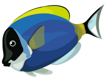
Chapter 20
Stochastic Calculus for Jump Processes

Jump processes are stochastic processes whose trajectories have discontinuities called jumps, that can occur at random times. This chapter presents the construction of jump processes with independent increments, such as the Poisson and compound Poisson processes, followed by an introduction to stochastic integrals and stochastic calculus with jumps. We also present the Girsanov Theorem for jump processes, which will be used for the construction of risk-neutral probability measures in Chapter 21 for option pricing and hedging in markets with jumps, in relation with market incompleteness.

20.1 The Poisson Process

The most elementary and useful jump process is the *standard Poisson process* $(N_t)_{t\in\mathbb{R}_+}$ which is a *counting process*, *i.e.* $(N_t)_{t\in\mathbb{R}_+}$ has jumps of size $+1$ only and its paths are constant in between two jumps, with $N_0 := 0$.



The counting process $(N_t)_{t\in\mathbb{R}_+}$ that can be used to model discrete arrival times such as claim dates in insurance, or connection logs.

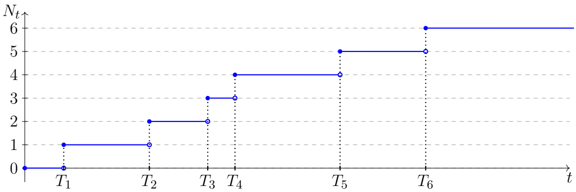


Fig. 20.1: Sample path of a counting process $(N_t)_{t\in\mathbb{R}_+}$.

Using the indicator functions

$$\mathbb{1}_{[T_k,\infty)}(t) = \begin{cases} 1 \text{ if } t \geqslant T_k, \\ 0 \text{ if } 0 \leqslant t < T_k, \quad k \geqslant 1, \end{cases}$$

the value of N_t at time t can be written as*

$$N_t = \sum_{k\geqslant 1} \mathbb{1}_{[T_k,\infty)}(t), \qquad t \geqslant 0, \tag{20.1}$$

where and $(T_k)_{k\geqslant 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty.$$

The operation defined in (20.1) can be implemented in R using the following code.

```
T=10; Tn=c(1,3,4,7,9); dev.new(width=T, height=5)
plot(stepfun(Tn,c(0,1,2,3,4,5)),xlim =c(0,T),xlab="t",ylab=expression('N'[t]),pch=1, cex=0.8,
      col='blue', lwd=2, main="", cex.axis=1.2, cex.lab=1.4,xaxs='i'); grid()
```

In order for the counting process $(N_t)_{t\in\mathbb{R}_+}$ to be a Poisson process, it has to satisfy the following conditions:

1. Independence of increments: for all $0 \leqslant t_0 < t_1 < \cdots < t_n$ and $n \geqslant 1$ the increments

$$N_{t_1} - N_{t_0}, \ldots, \ N_{t_n} - N_{t_{n-1}},$$

are mutually independent random variables.

2. Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leqslant s \leqslant t$.

The meaning of the above stationarity condition is that for all fixed $k \geqslant 0$ we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = k) = \mathbb{P}(N_t - N_s = k),$$

* The notation N_t is not to be confused with the notation used for numéraire processes in Chapter 16.

for all $h > 0$, *i.e.*, the value of the probability

$$\mathbb{P}(N_{t+h} - N_{s+h} = k)$$

does not depend on $h > 0$, for all fixed $0 \leqslant s \leqslant t$ and $k \geqslant 0$.

Based on the above assumption, given $T > 0$ a time value, a natural question arises:

what is the probability distribution of the random variable N_T?

We already know that N_t takes values in $\mathbb{N}$ and therefore it has a discrete distribution for all $t \in \mathbb{R}_+$.

It is a remarkable fact that the distribution of the increments of $(N_t)_{t\in\mathbb{R}_+}$, can be completely determined from the above conditions, as shown in the following theorem.

As seen in the next result, cf. Theorem 4.1 in Bosq and Nguyen (1996), the Poisson increment $N_t - N_s$ has the Poisson distribution with parameter $(t-s)\lambda$.

