}_t^{(2)}$ can be computed using integral expressions, see Yor (1992) and Proposition 13.5 when $(v_t)_{t\in\mathbb{R})_+}$ is a geometric Brownian motion, and Lemma 9 in Feller (1951) or Corollary 24 in Albanese and Lawi (2005) and (17.6) when $(v_t)_{t\in\mathbb{R})_+}$ is the CIR process.

Two-factor stochastic volatility model

Evidence based on financial market data, see Figure 9.16, Figure 1 of Papanicolaou and Sircar (2014) or § 2.3.1 in Fouque et al. (2011), shows that the variations in volatility tend to be negatively correlated with the variations of underlying asset prices. For this reason we need to consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ and a stochastic volatility process $(v_t)_{t\in\mathbb{R}_+}$ driven by

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dB_t^{(1)} \\ \\ dv_t = \mu(t, v_t)dt + \beta(t, v_t)dB_t^{(2)}, \end{cases}$$

Here, $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are possibly correlated Brownian motions, with

$$\operatorname{Cov}\big(B_t^{(1)}, B_t^{(2)}\big) = \rho t \quad \text{and} \quad dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt,$$

where the correlation parameter ρ satisfies $-1 \leqslant \rho \leqslant 1$, and the coefficients $\mu(t,x)$ and $\beta(t,x)$ can be chosen *e.g.* from mean-reverting models (CIR) or geometric Brownian models, as follows. Note that the observed correlation coefficient ρ is usually negative, cf. *e.g.* § 2.1 in Papanicolaou and Sircar (2014) and Figures 9.16 and 9.17.

The Heston model

In the Heston (1993) model, the stochastic volatility $(v_t)_{t\in\mathbb{R}_+}$ is chosen to be a Cox et al. (1985) (CIR) process, *i.e.* we have

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{v_t}dB_t^{(1)} \\ \\ dv_t = -\lambda(v_t - m)dt + \eta\sqrt{v_t}dB_t^{(2)}, \end{cases}$$

and $\mu(t,v) = -\lambda(v-m)$, $\beta(t,v) = \eta\sqrt{v}$, where λ, m, $\eta > 0$.

Option pricing formulas can be derived in the Heston model using Fourier inversion and complex integrals, cf. (8.29) below.

The SABR model

In the Sigma-Alpha-Beta-Rho (σ-α-β-ρ-SABR) model Hagan et al. (2002), based on the parameters (α, β, ρ), the stochastic volatility process $(\sigma_t)_{t\in\mathbb{R}_+}$

is modeled as a geometric Brownian motion with

$$\begin{cases} dF_t = \sigma_t F_t^\beta dB_t^{(1)} \\ \\ d\sigma_t = \alpha\sigma_t dB_t^{(2)}, \end{cases}$$

where $(F_t)_{t\in\mathbb{R}_+}$ typically models a forward interest rate. Here, we have $\alpha > 0$ and $\beta \in (0,1]$, and $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$, $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard Brownian motions with the correlation

$$dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt.$$

This setting is typically used for the modeling of LIBOR rates and is *not* mean-reverting, hence it is preferably used with a short time horizon. It allows in particular for short time asymptotics of Black implied volatilities that can be used for pricing by inputting them into the Black pricing formula, cf. § 3.3 in Rebonato (2009).

8.2 Realized Variance Swaps

Another look at historical volatility

In this section, given $T > 0$ and $N \geqslant 1$, we let

$$t_k := k\frac{T}{N}, \qquad k = 0, 1, \ldots, N.$$

a natural estimator for the trend parameter μ can be written in terms of actual returns as

$$\widehat{\mu}_N := \frac{1}{N}\sum_{k=1}^N \frac{1}{t_k - t_{k-1}} \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}},$$

or in terms of log-returns as

$$\begin{aligned} \widehat{\mu}_N &:= \frac{1}{N}\sum_{k=1}^N \frac{1}{t_k - t_{k-1}} \log\frac{S_{t_k}}{S_{t_{k-1}}} \\ &= \frac{1}{T}\sum_{k=1}^N \big(\log(S_{t_k}) - \log(S_{t_{k-1}})\big) \\ &= \frac{1}{T}\log\frac{S_T}{S_0}. \end{aligned}$$

Similarly, one can use the squared volatility (or realized variance) estimator

$$\widehat{\sigma}_N^2 := \frac{1}{N-1}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}-(t_{k+1}-t_k)\widehat{\mu}_N\right)^2$$
$$= \frac{1}{N-1}\sum_{k=1}^{N}\frac{1}{t_k-t_{k-1}}\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2-\frac{T}{N-1}(\widehat{\mu}_N)^2$$

using actual returns, or, using log-returns,*

$$\widehat{\sigma}_N^2 := \frac{1}{N-1}\sum_{k=1}^{N}\frac{1}{t_k-t_{k-1}}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}-(t_k-t_{k-1})\widehat{\mu}_N\right)^2$$
$$= \frac{1}{N-1}\sum_{k=1}^{N}\frac{1}{t_k-t_{k-1}}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2-\frac{T}{N-1}(\widehat{\mu}_N)^2. \tag{8.3}$$

Realized variance swaps

Realized variance swaps are forward contracts that allow for the exchange of the estimated volatility (8.3) against a fixed value κ_σ. They can be priced using log-returns and expected value as

$$\mathbb{E}\big[\widehat{\sigma}_N^2\big] = \frac{1}{T}\mathbb{E}\left[\sum_{k=1}^{N}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2-\frac{1}{N-1}\left(\log\frac{S_T}{S_0}\right)^2\right]-\kappa_\sigma^2$$

of their payoff

$$\frac{1}{T}\left(\sum_{k=1}^{N}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2-\frac{1}{N-1}\left(\log\frac{S_T}{S_0}\right)^2\right)-\kappa_\sigma^2,$$

where κ_σ is the volatility level. Note that the above payoff has to be multiplied by the *Vega notional*, which is part of the contract, in order to convert it into currency units.

Heston model

Consider the Heston (1993) model driven by the stochastic differential equation

$$dv_t = (a-bv_t)dt+\sigma\sqrt{v_t}dW_t,$$

where $a, b, \sigma > 0$. We have

$$\mathbb{E}[v_T] = v_0e^{-bT}+\frac{a}{b}\left(1-\mathrm{e}^{-bT}\right),$$

* We apply the identity $\sum_{k=1}^{n}\left(a_k-\sum_{l=1}^{n}a_l\right)^2=\sum_{k=1}^{n}a_k^2-\left(\sum_{l=1}^{n}a_l\right)^2$.

see Exercise 4.20-(b), and Exercise 8.2-(a), from which it follows that the realized variance $R_{0,T}^2 := \int_0^T v_t dt$ can be averaged as

$$\begin{aligned}\mathbb{E}\big[R_{0,T}^2\big] &= \mathbb{E}\left[\int_0^T v_t dt\right] \\ &= \int_0^T \mathbb{E}[v_t] dt \\ &= v_0 \frac{1-\mathrm{e}^{-bT}}{b} + a\frac{\mathrm{e}^{-bT}+bT-1}{b^2},\end{aligned} \tag{8.4}$$

and the variance swap with strike level κ_σ^2 and payoff $R_{0,T}^2 - \kappa_\sigma^2$ can be priced as

$$\mathbb{E}\big[R_{0,T}^2 - \kappa_\sigma^2\big] = v_0 \frac{1-\mathrm{e}^{-bT}}{b} + a\frac{\mathrm{e}^{-bT}+bT-1}{b^2} - \kappa_\sigma^2.$$

We can also express the variances

$$\mathrm{Var}[v_T] = v_0 \frac{\sigma^2}{b}\big(\mathrm{e}^{-bT} - \mathrm{e}^{-2bT}\big) + \frac{a\sigma^2}{2b^2}\big(1-\mathrm{e}^{-bT}\big)^2,$$

cf. Exercise 4.20-(e), and

$$\begin{aligned}\mathrm{Var}\left[R_{0,T}^2\right] &= v_0\sigma^2 \frac{1-2bTe^{-bT} - \mathrm{e}^{-2bT}}{b^3} \\ &\quad + a\sigma^2 \frac{\mathrm{e}^{-2bT} + 2bT + 4(bT+1)\,\mathrm{e}^{-bT} - 5}{2b^4},\end{aligned} \tag{8.5}$$

see *e.g.* Relation (3.3) in Prayoga and Privault (2017).

Stochastic volatility

In what follows, we assume that the risky asset price process is given by

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t, \tag{8.6}$$

under the risk-neutral probability measure $\mathbb{P}^*$, *i.e.*

$$S_t = S_0 \exp\left(rt + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right), \quad t \geqslant 0, \tag{8.7}$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ is a stochastic volatility process. In this setting, we have the following proposition.

Proposition 8.1. *Denoting by* $F_0 := \mathrm{e}^{rT} S_0$ *the futures contract price on* S_T*, we have the relation*

$$\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2\mathbb{E}^*\left[\log \frac{F_0}{S_T}\right]. \tag{8.8}$$

Proof. From (8.7), we have

$$\begin{aligned}
\mathbb{E}^*\left[\log \frac{S_T}{F_0}\right] &= \mathbb{E}^*\left[\log \frac{S_T}{S_0}\right] - rT \\
&= \mathbb{E}^*\left[\int_0^T \sigma_t dB_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right] \\
&= -\frac{1}{2}\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right].
\end{aligned}$$

□

Independent stochastic volatility

In this subsection, we assume that the stochastic volatility process $(\sigma_t)_{t\in\mathbb{R}_+}$ in (8.6) is *independent* of the Brownian motion $(B_t)_{t\in\mathbb{R}_+}$.

Lemma 8.2. *(Carr and Lee (2008), Proposition 5.1) Assume that $(\sigma_t)_{t\in\mathbb{R}_+}$ is* independent *of $(B_t)_{t\in\mathbb{R}_+}$, and let*

$$p_\lambda^\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}.$$

Then, for every $\lambda > 0$ we have

$$\mathbb{E}^*\left[\exp\left(\lambda\int_0^T \sigma_t^2 dt\right)\right] = \mathrm{e}^{-rp_\lambda^\pm T}\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda^\pm}\right]. \tag{8.9}$$

Proof. Letting $(\mathcal{F}_t^\sigma)_{t\in\mathbb{R}_+}$ denote the filtration generated by the process $(\sigma_t)_{t\in\mathbb{R}_+}$, we have

$$\begin{aligned}
\mathrm{e}^{-rp_\lambda T}\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda}\middle|\mathcal{F}_T^\sigma\right] &= \mathbb{E}^*\left[\exp\left(p_\lambda\int_0^T \sigma_t dB_t - \frac{p_\lambda}{2}\int_0^T \sigma_t^2 dt\right)\middle|\mathcal{F}_T^\sigma\right] \\
&= \exp\left(-\frac{p_\lambda}{2}\int_0^T \sigma_t^2 dt\right)\mathbb{E}^*\left[\exp\left(p_\lambda\int_0^T \sigma_t dB_t\right)\middle|\mathcal{F}_T^\sigma\right] \\
&= \exp\left(-\frac{p_\lambda}{2}\int_0^T \sigma_t^2 dt\right)\exp\left(\frac{p_\lambda^2}{2}\int_0^T \sigma_t^2 dt\right) \\
&= \exp\left(\frac{p_\lambda}{2}(p_\lambda - 1)\int_0^T \sigma_t^2 dt\right) \\
&= \exp\left(\lambda\int_0^T \sigma_t^2 dt\right),
\end{aligned}$$

provided that $\lambda = p_\lambda(p_\lambda - 1)/2$, and in this case we have

$$
\begin{aligned}
\mathrm{e}^{-rp_\lambda T}\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda}\right] &= \mathrm{e}^{-rp_\lambda T}\mathbb{E}^*\left[\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda}\,\Big|\,\mathcal{F}_T^\sigma\right]\right] \\
&= \mathbb{E}^*\left[\exp\left(\lambda\int_0^T \sigma_t^2 dt\right)\right].
\end{aligned}
$$

It remains to note that the equation $\lambda = p_\lambda(p_\lambda - 1)/2$, *i.e.* $p_\lambda^2 - p_\lambda - 2\lambda = 0$, has for solutions

$$
p_\lambda^\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda},
$$

with $p_\lambda^- < 0 < p_\lambda^+$ when $\lambda > 0$. □

By differentiating the moment generating function computed in Lemma 8.2 with respect to $\lambda > 0$, we can compute the first moment of the realized variance $R_{0,T}^2 = \int_0^T \sigma_t^2 dt$ in the following corollary.

Corollary 8.3. *Assume that* $(\sigma_t)_{t\in\mathbb{R}_+}$ *is* independent *of* $(B_t)_{t\in\mathbb{R}_+}$. *Denoting by* $F_0 := \mathrm{e}^{rT}S_0$ *the futures contract price on* S_T*, we have*

$$
\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2\mathbb{E}^*\left[\frac{S_T}{F_0}\log\frac{S_T}{F_0}\right].
$$

Proof. Rewriting (8.9) as

$$
\mathbb{E}^*\left[\exp\left(\lambda\int_0^T \sigma_t^2 dt\right)\right] = \mathbb{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm \log\frac{S_T}{S_0}\right)\right]
$$

and differentiating this relation with respect to λ, we get

$$
\begin{aligned}
\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt \exp\left(\lambda\int_0^T \sigma_t^2 dt\right)\right] &= -rp_\lambda' T\mathbb{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm \log\frac{S_T}{S_0}\right)\right] \\
&\quad + p_\lambda'\mathbb{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm\log\frac{S_T}{S_0}\right)\log\frac{S_T}{S_0}\right] \\
&= \mp\frac{rT}{\sqrt{2\lambda+1/4}}\mathbb{E}^*\left[\exp\left(-rp_\lambda^\pm T\right)\left(\frac{S_T}{S_0}\right)^{p_\lambda^\pm}\right] \\
&\quad \pm\frac{1}{\sqrt{2\lambda+1/4}}\mathbb{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm\log\frac{S_T}{S_0}\right)\log\frac{S_T}{S_0}\right],
\end{aligned}
$$

which, when $\lambda = 0$, recovers (8.8) in Proposition 8.1 as

$$
\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2rT - 2\mathbb{E}^*\left[\log\frac{S_T}{S_0}\right] = -2\mathbb{E}^*\left[\int_0^T \sigma_t dB_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right]
$$

if $p_0^- = 0$, and yields

$$\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2\,\mathrm{e}^{-rT}\mathbb{E}^*\left[\frac{S_T}{S_0}\log\frac{\mathrm{e}^{-rT}S_T}{S_0}\right]$$

for $p_0^+ = 1$. □

8.3 Realized Variance Options

In this section, we consider the realized variance call option with payoff

$$\left(\int_0^T \sigma_t^2 dt - \kappa_\sigma^2\right)^+.$$

Proposition 8.4. *Under the condition $\int_0^t \sigma_u^2 du \geqslant \kappa_\sigma^2$, the price of the realized variance call option in the money is given by*

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\int_0^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \middle| \mathcal{F}_t\right] \\
&\quad = \mathrm{e}^{-(T-t)r}\int_0^t \sigma_u^2 du - \mathrm{e}^{-(T-t)r}\kappa_\sigma^2 + \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right].
\end{aligned}$$

Proof. In case $\int_0^t \sigma_u^2 du \geqslant \kappa_\sigma^2$, we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\int_0^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \middle| \mathcal{F}_t\right] \\
&\quad = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \middle| \mathcal{F}_t\right]_{x=\int_0^t \sigma_u^2 du} \\
&\quad = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2 \middle| \mathcal{F}_t\right]_{x=\int_0^t \sigma_u^2 du} \\
&\quad = \mathrm{e}^{-(T-t)r}\int_0^t \sigma_u^2 du - \mathrm{e}^{-(T-t)r}\kappa_\sigma^2 + \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right].
\end{aligned}$$

□

In Proposition 8.4, the futures contract price $\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right]$ can be computed from Proposition 8.1.

Lognormal approximation

When $R_{0,t}^2 := \int_0^t \sigma_u^2 du < \kappa_\sigma^2$, in order to estimate the price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(x+\int_t^T\sigma_u^2du-\kappa_\sigma^2\right)^+\,\middle|\,\mathcal{F}_t\right]_{x=\int_0^t\sigma_u^2du}, \tag{8.10}$$

of the realized variance call option out of the money, we can approximate $R_{t,T} := \sqrt{\int_t^T \sigma_u^2 du}$ by a lognormal random variable

$$R_{t,T} = \sqrt{\int_t^T \sigma_u^2 du} \simeq \mathrm{e}^{\tilde{\mu}_{t,T}+\widetilde{\sigma}_{t,T}X}$$

with mean $\tilde{\mu}_{t,T}$ and variance $\eta_{t,T}^2$, where $X \simeq \mathcal{N}(0, T-t)$ is a centered Gaussian random variable with variance $T-t$.

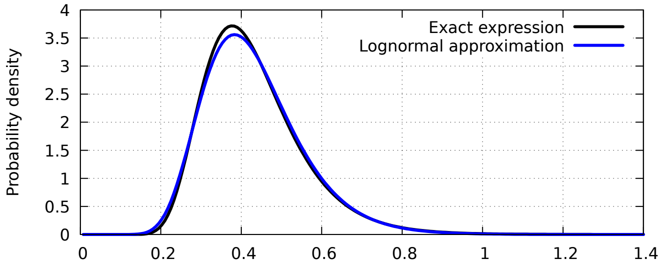

Fig. 8.2: Fitting of a lognormal probability density function (example).

Proposition 8.5. *(Lognormal approximation by volatility swap moment matching). The probability density function $\varphi_{R_{t,T}}$ of $R_{t,T} := \sqrt{\int_t^T \sigma_u^2 du}$ can be approximated as*

$$\varphi_{R_{t,T}}(x) \approx \frac{1}{x\widetilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}}\exp\left(-\frac{(\tilde{\mu}_{t,T}-\log x)^2}{2(T-t)\widetilde{\sigma}_{t,T}^2}\right), \quad x>0, \tag{8.11}$$

where

$$\tilde{\mu}_{t,T} := \log\left(\frac{\left(\mathbb{E}[R_{t,T}]\right)^2}{\sqrt{\mathbb{E}\left[R_{t,T}^2\right]}}\right) \quad \text{and} \quad \widetilde{\sigma}_{t,T}^2 := \frac{2}{T-t}\log\left(\frac{\sqrt{\mathbb{E}\left[R_{t,T}^2\right]}}{\mathbb{E}[R_{t,T}]}\right), \tag{8.12}$$

and $\mathbb{E}\left[R_{t,T}^2\right]$, $\mathbb{E}[R_{t,T}]$ can be estimated from realized variance and volatility swap prices.