Theorem 20.1. *Assume that the counting process $(N_t)_{t\in\mathbb{R}_+}$ satisfies the above independence and stationarity Conditions 1 and 2 on page 726. Then, for all fixed $0 \leqslant s \leqslant t$ the increment $N_t - N_s$ follows the Poisson distribution with parameter $(t-s)\lambda$, i.e. we have*

$$\boxed{\mathbb{P}(N_t - N_s = k) = \mathrm{e}^{-(t-s)\lambda}\frac{((t-s)\lambda)^k}{k!}, \qquad k \geqslant 0,} \tag{20.2}$$

for some constant $\lambda > 0$.

The parameter $\lambda > 0$ is called the <u>intensity</u> of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ and it is given by

$$\lambda := \lim_{h\to 0} \frac{1}{h}\mathbb{P}(N_h = 1). \tag{20.3}$$

The proof of the above Theorem 20.1 is technical and not included here, cf. *e.g.* Bosq and Nguyen (1996) for details, and we could in fact take this distribution property (20.2) as one of the hypotheses that define the Poisson process.

Precisely, we could restate the definition of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ as being a stochastic process defined by (20.1), which is assumed to have independent increments distributed according to the Poisson distribution, in the sense that for all $0 \leqslant t_0 \leqslant t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$((t_1 - t_0)\lambda, \ldots, (t_n - t_{n-1})\lambda).$$

In particular, N_t has the Poisson distribution with parameter λt, *i.e.*,

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}, \qquad t > 0.$$

The *expected value* $\mathbb{E}[N_t]$ and the variance of N_t can be computed as

$$\mathbb{E}[N_t] = \mathrm{Var}[N_t] = \lambda t, \tag{20.4}$$

see Exercise A.1. As a consequence, the *dispersion index* of the Poisson process is

$$\frac{\mathrm{Var}[N_t]}{\mathbb{E}[N_t]} = 1, \qquad t \geqslant 0. \tag{20.5}$$

Short time behaviour

From (20.3) above we deduce the *short time asymptotics*[*]

$$\begin{cases} \mathbb{P}(N_h = 0) = \mathrm{e}^{-\lambda h} = 1 - \lambda h + o(h), \qquad h \to 0, \\ \mathbb{P}(N_h = 1) = \lambda h \, \mathrm{e}^{-\lambda h} \simeq \lambda h, \qquad h \to 0. \end{cases}$$

By stationarity of the Poisson process we also find more generally that

$$\begin{cases} \mathbb{P}(N_{t+h} - N_t = 0) = \mathrm{e}^{-\lambda h} = 1 - \lambda h + o(h), \qquad h \to 0, \\ \mathbb{P}(N_{t+h} - N_t = 1) = \lambda h \, \mathrm{e}^{-\lambda h} \simeq \lambda h, \qquad h \to 0, \\ \mathbb{P}(N_{t+h} - N_t = 2) \simeq h^2 \dfrac{\lambda^2}{2} = o(h), \qquad h \to 0, \qquad t > 0, \end{cases} \tag{20.6}$$

for all $t > 0$. This means that within a "short" time interval $[t, t+h]$ of length h, the increment $N_{t+h} - N_t$ behaves like a Bernoulli random variable with parameter λh. This fact can be used for the random simulation of Poisson process paths.

The next R code and Figure 20.2 present a simulation of the standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ according to its short time behavior (20.6).

[*] The notation $f(h) = o(h^k)$ means $\lim_{h \to 0} f(h)/h^k = 0$, and $f(h) \simeq h^k$ means $\lim_{h \to 0} f(h)/h^k = 1$.

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https://personal.ntu.edu.sg/nprivault/indext.html

```
lambda = 0.6;T=10;N=1000*lambda;h=T*1.0/N
t=0;s=c();for (k in 1:N) {if (runif(1)<lambda*h) {s=c(s,t)};t=t+h}
dev.new(width=T, height=5)
plot(stepfun(s,cumsum(c(0,rep(1,length(s))))),xlim
    =c(0,T),xlab="t",ylab=expression('N'[t]),pch=1, cex=0.8, col='blue', lwd=2, main="",
    cex.axis=1.2, cex.lab=1.4,xaxs='i'); grid()
```