Proof. The parameters $\tilde{\mu}_{t,T}$ and $\widetilde{\sigma}_{t,T}$ are estimated by matching the first and second moments $\mathbb{E}\left[R_{t,T}\right]$ and $\mathbb{E}\left[R_{t,T}^2\right]$ of $R_{t,T}$ to those of the lognormal distribution with mean $\tilde{\mu}_{t,T}$ and variance $(T-t)\widetilde{\sigma}_{t,T}^2$, which yields

$$\mathbb{E}[R_{t,T}] = \mathrm{e}^{\tilde{\mu}_{t,T}+(T-t)\widetilde{\sigma}^2_{t,T}/2} \quad \text{and} \quad \mathbb{E}\big[R^2_{t,T}\big] = \mathrm{e}^{2(\tilde{\mu}_{t,T}+(T-t)\widetilde{\sigma}^2_{t,T})},$$

and (8.12). □

By (8.12), the parameters $\tilde{\mu}_{t,T}$ and $\widetilde{\sigma}^2_{t,T}$ can be estimated from the realized volatility swap price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[R_{t,T} \mid \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\sqrt{\int_t^T \sigma_u^2 du}\,\middle|\,\mathcal{F}_t\right],$$

and from the realized variance swap price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[R^2_{t,T} \mid \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\int_t^T \sigma_u^2 du\,\middle|\,\mathcal{F}_t\right].$$

By Proposition 8.7, we can estimate the price (8.10) of the realized variance call option by approximating $R^2_{t,T} = \int_t^T \sigma_u^2 du$ by a lognormal random variable. We refer to § 8.4 in Friz and Gatheral (2005) or to Relation (11.15) page 152 of Gatheral (2006) for the following result.

Proposition 8.6. *Under the lognormal approximation (8.11), the price*

$$\mathrm{VC}_{t,T}(\kappa_\sigma) = \mathrm{e}^{-(T-t)r}\mathbb{E}\big[\big(x + R^2_{t,T} - \kappa_\sigma^2\big)^+\big]_{x=R^2_{0,t}}$$

of the realized variance call option can be approximated as

$$\mathrm{VC}_{t,T}(\kappa_\sigma) \approx \mathrm{e}^{-(T-t)r}\mathbb{E}\big[R^2_{t,T}\big]\Phi(d_+) - \mathrm{e}^{-(T-t)r}\big(\kappa_\sigma^2 - R^2_{0,t}\big)\Phi(d_-), \quad (8.13)$$

where

$$\begin{aligned} d_+ &:= \frac{\log\big((\mathbb{E}[R_{t,T}])^2/\big(\kappa_\sigma^2 - R^2_{0,t}\big)\big)}{2\widetilde{\sigma}_{t,T}\sqrt{T-t}} + 2\widetilde{\sigma}_{t,T}\sqrt{T-t} \\ &= \frac{-\log\big(\kappa_\sigma^2 - R^2_{0,t}\big) + 2\tilde{\mu}_{t,T} + 4(T-t)\widetilde{\sigma}^2_{t,T}}{2\widetilde{\sigma}_{t,T}\sqrt{T-t}}, \end{aligned}$$

and

$$d_- := d_+ - 2\widetilde{\sigma}_{t,T}\sqrt{T-t} = \frac{2\tilde{\mu}_{t,T} - \log\big(\kappa_\sigma^2 - R^2_{0,t}\big)}{2\widetilde{\sigma}_{t,T}\sqrt{T-t}},$$

and Φ denotes the standard Gaussian cumulative distribution function.

Proof. The lognormal approximation (8.13) by realized variance moment matching states that

$$\varphi_{R_{t,T}}(x) \approx \frac{1}{x\widetilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}}\,\mathrm{e}^{-(-\tilde{\mu}_{t,T}+\log x)^2/(2(T-t)\widetilde{\sigma}^2_{t,T})}, \quad x > 0,$$

or equivalently

$$\begin{aligned}\varphi_{R^2_{t,T}}(x) &= \frac{1}{2\sqrt{x}}\varphi_{R_{t,T}}(\sqrt{x}) \\ &\approx \frac{1}{2x\widetilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}}\mathrm{e}^{-(-2\tilde{\mu}_{t,T}+\log x)^2/(2(T-t)(2\widetilde{\sigma}_{t,T})^2)}, \quad x>0.\end{aligned}$$

In other words, the distribution of $R^2_{t,T}$ is approximately that of $\mathrm{e}^{2\tilde{\mu}_{t,T}+2\widetilde{\sigma}_{t,T}X}$ where $X \simeq \mathcal{N}(0, T-t)$, hence by *e.g.* Lemma 7.7 we have

$$\begin{aligned}\mathrm{VC}_{t,T}(\kappa_\sigma) &= \mathrm{e}^{-(T-t)r}\mathbb{E}\big[\big(x+R^2_{t,T}-\kappa^2_\sigma\big)^+\big]_{x=R^2_{0,t}} \\ &= \mathrm{e}^{-(T-t)r}\int_{\kappa_\sigma}^\infty \big(y-\big(\kappa^2_\sigma-R^2_{0,t}\big)\big)^+\varphi_{R^2_{t,T}}(y)dy \\ &\approx \mathrm{e}^{-(T-t)r}\mathbb{E}\big[\big(\mathrm{e}^{2\tilde{\mu}_{t,T}+2\widetilde{\sigma}_{t,T}X}-\big(\kappa^2_\sigma-x\big)\big)^+\big]_{x=R^2_{0,t}} \\ &= \mathrm{e}^{-(T-t)r}\big(\mathrm{e}^{2\tilde{\mu}_{t,T}+2(T-t)\widetilde{\sigma}^2_{t,T}}\Phi(d_+)-\big(\kappa^2_\sigma-R^2_{0,t}\big)\Phi(d_-)\big) \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}\big[R^2_{t,T}\big]\Phi(d_+)-\mathrm{e}^{-(T-t)r}\big(\kappa^2_\sigma-R^2_{0,t}\big)\Phi(d_-), \quad (8.14)\end{aligned}$$

see Lemma 7.7. □

In order to estimate the price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(x+\int_t^T\sigma^2_u du-\kappa^2_\sigma\right)^+ \,\middle|\, \mathcal{F}_t\right]_{x=\int_0^t\sigma^2_u du},$$

of the realized variance call option when $R_{0,t} := \sqrt{\int_0^t \sigma^2_u du} < \kappa_\sigma$, we can also approximate $R^2_{t,T} := \int_t^T \sigma^2_u du$ by a lognormal random variable

$$R^2_{t,T} = \int_t^T \sigma^2_u du \simeq \mathrm{e}^{\tilde{\mu}_{t,T}+\widetilde{\sigma}_{t,T}X}$$

with mean $\tilde{\mu}_{t,T}$ and variance $\sigma^2_{t,T}$, where $X \simeq \mathcal{N}(0,1)$ is a standard normal random variable.

Proposition 8.7. *(Lognormal approximation by realized variance moment matching). Under the lognormal approximation, the probability density function* $\varphi_{R^2_{t,T}}$ *of* $R^2_{t,T} := \int_t^T \sigma^2_u du$ *can be approximated as*

$$\varphi_{R^2_{t,T}}(x) \approx \frac{1}{x\widetilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}}\exp\left(-\frac{(\tilde{\mu}_{t,T}-\log x)^2}{2(T-t)\widetilde{\sigma}^2_{t,T}}\right), \quad x>0, \quad (8.15)$$

where

$$\tilde{\mu}_{t,T} := -(T-t)\frac{\widetilde{\sigma}^2_{t,T}}{2}+\log\mathbb{E}[R^2_{t,T}], \quad (8.16)$$

and

$$\widetilde{\sigma}_{t,T}^2 = \frac{1}{T-t}\log\left(1+\frac{\mathrm{Var}[R_{t,T}^2]}{(\mathbb{E}[R_{t,T}^2])^2}\right). \tag{8.17}$$

Proof. The parameters $\tilde{\mu}_{t,T}$ and $\widetilde{\sigma}_{t,T}$ are estimated by matching the first and second moments $\mathbb{E}\big[R_{t,T}^2\big]$ and $\mathbb{E}\big[R_{t,T}^4\big]$ of $R_{t,T}^4$ to those of the lognormal distribution with mean $\tilde{\mu}_{t,T}$ and variance $(T-t)\widetilde{\sigma}_{t,T}^2$, which yields

$$\mathbb{E}\big[R_{t,T}^2\big] = \mathrm{e}^{\tilde{\mu}_{t,T}+(T-t)\widetilde{\sigma}_{t,T}^2/2}, \qquad \mathbb{E}\big[R_{t,T}^4\big] = \mathrm{e}^{2(\tilde{\mu}_{t,T}+(T-t)\widetilde{\sigma}_{t,T}^2)},$$

and

$$\tilde{\mu}_{t,T} = -(T-t)\frac{\widetilde{\sigma}_{t,T}^2}{2} + \log\mathbb{E}\big[R_{t,T}^2\big] \text{ and } \widetilde{\sigma}_{t,T}^2 := \frac{1}{T-t}\log\left(\frac{\mathbb{E}\big[R_{t,T}^4\big]}{\big(\mathbb{E}\big[R_{t,T}^2\big]\big)^2}\right).$$

□

By (8.16)-(8.17), the parameters $\tilde{\mu}_{t,T}$ and $\widetilde{\sigma}_{t,T}^2$ can be estimated from the realized variance swap price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[R_{t,T}^2 \mid \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \,\middle|\, \mathcal{F}_t\right],$$

and from the realized variance power option price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[R_{t,T}^4 \mid \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\int_t^T \sigma_u^2 du\right)^2 \,\middle|\, \mathcal{F}_t\right].$$

The next proposition is obtained by the same argument as in the proof of Proposition 8.6.

Proposition 8.8. *Under the lognormal approximation* (8.15)*, the price*

$$\mathrm{VC}_{t,T}(\kappa_\sigma) = \mathrm{e}^{-(T-t)r}\mathbb{E}\big[\big(x+R_{t,T}^2-\kappa_\sigma^2\big)^+\big]_{x=R_{0,t}}$$

of the realized variance call option can be approximated as

$$\mathrm{VC}_{t,T}(\kappa_\sigma) \approx \mathrm{e}^{-(T-t)r}\mathbb{E}\big[R_{t,T}^2\big]\Phi(d_+) - \mathrm{e}^{-(T-t)r}\big(\kappa_\sigma^2 - R_{0,t}^2\big)\Phi(d_-), \tag{8.18}$$

where

$$\begin{aligned} d_+ &:= \frac{\log\big(\mathbb{E}\big[R_{t,T}^2\big]/\big(\kappa_\sigma^2 - R_{0,t}^2\big)\big)}{\widetilde{\sigma}_{t,T}\sqrt{T-t}} + \widetilde{\sigma}_{t,T}\frac{\sqrt{T-t}}{2} \\ &= \frac{-\log\big(\kappa_\sigma^2 - R_{0,t}^2\big) + \tilde{\mu}_{t,T} + (T-t)\widetilde{\sigma}_{t,T}^2}{\widetilde{\sigma}_{t,T}\sqrt{T-t}}, \end{aligned}$$

and

$$d_- := d_+ - \widetilde{\sigma}_{t,T}\sqrt{T-t} = \frac{\tilde{\mu}_{t,T} - \log\left(\kappa_\sigma^2 - R_{0,t}^2\right)}{\widetilde{\sigma}_{t,T}\sqrt{T-t}},$$

and Φ denotes the standard Gaussian cumulative distribution function.

Note that, using the integral identity

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty (1 - \mathrm{e}^{-\lambda x})\frac{d\lambda}{\lambda^{3/2}},$$

see *e.g.* Relation 3.434.1 in Gradshteyn and Ryzhik (2007) for $\rho = 1/2$ and Exercise 9.11-(a), the realized volatility swap price $\mathbb{E}[R_{t,T}]$ can be expressed as

$$\mathbb{E}[R_{t,T}] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - \mathbb{E}\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]\right)\frac{d\lambda}{\lambda^{3/2}}, \tag{8.19}$$

see § 3.1 in Friz and Gatheral (2005), where $\mathbb{E}\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]$ can be expressed from Lemma 8.2. In particular, by *e.g.* Relation (3.25) in Brigo and Mercurio (2006), in the Cox et al. (1985) (CIR)

$$dv_t = (a - bv_t)dt + \eta\sqrt{v_t}dW_t$$

variance model with $v_t = \sigma_t^2$, we have

$$\mathbb{E}\left[\mathrm{e}^{-\lambda R_{0,T}^2}\right] = \exp\left(-\frac{2v_0\lambda(1-e^{-\bar{b}T})}{\bar{b}+b+(\bar{b}-b)\,\mathrm{e}^{-\bar{b}T}} - \frac{a}{\eta^2}(\bar{b}-b)T - \frac{2a}{\eta^2}\log\frac{\bar{b}+b+(\bar{b}-b)e^{-\bar{b}T}}{2\bar{b}}\right),$$

where $\bar{b} := \sqrt{b^2 + 2\lambda\eta^2}$.

Gamma approximation

In case $R_{0,t}^2 = \int_0^t \sigma_u^2 du < \kappa_\sigma^2$, the realized variance call option price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \,\middle|\, \mathcal{F}_t\right]_{x=\int_0^t \sigma_u^2 du}$$

can be estimated by approximating $R_{t,T}^2 = \int_t^T \sigma_u^2 du$ by a gamma random variable as in the probability density graph of Figure 8.3.

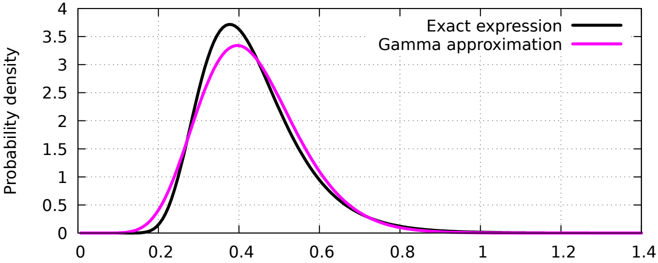

Fig. 8.3: Fitting of a gamma probability density function (example).

Proposition 8.9. *(Gamma approximation). Under the gamma approximation the probability density function* $\varphi_{R^2_{t,T}}$ *of* $R^2_{t,T} := \int_t^T \sigma_u^2 du$ *can be approximated as*

$$\varphi_{R^2_{t,T}}(x) \approx \frac{(x/\theta_{t,T})^{-1+\nu_{t,T}}}{\theta_{t,T}\Gamma(\nu_{t,T})} \mathrm{e}^{-x/\theta_{t,T}}, \quad x > 0, \tag{8.20}$$

where

$$\theta_{t,T} = \frac{\mathrm{Var}\left[R^2_{t,T}\right]}{\mathbb{E}\left[R^2_{t,T}\right]} \quad \textit{and} \quad \nu_{t,T} = \frac{\mathbb{E}\left[R^2_{t,T}\right]}{\theta_{t,T}} = \frac{\left(\mathbb{E}\left[R^2_{t,T}\right]\right)^2}{\mathrm{Var}\left[R^2_{t,T}\right]}. \tag{8.21}$$

Proof. The parameters $\theta_{t,T}$, $\nu_{t,T}$ are estimated by matching the first and second moments of $R^2_{t,T}$ to those of the gamma distribution with scale and shape parameters $\theta_{t,T}$ and $\nu_{t,T}$, which yields

$$\mathbb{E}\left[R^2_{t,T}\right] = \nu_{t,T}\theta_{t,T} \quad \text{and} \quad \mathrm{Var}\left[R^2_{t,T}\right] = \nu_{t,T}\theta^2_{t,T},$$

and (8.21). □

Proposition 8.10. *Under the gamma approximation* (8.20)*, the price*

$$\mathrm{EA}(\kappa_\sigma, T) = \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\left(x + R^2_{t,T} - \kappa^2_\sigma\right)^+\right]_{x=R^2_{0,t}}$$

of the realized variance call option can be approximated as

$$\mathrm{EA}(\kappa_\sigma, T) = \mathrm{e}^{-(T-t)r}\left(\mathbb{E}\left[R^2_{t,T}\right]Q\left(1+\nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right)\right), \tag{8.22}$$

where

$$Q(\lambda, z) := \frac{1}{\Gamma(\lambda)}\int_z^\infty t^{\lambda-1}\mathrm{e}^{-t}dt, \qquad z > 0,$$

is the (normalized) upper incomplete gamma function.

Proof. Using the gamma approximation

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$$\varphi_{R^2_{t,T}}(x) \approx \frac{\mathrm{e}^{-x/\theta_{t,T}}}{\Gamma(\nu_{t,T})} \frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}}, \tag{8.23}$$

where $\theta_{t,T}$ and $\nu_{t,T}$ are given by (8.21), we have

$$\begin{aligned}
&\mathbb{E}\big[\big(R^2_{t,T} - \kappa^2_\sigma\big)^+\big] = \int_{\kappa^2_\sigma}^\infty (x - \kappa^2_\sigma)^+ \varphi_{R^2_{t,T}}(x) dx \\
&\approx \frac{1}{\Gamma(\nu_{t,T})} \int_{\kappa^2_\sigma}^\infty (x - \kappa^2_\sigma) \frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}} \mathrm{e}^{-x/\theta_{t,T}} dx \\
&= \frac{1}{\Gamma(\nu_{t,T})} \int_{\kappa^2_\sigma}^\infty (x/\theta_{t,T})^{\nu_{t,T}} \mathrm{e}^{-x/\theta_{t,T}} dx - \frac{\kappa^2_\sigma}{\Gamma(\nu_{t,T})} \int_{\kappa^2_\sigma}^\infty \frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}} \mathrm{e}^{-x/\theta_{t,T}} dx \\
&= \frac{\theta_{t,T}}{\Gamma(\nu_{t,T})} \int_{\kappa^2_\sigma/\theta_{t,T}}^\infty x^{\nu_{t,T}} \mathrm{e}^{-x} dx - \frac{\kappa^2_\sigma}{\Gamma(\nu_{t,T})} \int_{\kappa^2_\sigma/\theta_{t,T}}^\infty x^{-1+\nu_{t,T}} \mathrm{e}^{-x} dx \\
&= \theta_{t,T}\nu_{t,T} Q\left(1 + \nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right),
\end{aligned}$$

where

$$Q(\lambda, z) := \frac{1}{\Gamma(\lambda)} \int_z^\infty t^{\lambda-1} \mathrm{e}^{-t} dt, \qquad z > 0,$$

is the (normalized) upper incomplete gamma function, which yields

$$\begin{aligned}
\mathrm{EA}(\kappa_\sigma, T) &= \mathrm{e}^{-(T-t)r} \mathbb{E}\big[\big(x + R^2_{t,T} - \kappa^2_\sigma\big)^+\big] \\
&\approx \mathrm{e}^{-(T-t)r} \left(\nu_{t,T}\theta_{t,T} Q\left(1 + \nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right)\right) \qquad (8.24) \\
&= \mathrm{e}^{-(T-t)r} \left(\mathbb{E}\big[R^2_{t,T}\big] Q\left(1 + \nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T}, \frac{\kappa^2_\sigma}{\theta_{t,T}}\right)\right).
\end{aligned}$$

□

Realized variance options in the Heston model

Taking $r = 0$, $t = 0$ and $R_{0,0} = 0$, and using the parameters

$$\sigma = 0.39,\ b = 1.15,\ a = 0.04 \times b,\ v_0 = 0.04,\ T = 1$$

in the Heston stochastic differential equation

$$dv_t = (a - bv_t)dt + \sigma\sqrt{v_t}dW_t,$$

in Figures 8.4-8.5 we plot the graphs of the lognormal volatility swap and realized variance moment matching approximations (8.13), (8.18), and of the gamma approximation (8.22) for realized variance call option prices with $\kappa^2_\sigma \in [0, 0.2]$, based on the expressions (8.4)-(8.5) of $\mathbb{E}\big[R^2_{0,T}\big]$ and $\mathrm{Var}\big[R^2_{0,T}\big]$.