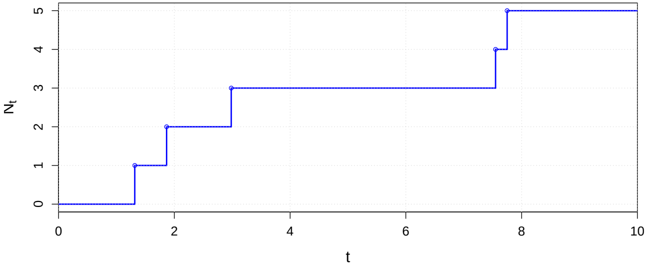

Fig. 20.2: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

More generally, for $k \geqslant 1$ we have

$$\mathbb{P}(N_{t+h} - N_t = k) \simeq h^k \frac{\lambda^k}{k!}, \qquad h \to 0, \qquad t > 0.$$

Time-dependent intensity

The intensity of the Poisson process can in fact be made time-dependent (*e.g.* by a time change), in which case we have

$$\mathbb{P}(N_t - N_s = k) = \exp\left(-\int_s^t \lambda(u)du\right) \frac{\left(\int_s^t \lambda(u)du\right)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Assuming that $\lambda(t)$ is a continuous function of time t we have in particular, as h tends to zero,

$$\begin{aligned}
&\mathbb{P}(N_{t+h} - N_t = k) \\
&= \begin{cases}
\exp\left(-\int_t^{t+h} \lambda(u)du\right) = 1 - \lambda(t)h + o(h), & k = 0, \\
\exp\left(-\int_t^{t+h} \lambda(u)du\right) \int_t^{t+h} \lambda(u)du = \lambda(t)h + o(h), & k = 1, \\
o(h), & k \geqslant 2.
\end{cases}
\end{aligned}$$

The intensity process $(\lambda(t))_{t\in\mathbb{R}_+}$ can also be made random, as in the case of Cox processes.

Poisson process jump times

In order to determine the distribution of the first jump time T_1 we note that we have the equivalence

$$\{T_1 > t\} \iff \{N_t = 0\},$$

which implies

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \geqslant 0,$$

i.e., T_1 has an exponential distribution with parameter $\lambda > 0$.

In order to prove the next proposition we note that more generally, we have the equivalence

$$\{T_n > t\} \iff \{N_t \leqslant n-1\},$$

for all $n \geqslant 1$. This allows us to compute the distribution of the random jump time T_n with its probability density function. It coincides with the *gamma* distribution with integer parameter $n \geqslant 1$, also known as the Erlang distribution in queueing theory.

Proposition 20.2. *For all $n \geqslant 1$, the probability distribution of T_n has the gamma probability density function*

$$t \longmapsto \lambda^n \mathrm{e}^{-\lambda t} \frac{t^{n-1}}{(n-1)!}$$

with shape parameter $n \geqslant 1$ and scaling parameter $\lambda > 0$ on $\mathbb{R}_+$, i.e., for all $t > 0$ the probability $\mathbb{P}(T_n \geqslant t)$ is given by

$$\mathbb{P}(T_n \geqslant t) = \lambda^n \int_t^\infty \mathrm{e}^{-\lambda s} \frac{s^{n-1}}{(n-1)!} ds.$$

Proof. We have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \geqslant 0,$$

and by induction, assuming that

$$\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \geqslant 2,$$

we obtain

$$\begin{aligned}\mathbb{P}(T_n > t) &= \mathbb{P}(T_n > t \geqslant T_{n-1}) + \mathbb{P}(T_{n-1} > t)\\ &= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t)\\ &= \mathrm{e}^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty \mathrm{e}^{-\lambda s}\frac{(\lambda s)^{n-2}}{(n-2)!}ds\\ &= \lambda \int_t^\infty \mathrm{e}^{-\lambda s}\frac{(\lambda s)^{n-1}}{(n-1)!}ds, \qquad t \geqslant 0,\end{aligned}$$

where we applied an integration by parts to derive the last line. □

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

i.e., $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geqslant 1$, is the probability density function of the random jump time T_n.

In addition to Proposition 20.2 we could show the following proposition which relies on the *strong Markov property*, see *e.g.* Theorem 6.5.4 of Norris (1998).