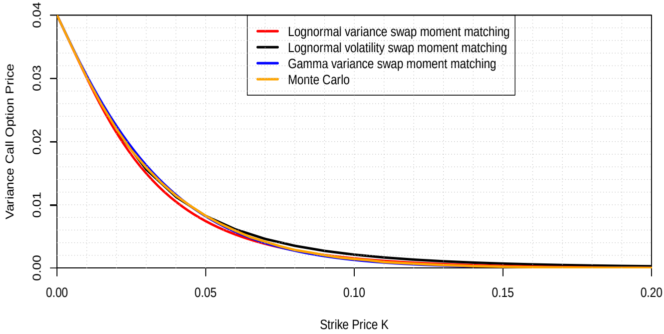

Fig. 8.4: One-year variance call option prices with $b = 0.15$.

The graphs of Figures 8.4-8.5 are obtained using this **code** and **data file**.

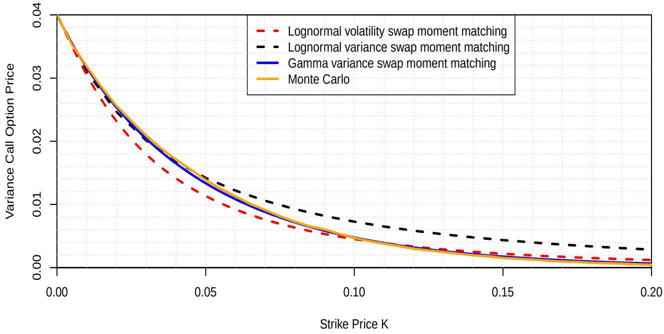

Fig. 8.5: One-year variance call option prices with $b = -0.05$.

As can be checked from in Figure 8.5 with

$$\sigma = 0.39, \ b = 1.15, \ a = 0.04 \times b, \ v_0 = 0.04, \ T = 1,$$

the gamma approximation (8.22) appears to be more accurate than the log-normal approximations for large values of κ_σ^2, which can be consistent with the fact that the long run distribution of the CIR-Heston process has the gamma probability density function

$$f(x) = \frac{1}{\Gamma(2a/\sigma^2)} \left(\frac{2b}{\sigma^2} \right)^{2a/\sigma^2} x^{-1+2a/\sigma^2} \mathrm{e}^{-2bx/\sigma^2}, \quad x > 0.$$

with shape parameter $2a/\sigma^2$ and scale parameter $\sigma^2/(2b)$, which is also the *invariant distribution* of v_t.

8.4 European Options - PDE Method

In what follows we consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ in the stochastic volatility model

$$dS_t = rS_t dt + S_t\sqrt{v_t}dB_t^{(1)}$$

under the risk-neutral probability measure $\mathbb{P}^*$, where $(v_t)_{t\in\mathbb{R}_+}$ is a squared volatility (or variance) process satisfying a stochastic differential equation of the form

$$dv_t = \mu(t, v_t)dt + \beta(t, v_t)dB_t^{(2)}.$$

Here, $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are correlated standard Brownian motions started at 0 with correlation $\mathrm{Corr}\big(B_s^{(1)}, B_t^{(2)}\big) = \rho\min(s,t)$ under the risk-neutral probability measure $\mathbb{P}^*$, *i.e.* $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$.

Proposition 8.11. *Assume that* $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ *is also a standard Brownian motion under the risk-neutral probability measure** $\mathbb{P}^*$. *Consider a vanilla option with payoff* $h(S_T)$ *priced as*

$$V_t = f(t, v_t, S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[h(S_T) \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T.$$

The function $f(t,y,x)$ *satisfies the PDE*

$$\frac{\partial f}{\partial t}(t,v,x) + rx\frac{\partial f}{\partial x}(t,v,x) + \frac{1}{2}vx^2\frac{\partial^2 f}{\partial x^2}(t,v,x) \tag{8.25}$$
$$+ \mu(t,v)\frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x)$$
$$= rf(t,v,x),$$

under the terminal condition $f(T,v,x) = h(x)$.

Proof. By Itô calculus with respect to the correlated Brownian motions $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$, the portfolio value $f(t,v_t,S_t)$ can be differentiated as follows:

$$df(t,v_t,S_t) \tag{8.26}$$
$$= \frac{\partial f}{\partial t}(t,v_t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,v_t,S_t)dt + \sqrt{v_t}S_t\frac{\partial f}{\partial x}(t,v_t,S_t)dB_t^{(1)}$$

* When this condition is not satisfied, we need to introduce a drift that will yield a market price of volatility.

$$
\begin{aligned}
&+\frac{1}{2}v_tS_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt+\mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dt \\
&+\beta(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dB_t^{(2)}+\frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt \\
&+\beta(t,v_t)\sqrt{v_t}S_t\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t)dB_t^{(1)}\bullet dB_t^{(2)} \\
=\ &\frac{\partial f}{\partial t}(t,v_t,S_t)dt+rS_t\frac{\partial f}{\partial x}(t,v_t,S_t)dt+\sqrt{v_t}S_t\frac{\partial f}{\partial x}(t,v_t,S_t)dB_t^{(1)} \\
&+\frac{1}{2}v_tS_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt+\mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dt \\
&+\beta(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dB_t^{(2)}+\frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt \\
&+\rho\beta(t,v_t)\sqrt{v_t}S_t\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t)dt.
\end{aligned}
$$

Knowing that the discounted portfolio value process $(\mathrm{e}^{-rt}f(t,v_t,S_t))_{t\in\mathbb{R}_+}$ is also a martingale under $\mathbb{P}^*$, from the relation

$$
d(\mathrm{e}^{-rt}f(t,v_t,S_t))=-r\,\mathrm{e}^{-rt}f(t,v_t,S_t)dt+\mathrm{e}^{-rt}df(t,v_t,S_t),
$$

we obtain

$$
\begin{aligned}
&-rf(t,v_t,S_t)dt+\frac{\partial f}{\partial t}(t,v_t,S_t)dt+rS_t\frac{\partial f}{\partial x}(t,v_t,S_t)dt+\frac{1}{2}v_tS_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt \\
&+\mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dt+\frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt \\
&+\rho\beta(t,v_t)S_t\sqrt{v_t}\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t)dt \\
&=0,
\end{aligned}
$$

and the pricing PDE (8.25). □

Heston model

In the Heston model with $\mu(t,v)=-\lambda(v-m)$ and $\beta(t,v)=\eta\sqrt{v}$, from (8.25) we find the Heston PDE

$$
\begin{aligned}
&\frac{\partial f}{\partial t}(t,v,x)+rx\frac{\partial f}{\partial x}(t,v,x)+\frac{1}{2}vx^2\frac{\partial^2 f}{\partial x^2}(t,v,x) \\
&-\lambda(v-m)\frac{\partial f}{\partial v}(t,v,x)+\frac{1}{2}\eta^2v\frac{\partial^2 f}{\partial v^2}(t,v,x)+\rho\eta xv\frac{\partial^2 f}{\partial v\partial x}(t,v,x)=rf(t,v,x).
\end{aligned}
\tag{8.27}
$$

The solution of this PDE has been expressed in Heston (1993) as a complex integral by inversion of a characteristic function.

Using the change of variable $y = \log x$ where with $g(t,v,y) = f(t,v,e^y)$, the PDE (8.27) is transformed into

$$\frac{\partial g}{\partial t}(t,v,y) + \frac{1}{2}v\frac{\partial^2 g}{\partial y^2}(t,v,y) + \left(r - \frac{v}{2}\right)\frac{\partial g}{\partial y}(t,v,x)$$
$$+ \lambda(m-v)\frac{\partial g}{\partial v}(t,v,y) + v\frac{\eta^2}{2}\frac{\partial^2 g}{\partial v^2}(t,v,y) + \rho\eta v\frac{\partial^2 g}{\partial v \partial y}(t,v,y) = rg(t,v,y).$$

The following proposition shows that the Fourier transform of $g(t,v,y)$ satisfies an affine PDE with respect to the variable v, when z is regarded as a constant parameter.

Proposition 8.12. *Assume that $\rho = 0$. The Fourier transform*

$$\widehat{g}(t,v,z) := \int_{-\infty}^{\infty} \mathrm{e}^{-iyz} g(t,v,y)dy$$

satisfies the partial differential equation

$$\frac{\partial \widehat{g}}{\partial t}(t,v,z) + \left(irz - \frac{1}{2}vz^2\right)\widehat{g}(t,v,z) - iz\frac{1}{2}v\widehat{g}(t,v,z) \tag{8.28}$$
$$+ (\lambda(m-v) + i\rho\eta z v)\frac{\partial \widehat{g}}{\partial v}(t,v,z) + v\frac{\eta^2}{2}\frac{\partial^2 \widehat{g}}{\partial v^2}(t,v,z) = r\widehat{g}(t,v,z).$$

Proof. We apply the relations $i^2 = -1$ and

$$iz\widehat{g}(t,v,z) = \int_{-\infty}^{\infty} \mathrm{e}^{-iyz}\frac{\partial g}{\partial y}(t,v,y)dy.$$

□

The equation (8.28) can be solved in closed form, and the final solution $g(t,v,y)$ can then be obtained by the Fourier inversion relation

$$g(t,v,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{izy}\widehat{g}(t,v,z)dz, \tag{8.29}$$

see Heston (1993), Attari (2004), Albrecher et al. (2007), and Rouah (2013) for details.

Delta hedging in the Heston model

Consider a portfolio of the form

$$V_t = \eta_t \, \mathrm{e}^{rt} + \xi_t S_t$$

based on the riskless asset $A_t = \mathrm{e}^{rt}$ and on the risky asset S_t. When this portfolio is self-financing we have

$$\begin{aligned} dV_t &= df(t, v_t, S_t) \\ &= r\eta_t \, \mathrm{e}^{rt} dt + \xi_t dS_t \\ &= r\eta_t \, \mathrm{e}^{rt} dt + \xi_t \big(rS_t dt + S_t \sqrt{v_t} dB_t^{(1)} \big) \\ &= rV_t dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} \\ &= rf(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)}. \end{aligned} \tag{8.30}$$

However, trying to match (8.26) to (8.30) yields

$$\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)} = \xi_t S_t \sqrt{v_t} dB_t^{(1)}, \tag{8.31}$$

which admits no solution unless $\beta(t, v) = 0$, *i.e.* when volatility is deterministic. A solution to that problem is to consider instead a portfolio

$$V_t = f(t, v_t, S_t) = \eta_t \, \mathrm{e}^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t)$$

that includes an additional asset with price $P(t, v_t, S_t)$, which can be an option depending on the volatility v_t.

Proposition 8.13. *Assume that $\rho = 0$. The self-financing portfolio allocation $(\xi_t, \zeta_t)_{t \in [0,T]}$ in the assets $\big(\mathrm{e}^{rt}, S_t, P(t, v_t, S_t) \big)_{t \in [0,T]}$ with portfolio value*

$$V_t = f(t, v_t, S_t) = \eta_t \, \mathrm{e}^{rt} + \xi_t S_t + \zeta_t P(t, v_t, S_t) \tag{8.32}$$

is given by

$$\zeta_t = \frac{\dfrac{\partial f}{\partial v}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)}, \tag{8.33}$$

and

$$\xi_t = \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\dfrac{\partial P}{\partial x}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)}. \tag{8.34}$$

Proof. Using (8.32), we replace (8.30) with the self-financing condition

$$dV_t = df(t, v_t, S_t)$$

$$
\begin{aligned}
&= r\eta_t \,\mathrm{e}^{rt} dt + \xi_t dS_t + \zeta_t dP(t, v_t, S_t)\\
&= r\eta_t \,\mathrm{e}^{rt} dt + \xi_t (rS_t dt + S_t\sqrt{v_t} dB_t^{(1)}) + r\zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt\\
&\quad + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2}\zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt\\
&\quad + \frac{1}{2}\zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho\zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt\\
&\quad + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)},\\
&= (V_t - \zeta_t P(t, v_t, S_t)) r dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r\zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt\\
&\quad + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2}\zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt\\
&\quad + \frac{1}{2}\zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho\zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt\\
&\quad + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}\\
&= r f(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r\zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt\\
&\quad + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2}\zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt\\
&\quad + \frac{1}{2}\zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho\zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt\\
&\quad + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}, \qquad (8.35)
\end{aligned}
$$

and by matching (8.35) to (8.26), the equation (8.31) now becomes

$$
\begin{aligned}
&\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)}\\
&= \xi_t S_t \sqrt{v_t} dB_t^{(1)} + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}.
\end{aligned}
$$

This leads to the equations

$$
\begin{cases}
\sqrt{v_t} S_t \dfrac{\partial f}{\partial x}(t, v_t, S_t) = \xi_t S_t \sqrt{v_t} + \zeta_t S_t \sqrt{v_t} \dfrac{\partial P}{\partial x}(t, v_t, S_t),\\[2ex]
\beta(t, v_t) \dfrac{\partial f}{\partial v}(t, v_t, S_t) = \zeta_t \beta(t, v_t) \dfrac{\partial P}{\partial v}(t, v_t, S_t),
\end{cases}
$$

which show that

$$
\zeta_t = \frac{\dfrac{\partial f}{\partial v}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)},
$$

and

$$\begin{aligned}\xi_t &= \frac{1}{S_t\sqrt{v_t}}\left(\sqrt{v_t}S_t\frac{\partial f}{\partial x}(t,v_t,S_t) - \zeta_t S_t\sqrt{v_t}\frac{\partial P}{\partial x}(t,v_t,S_t),\right) \\ &= \frac{\partial f}{\partial x}(t,v_t,S_t) - \zeta_t\frac{\partial P}{\partial x}(t,v_t,S_t) \\ &= \frac{\partial f}{\partial x}(t,v_t,S_t) - \frac{\partial f}{\partial v}(t,v_t,S_t)\frac{\frac{\partial P}{\partial x}(t,v_t,S_t)}{\frac{\partial P}{\partial v}(t,v_t,S_t)}.\end{aligned}$$

□

We note in addition that identifying the "dt" terms when equating (8.35) to (8.26) would now lead to the more complicated PDE

$$\begin{aligned}&(f(t,v_t,S_t) - \zeta_t P(t,v_t,S_t))r + r\zeta_t S_t\frac{\partial P}{\partial x}(t,v_t,S_t) + \zeta_t\mu(t,v_t)\frac{\partial P}{\partial v}(t,v_t,S_t) \\ &\quad +\zeta_t\frac{\partial P}{\partial t}(t,v_t,S_t) + \frac{1}{2}\zeta_t S_t^2 v_t\frac{\partial^2 P}{\partial x^2}(t,v_t,S_t) + \frac{1}{2}\zeta_t\beta^2(t,v_t)\frac{\partial^2 P}{\partial v^2}(t,v_t,S_t) \\ &\quad +\rho\zeta_t\beta(t,v_t)S_t\sqrt{v_t}\frac{\partial^2 P}{\partial x\partial v}(t,v_t,S_t) \\ &= \frac{\partial f}{\partial t}(t,v_t,S_t) + rS_t\frac{\partial f}{\partial x}(t,v_t,S_t) + \frac{1}{2}v_t S_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t) + \mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t) \\ &\quad +\frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t) + \rho\beta(t,v_t)S_t\sqrt{v_t}\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t),\end{aligned}$$

which can be rewritten using (8.33) as

$$\begin{aligned}&\frac{\partial f}{\partial v}(t,v,x)\left(-rP(t,v,x) + \frac{\partial P}{\partial t}(t,v,x) + rx\frac{\partial P}{\partial x}(t,v,x) + \mu(t,v)\frac{\partial P}{\partial v}(t,v,x)\right) \\ &+\frac{\partial f}{\partial v}(t,v,x)\left(\frac{x^2 v}{2}\frac{\partial^2 P}{\partial x^2}(t,v,x) + \frac{1}{2}\beta^2(t,v)\frac{\partial^2 P}{\partial v^2}(t,v,x) + \rho\beta(t,v)x\sqrt{v}\frac{\partial^2 P}{\partial x\partial v}(t,v,x)\right) \\ &= \frac{\partial P}{\partial v}(t,v,x)\left(-rf(t,v,x) + \frac{\partial f}{\partial t}(t,v,x) + rx\frac{\partial f}{\partial x}(t,v,x) + \frac{vx^2}{2}\frac{\partial^2 f}{\partial x^2}(t,v,x)\right) \\ &+\frac{\partial P}{\partial v}(t,v,x)\left(\mu(t,v)\frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x)\right).\end{aligned}$$

Therefore, dividing both sides by $\dfrac{\partial P}{\partial v}(t,v,x)$ and letting

$$\begin{aligned}&\lambda(t,v,x) \\ &:= \frac{1}{\frac{\partial P}{\partial v}(t,v,x)}\left(-rP(t,v,x) + rx\frac{\partial P}{\partial x}(t,v,x) + \frac{\partial P}{\partial t}(t,v,x)\right)\end{aligned} \tag{8.36}$$

$$+\frac{1}{\frac{\partial P}{\partial v}(t,v,x)}\left(\frac{x^2 v}{2}\frac{\partial^2 P}{\partial x^2}(t,v,x)+\frac{1}{2}\beta^2(t,v)\frac{\partial^2 P}{\partial v^2}(t,v,x)+\rho\beta(t,v)x\sqrt{v}\frac{\partial^2 P}{\partial x\partial v}(t,v,x)\right)$$

$$=\frac{1}{\frac{\partial f}{\partial v}(t,v,x)}\left(-rf(t,v,x)+\frac{\partial f}{\partial t}(t,v,x)+rx\frac{\partial f}{\partial x}(t,v,x)+\frac{vx^2}{2}\frac{\partial^2 f}{\partial x^2}(t,v,x)\right) \tag{8.37}$$

$$+\frac{1}{\frac{\partial f}{\partial v}(t,v,x)}\left(\frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x)+\rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x)\right) \tag{8.38}$$

defines a function $\lambda(t,v,x)$ that depends only on the parameters (t,v,x) and not on P, without requiring $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ to be a standard Brownian motion under $\mathbb{P}$. The function $\lambda(t,v,x)$ is linked to the market price of volatility risk, cf. Chapter 1 of Gatheral (2006) § 2.4.1 in Fouque et al. (2000; 2011) for details.

Combining (8.36)-(8.38) allows us to rewrite the pricing PDE as

$$\frac{\partial f}{\partial t}(t,v,x)+rx\frac{\partial f}{\partial x}(t,v,x)+\frac{vx^2}{2}\frac{\partial^2 f}{\partial x^2}(t,v,x)+\frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x)$$
$$+\rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x)=rf(t,v,x)+\lambda(t,v,x)\frac{\partial f}{\partial v}(t,v,x),$$

and (8.25) corresponds to the choice $\lambda(t,v,x)=-\mu(t,v)$, which corresponds to a vanishing "market price of volatility risk".