Proposition 20.3. *The (random) interjump times*

$$\tau_k := T_{k+1} - T_k$$

spent at state $k \geqslant 0$, with $T_0 = 0$, form a sequence of independent identically distributed random variables having the exponential distribution with parameter $\lambda > 0$, i.e.,

$$\mathbb{P}(\tau_0 > t_0, \ldots, \tau_n > t_n) = \mathrm{e}^{-(t_0+t_1+\cdots+t_n)\lambda}, \qquad t_0, t_1, \ldots, t_n \geqslant 0.$$

As the expectation of the exponentially distributed random variable τ_k with parameter $\lambda > 0$ is given by

$$\mathbb{E}[\tau_k] = \lambda \int_0^\infty x\,\mathrm{e}^{-\lambda x}dx = \frac{1}{\lambda},$$

we can check that the *nth* jump time $T_n = \tau_0 + \cdots + \tau_{n-1}$ has the mean

$$\mathbb{E}[T_n] = \frac{n}{\lambda}, \qquad n \geqslant 1.$$

Consequently, the higher the intensity $\lambda > 0$ is (*i.e.*, the higher the probability of having a jump within a small interval), the smaller the time spent in each state $k \geqslant 0$ is on average.

As a consequence of Proposition 20.2, random samples of Poisson process jump times can be generated from Poisson jump times using the following R code according to Proposition 20.3.

```
lambda = 0.6;T=10;Tn=c();n=0;
S=0; while (S<T) {S=S+rexp(1,rate=lambda); Tn=c(Tn,S); n=n+1}
Z<-cumsum(c(0,rep(1,n))); dev.new(width=T, height=5)
plot(stepfun(Tn,Z),xlim =c(0,T),ylim=c(0,8),xlab="t",ylab=expression('N'[t]),pch=1, cex=1,
    col="blue", lwd=2, main="", las = 1, cex.axis=1.2, cex.lab=1.4,xaxs='i',yaxs='i'); grid()
```

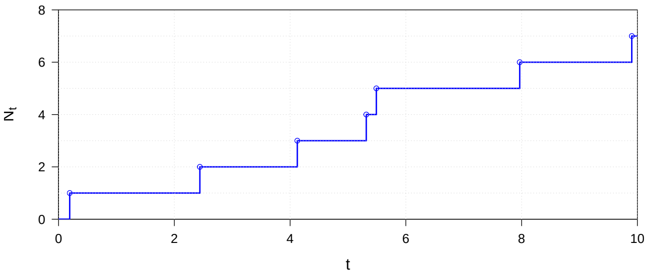

Fig. 20.3: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

In addition, conditionally to $\{N_T = n\}$, the n jump times on $[0,T]$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are independent uniformly distributed random variables on $[0,T]^n$, cf. *e.g.* § 11.1 in Privault (2018). This fact can also be useful for the random simulation of Poisson process paths.

```
lambda = 0.6;T=10;n = rpois(1,lambda*T);Tn <- sort(runif(n,0,T)); Z<-cumsum(c(0,rep(1,n)));
    dev.new(width=T, height=5)
plot(stepfun(Tn,Z),xlim =c(0,T),ylim=c(0,8),xlab="t",ylab=expression('N'[t]),pch=1, cex=1,
    col="blue", lwd=2, main="", las = 1, cex.axis=1.2, cex.lab=1.4,xaxs='i',tick.ratio = 0.5);
    grid()
```

Compensated Poisson martingale

From (20.4) above we deduce that

$$\mathbb{E}[N_t - \lambda t] = 0, \tag{20.7}$$

i.e., the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ has *centered increments*.

```
lambda = 0.6;T=10;Tn=c();S=0;n=0;
while (S<T) {S=S+rexp(1,rate=lambda); Tn=c(Tn,S); n=n+1}
Z<-cumsum(c(0,rep(1,n)));
N <- function(t) {return(stepfun(Tn,Z)(t))};t <- seq(0,10,0.01)
dev.new(width=T, height=5)
plot(t,N(t)-lambda*t,xlim = c(0,10),ylim =
      c(-2,2),xlab="t",ylab=expression(paste('N'[t],'-t')),type="l",lwd=2,col="blue",main="",
      xaxs = "i", yaxs = "i", xaxs = "i", yaxs = "i", las = 1, cex.axis=1.2, cex.lab=1.4)
abline(h = 0, col="black", lwd =2)
points(Tn,N(Tn)-lambda*Tn,pch=1,cex=0.8,col="blue",lwd=2)
```

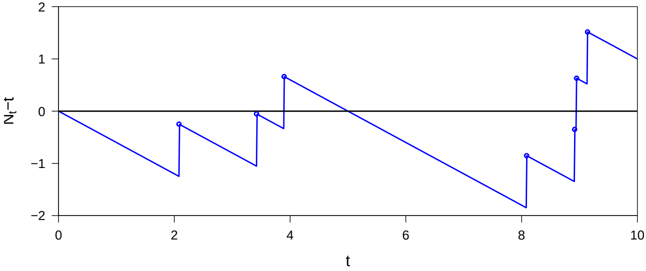

Fig. 20.4: Sample path of the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$.