8.5 Perturbation Analysis

We refer to Chapter 4 of Fouque et al. (2011) for the contents of this section. Consider the time-rescaled model

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{v_{t/\varepsilon}}\,dB_t^{(1)} \\ \\ dv_t = \mu(v_t)dt + \beta(v_t)dB_t^{(2)}. \end{cases} \tag{8.39}$$

We note that $v_t^{(\varepsilon)} := v_{t/\varepsilon}$ satisfies the SDE

$$\begin{aligned} dv_t^{(\varepsilon)} &= dv_{t/\varepsilon} \\ &\simeq v_{(t+dt)/\varepsilon} - v_{t/\varepsilon} \\ &= v_{t/\varepsilon+dt/\varepsilon} - v_{t/\varepsilon} \\ &= \frac{1}{\varepsilon}\mu(v_{t/\varepsilon})dt + \beta(v_{t/\varepsilon})dB_{t/\varepsilon}^{(2)}, \end{aligned}$$

with

$$\left(dB^{(2)}_{t/\varepsilon}\right)^2 \simeq \frac{dt}{\varepsilon} \simeq \frac{1}{\varepsilon}\left(dB^{(2)}_t\right)^2 \simeq \left(\frac{1}{\sqrt{\varepsilon}} dB^{(2)}_t\right)^2,$$

hence the SDE for $v^{(\varepsilon)}_t$ can be rewritten as the slow-fast system

$$dv^{(\varepsilon)}_t = \frac{1}{\varepsilon}\mu(v^{(\varepsilon)}_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(v^{(\varepsilon)}_t)dB^{(2)}_t.$$

In other words, $\varepsilon \to 0$ corresponds to fast mean-reversion and (8.39) can be rewritten as

$$\begin{cases} dS_t = rS_t dt + \sqrt{v^{(\varepsilon)}_t} S_t dB^{(1)}_t \\ dv^{(\varepsilon)}_t = \dfrac{1}{\varepsilon}\mu\big(v^{(\varepsilon)}_t\big)dt + \dfrac{1}{\sqrt{\varepsilon}}\beta\big(v^{(\varepsilon)}_t\big)dB^{(2)}_t, \qquad \varepsilon > 0. \end{cases}$$

The perturbed PDE

$$\frac{\partial f_\varepsilon}{\partial t}(t,v,x) + rx\frac{\partial f_\varepsilon}{\partial x}(t,v,x) + \frac{vx^2}{2}\frac{\partial^2 f_\varepsilon}{\partial x^2}(t,v,x) + \frac{1}{\varepsilon}\mu(v)\frac{\partial f_\varepsilon}{\partial v}(t,v,x)$$
$$+ \frac{1}{2\varepsilon}\beta^2(v)\frac{\partial^2 f_\varepsilon}{\partial v^2}(t,v,x) + \frac{\rho}{\sqrt{\varepsilon}}\beta(v)x\sqrt{v}\frac{\partial^2 f_\varepsilon}{\partial v \partial x}(t,v,x) = rf_\varepsilon(t,v,x)$$

with terminal condition $f_\varepsilon(T,v,x) = (x-K)^+$ rewrites as

$$\frac{1}{\varepsilon}\mathcal{L}_0 f_\varepsilon(t,v,x) + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 f_\varepsilon(t,v,x) + \mathcal{L}_2 f_\varepsilon(t,v,x) = rf_\varepsilon(t,v,x), \qquad (8.40)$$

where

$$\begin{cases} \mathcal{L}_0 f_\varepsilon(t,v,x) := \dfrac{1}{2}\beta^2(v)\dfrac{\partial^2 f_\varepsilon}{\partial v^2}(t,v,x) + \mu(v)\dfrac{\partial f_\varepsilon}{\partial v}(t,v,x), \\ \mathcal{L}_1 f_\varepsilon(t,v,x) := \rho x\beta(v)\sqrt{v}\dfrac{\partial^2 f_\varepsilon}{\partial v \partial x}(t,v,x), \\ \mathcal{L}_2 f_\varepsilon(t,v,x) := \dfrac{\partial f_\varepsilon}{\partial t}(t,v,x) + rx\dfrac{\partial f_\varepsilon}{\partial x}(t,v,x) + \dfrac{vx^2}{2}\dfrac{\partial^2 f_\varepsilon}{\partial x^2}(t,v,x). \end{cases}$$

Note that

- $\mathcal{L}_0$ is the infinitesimal generator of the process $\big(v^1_s\big)_{s\in\mathbb{R}_+}$, see (8.44) below,

  and

- $\mathcal{L}_2$ is the Black-Scholes operator, *i.e.* $\mathcal{L}_2 f = rf$ is the Black-Scholes PDE.

The solution $f_\varepsilon(t,v,x)$ will be expanded as

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$$f_\varepsilon(t,v,x) = f^{(0)}(t,v,x) + \sqrt{\varepsilon} f^{(1)}(t,v,x) + \varepsilon f^{(2)}(t,v,x) + \cdots \qquad (8.41)$$

with $f(T,v,x) = (x-K)^+$, $f^{(1)}(T,v,x) = 0$, and $f^{(2)}(T,v,x) = 0$. Since $\mathcal{L}_0$ contains only differentials with respect to v, we will choose $f^{(0)}(t,v,x)$ of the form

$$f^{(0)}(t,v,x) = f^{(0)}(t,x),$$

cf. § 4.2.1 of Fouque et al. (2011) for details, with

$$\mathcal{L}_0 f^{(0)}(t,x) = \mathcal{L}_1 f^{(0)}(t,x) = 0. \qquad (8.42)$$

Proposition 8.14. *(Fouque et al. (2011), § 3.2). The first-order term $f_0(t,v)$ in (8.41) satisfies the Black-Scholes PDE*

$$rf^{(0)}(t,x) = \frac{\partial f^{(0)}}{\partial t}(t,x) + rx\frac{\partial f^{(0)}}{\partial x}(t,x) + \frac{\eta^2}{2}\int_0^\infty v\phi(v)dv \frac{\partial^2 f^{(0)}}{\partial x^2}(t,x)$$

with the terminal condition $f^{(0)}(T,x) = (x-K)^+$, where $\phi(v)$ is the stationary (or invariant) probability density function of the process $\big(v_t^{(1)}\big)_{t\in\mathbb{R}_+}$.

Proof. By identifying the terms of order $1/\sqrt{\varepsilon}$ when plugging (8.41) in (8.40), we have

$$\mathcal{L}_0 f^{(1)}(t,v,x) + \mathcal{L}_1 f^{(0)}(t,x) = 0,$$

hence $\mathcal{L}_0 f^{(1)}(t,v,x) = 0$. Similarly, by identifying the terms that do not depend on ε in (8.40) and taking $f^{(1)}(t,v,x) = f^{(1)}(t,x)$, we have $\mathcal{L}_1 f^{(1)} = 0$ and

$$\mathcal{L}_0 f^{(2)}(t,v,x) + \mathcal{L}_2 f^{(0)}(t,x) = 0. \qquad (8.43)$$

Using the Itô formula, we have

$$\begin{aligned}\mathbb{E}\big[f^{(2)}(t,v_s^1,x)\big] &= f^{(2)}(t,v_0^1,x) + \mathbb{E}\left[\int_0^s \frac{\partial f^{(2)}}{\partial x}(t,v_\tau^1,x)dB_\tau^{(2)}\right] \\ &\quad + \mathbb{E}\left[\int_0^s \left(\mu(v_\tau^1)\frac{\partial f^{(2)}}{\partial v}(t,v_\tau^1,x) + \frac{1}{2}\beta^2(v_\tau^1)\frac{\partial^2 f^{(2)}}{\partial v^2}(t,v_\tau^1,x)\right)d\tau\right] \\ &= f^{(2)}(t,v_0^1,x) + \int_0^s \mathbb{E}\big[\mathcal{L}_0 f^{(2)}(t,v_\tau^1,x)\big]d\tau. \end{aligned} \qquad (8.44)$$

When the process $\big(v_t^{(1)}\big)_{t\in\mathbb{R}_+}$ is started under its stationary (or invariant) probability distribution with probability density function $\phi(v)$, we have

$$\mathbb{E}\big[f^{(2)}(t,v_\tau^1,x)\big] = \int_0^\infty f^{(2)}(t,v,x)\phi(v)dv, \qquad \tau \geqslant 0,$$

hence (8.44) rewrites as

$$\int_0^\infty f^{(2)}(t,v,x)\phi(v)dv = \int_0^\infty f^{(2)}(t,v,x)\phi(v)dv + \int_0^s \int_0^\infty \mathcal{L}_0 f^{(2)}(t,v,x)\phi(v)dvd\tau.$$

By differentiation with respect to $s > 0$ this yields

$$\int_0^\infty \mathcal{L}_0 f^{(2)}(t,v,x)\phi(v)dv = 0,$$

hence by (8.43) we find

$$\int_0^\infty \mathcal{L}_2 f^{(0)}(t,x)\phi(v)dv = 0,$$

cf. § 3.2 of Fouque et al. (2011), *i.e.* we find

$$\frac{\partial f^{(0)}}{\partial t}(t,x) + rx\frac{\partial f^{(0)}}{\partial x}(t,x) + \frac{\eta^2}{2}\int_0^\infty v\phi(v)dv\frac{\partial^2 f^{(0)}}{\partial x^2}(t,x) = rf^{(0)}(t,x),$$

with the terminal condition $f^{(0)}(T,x) = (x-K)^+$. □

As a consequence of Proposition 8.14, the first-order term $f^{(0)}(t,x)$ in the expansion (8.41) is the Black-Scholes function

$$f^{(0)}(t,x) = \mathrm{Bl}\left(S_t, K, r, T-t, \sqrt{\int_0^\infty v\phi(v)dv}\right),$$

with the averaged squared volatility

$$\int_0^\infty v\phi(v)dv = \mathbb{E}\left[v_\tau^1\right], \qquad \tau \geqslant 0, \tag{8.45}$$

under the stationary distribution of the process with infinitesimal generator $\mathcal{L}_0$, *i.e.* the stationary distribution of the solution to

$$dv_t^{(1)} = \mu(v_t^{(1)})dt + \beta(v_t^{(1)})dB_t^{(2)}.$$

Perturbation analysis in the Heston model

We have

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{v_t^{(\varepsilon)}}dB_t^{(1)} \\ \\ dv_t^{(\varepsilon)} = -\dfrac{\lambda}{\varepsilon}(v_t^{(\varepsilon)} - m)dt + \eta\sqrt{\dfrac{v_t^{(\varepsilon)}}{\varepsilon}}dB_t^{(2)}, \end{cases}$$

under the modified short mean-reversion time scale, and the SDE can be rewritten as

$$dv_t^{(\varepsilon)} = -\frac{\lambda}{\varepsilon}\big(v_t^{(\varepsilon)} - m\big)dt + \eta\sqrt{\frac{v_t^{(\varepsilon)}}{\varepsilon}}dB_t^{(2)}.$$

In other words, $\varepsilon \to 0$ corresponds to fast mean reversion, in which $v_t^{(\varepsilon)}$ becomes close to its mean (8.45).

Recall, cf. (17.7), that the CIR process $\big(v_t^{(1)}\big)_{t\in\mathbb{R}_+}$ has a gamma invariant (or stationary) distribution with shape parameter $2\lambda m/\eta^2$, scale parameter $\eta^2/(2\lambda)$, and probability density function ϕ given by

$$\phi(v) = \frac{1}{\Gamma(2\lambda m/\eta^2)(\eta^2/(2\lambda))^{2\lambda m/\eta^2}} v^{-1+2\lambda m/\eta^2}\,\mathrm{e}^{-2v\lambda/\eta^2}\mathbb{1}_{[0,\infty)}(v), \quad v > 0,$$

and mean

$$m = \int_0^\infty v\phi(v)dv.$$

Hence the first-order term $f^{(0)}(t,x)$ in the expansion (8.41) reads

$$f^{(0)}(t,x) = \mathrm{Bl}\left(S_t, K, r, T-t, \sqrt{m}\right),$$

with the averaged squared volatility

$$m = \int_0^\infty v\phi(v)dv = \mathbb{E}\left[v_\tau^1\right], \qquad \tau \geqslant 0,$$

under the stationary distribution of the process with infinitesimal generator $\mathcal{L}_0$, *i.e.* the stationary distribution of the solution to

$$dv_t^{(1)} = \mu\big(v_t^{(1)}\big)dt + \beta\big(v_t^{(1)}\big)dB_t^{(2)}.$$

In Figure 8.6, cf. Privault and She (2016), related approximations of put option prices are plotted against the value of v with correlation $\rho = -0.5$ and $\varepsilon = 0.01$ in the α-hypergeometric stochastic volatility model of Fonseca and Martini (2016), based on the series expansion of Han et al. (2013), and compared to a Monte Carlo curve requiring 300,000 samples and 30,000 time steps.

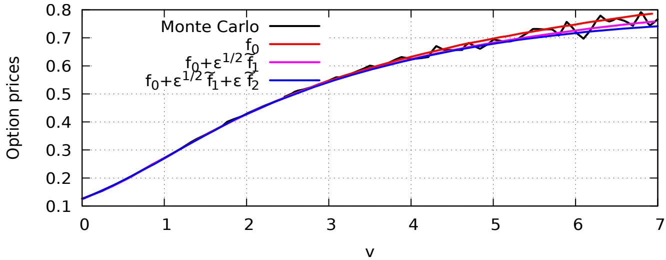

Fig. 8.6: Option price approximations plotted against v with $\rho = -0.5$.

Exercises

Exercise 8.1 (Gatheral (2006), Chapter 11). Compute the expected realized variance on the time interval $[0, T]$ in the Heston model, with

$$dv_t = -\lambda(v_t - m)dt + \eta\sqrt{v_t}dB_t, \qquad 0 \leqslant t \leqslant T.$$

Exercise 8.2 Compute the variance swap rate

$$\mathrm{VS}_T := \frac{1}{T}\mathbb{E}\left[\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{S_{kT/n} - S_{(k-1)T/n}}{S_{(k-1)T/n}}\right)^2\right] = \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}(dS_t)^2\right]$$

on the index whose level S_t is given in the following two models.

a) Heston (1993) model. Here, $(S_t)_{t\in\mathbb{R}_+}$ is given by the system of stochastic differential equations

$$\begin{cases} dS_t = (r - \alpha v_t)S_t dt + S_t\sqrt{\beta + v_t}dB_t^{(1)} \\ \\ dv_t = -\lambda(v_t - m)dt + \gamma\sqrt{v_t}dB_t^{(2)}, \end{cases}$$

where $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard Brownian motions with correlation $\rho \in [-1, 1]$ and $\alpha \geqslant 0$, $\beta \geqslant 0$, $\lambda > 0$, $m > 0$, $r > 0$, $\gamma > 0$.

b) SABR model with $\beta = 1$. The index level S_t is given by the system of stochastic differential equations

$$\begin{cases} dS_t = \sigma_t S_t dB_t^{(1)} \\ \\ d\sigma_t = \alpha\sigma_t dB_t^{(2)}, \end{cases}$$

where $\alpha > 0$ and $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard Brownian motions with correlation $\rho \in [-1,1]$.

Exercise 8.3 Convexity adjustment (§ 2.3 of Broadie and Jain (2008)).

a) Using Taylor's formula

$$\sqrt{x} = \sqrt{x_0} + \frac{x-x_0}{2\sqrt{x_0}} - \frac{(x-x_0)^2}{8x_0^{3/2}} + o((x-x_0)^2),$$

find an approximation of $R_{0,T} = \sqrt{R_{0,T}^2}$ using $\sqrt{\mathbb{E}\big[R_{0,T}^2\big]}$ and correction terms.

b) Find an (approximate) relation between the variance swap price $\mathbb{E}^*\big[R_{0,T}^2\big]$ and the volatility swap price $\mathbb{E}^*[R_{0,T}]$ up to a correction term.

Exercise 8.4 Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ with the log-return dynamics

$$d\log S_t = \mu dt + Z_{N_{t^-}} dN_t, \qquad t \geqslant 0,$$

i.e. $S_t := S_0 \,\mathrm{e}^{\mu t + Y_t}$ in a pure jump Merton model, where $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity $\lambda > 0$ and $(Z_k)_{k\geqslant 0}$ is a family of independent identically distributed Gaussian $\mathcal{N}(\delta,\eta^2)$ random variables. Compute the price of the log-return variance swap

$$\begin{aligned}
\mathbb{E}\left[\int_0^T (d\log S_t)^2 dN_t\right] &= \mathbb{E}\left[\int_0^T (\mu dt + Z_{N_{t^-}} dN_t)^2 dN_t\right] \\
&= \mathbb{E}\left[\int_0^T (Z_{N_{t^-}} dN_t)^2 dN_t\right] \\
&= \mathbb{E}\left[\int_0^T \left(\log\frac{S_t}{S_{t^-}}\right)^2 dN_t\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{N_T} \left(\log\frac{S_{T_k}}{S_{T_{k-1}}}\right)^2\right]
\end{aligned}$$

using the smoothing lemma Proposition 20.11.

Exercise 8.5 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{8.46}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r\in\mathbb{R}$ and $\sigma > 0$.

a) Write down the solution $(S_t)_{t\in\mathbb{R}_+}$ of Equation (8.46) in explicit form.
b) Show by a direct calculation that Corollary 8.3 is satisfied by $(S_t)_{t\in\mathbb{R}_+}$.

Exercise 8.6 (Carr and Lee (2008)) Consider an underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ given by $dS_t = rS_t dt + \sigma_t S_t dB_t$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $(\sigma_t)_{t\in\mathbb{R}_+}$ is an (adapted) stochastic volatility process. The riskless asset is priced $A_t := \mathrm{e}^{rt}$, $t \in [0,T]$. We consider a realized variance swap with payoff $R^2_{0,T} = \int_0^T \sigma_t^2 dt$.

a) Show that the payoff $\int_0^T \sigma_t^2 dt$ of the realized variance swap satisfies

$$\int_0^T \sigma_t^2 dt = 2\int_0^T \frac{dS_t}{S_t} - 2\log\frac{S_T}{S_0}. \tag{8.47}$$

b) Show that the price $V_t := \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt \,\middle|\, \mathcal{F}_t\right]$ of the variance swap at time $t \in [0,T]$ satisfies

$$V_t = L_t + 2(T-t)r\,\mathrm{e}^{-(T-t)r} + 2\,\mathrm{e}^{-(T-t)r}\int_0^t \frac{dS_u}{S_u}, \tag{8.48}$$

where

$$L_t := -2\,\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\log\frac{S_T}{S_0} \,\middle|\, \mathcal{F}_t\right]$$

is the price at time t of the log contract (see Neuberger (1994), Demeterfi et al. (1999)) with payoff $-2\log(S_T/S_0)$, see also Exercises 6.10 and 7.15.

c) Show that the portfolio made at time $t \in [0,T]$ of:

- one log contract priced L_t,
- $2\,\mathrm{e}^{-(T-t)r}/S_t$ in shares priced S_t,
- $2\,\mathrm{e}^{-rT}\left(\int_0^t \frac{dS_u}{S_u} + (T-t)r - 1\right)$ in the riskless asset $A_t = \mathrm{e}^{rt}$,

hedges the realized variance swap.

d) Show that the above portfolio is self-financing.

Exercise 8.7 Let $(S_t)_{t\in\mathbb{R}_+}$ denote the geometric Brownian motion

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right), \qquad t \geqslant 0,$$

solution of $dS_t = \sigma_t S_t dW_t$, where $(\sigma_t)_{t\in\mathbb{R}_+}$ denotes a stochastic volatility process. Show that the gamma swap (or entropy contract) with payoff $\int_0^T S_t d\langle \log S\rangle_t$ can be priced as in Corollary 8.3.

Exercise 8.8 Compute the moment $\mathbb{E}^*\left[R_{0,T}^4\right]$ from Lemma 8.2.