Since in addition $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ also has independent increments, we get the following proposition, see *e.g.* Example 2 page 272. We let

$$\mathcal{F}_t := \sigma\big(N_s \ : \ s \in [0,t]\big), \qquad t \geqslant 0,$$

denote the *filtration* generated by the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Proposition 20.4. *The compensated Poisson process*

$$(N_t - \lambda t)_{t\in\mathbb{R}_+}$$

is a martingale *with respect* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

The Poisson process belong to the family of *renewal processes*, which are counting processes of the form

$$N_t = \sum_{n\geqslant 1} \mathbb{1}_{[T_n,\infty)}(t), \qquad t \geqslant 0,$$

for which $\tau_k := T_{k+1} - T_k$, $k \geqslant 0$, is a sequence of independent identically distributed random variables.

20.2 Compound Poisson Process

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore there is some interest in considering jump processes that can have random jump sizes.

Let $(Z_k)_{k\geqslant 1}$ denote a sequence of independent, identically distributed (*i.i.d.*) square-integrable random variables, distributed as a common random variable Z with probability distribution $\nu(dy)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$. We have

$$\mathbb{P}(Z \in [a,b]) = \nu([a,b]) = \int_a^b \nu(dy), \quad -\infty < a \leqslant b < \infty, \quad k \geqslant 1,$$

and when the distribution $\nu(dy)$ admits a probability density $\varphi(y)$ on $\mathbb{R}$, we write $\nu(dy) = \varphi(y)dy$ and

$$\mathbb{P}(Z \in [a,b]) = \int_a^b \varphi(y)dy, \quad -\infty < a \leqslant b < \infty, \quad k \geqslant 1.$$

Figure 20.5 shows an example of Gaussian jump size distribution.

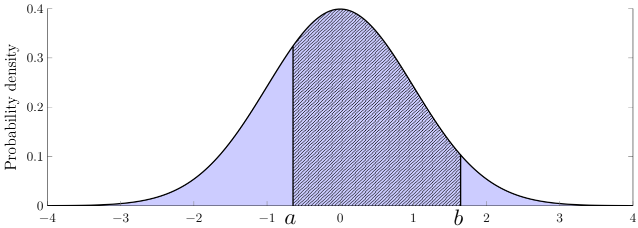

Fig. 20.5: Probability density function φ.

Definition 20.5. *The process $(Y_t)_{t\in\mathbb{R}_+}$ given by the random sum*

$$Y_t := Z_1 + Z_2 + \cdots + Z_{N_t} = \sum_{k=1}^{N_t} Z_k, \qquad t \geqslant 0, \tag{20.8}$$

*is called a compound Poisson process.**

Letting Y_{t^-} denote the left limit

$$Y_{t^-} := \lim_{s \nearrow t} Y_s, \qquad t > 0,$$

* We use the convention $\displaystyle\sum_{k=1}^{n} Z_k = 0$ if $n = 0$, so that $Y_0 = 0$.

we note that the jump size

$$\Delta Y_t := Y_t - Y_{t^-}, \qquad t \geqslant 0,$$

of $(Y_t)_{t\in\mathbb{R}_+}$ at time t is given by the relation

$$\Delta Y_t = Z_{N_t}\Delta N_t, \qquad t \geqslant 0, \tag{20.9}$$

where

$$\Delta N_t := N_t - N_{t^-} \in \{0,1\}, \qquad t \geqslant 0,$$

denotes the jump size of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and N_{t^-} is the left limit

$$N_{t^-} := \lim_{s\nearrow t} N_s, \qquad t > 0,$$

The next Figure 20.6 represents a sample path of a compound Poisson process, with here $Z_1 = 0.9$, $Z_2 = -0.7$, $Z_3 = 1.4$, $Z_4 = 0.6$, $Z_5 = -2.5$, $Z_6 = 1.5$, $Z_7 = -0.5$, with the relation

$$Y_{T_k} = Y_{T_k^-} + Z_k, \qquad k \geqslant 1.$$

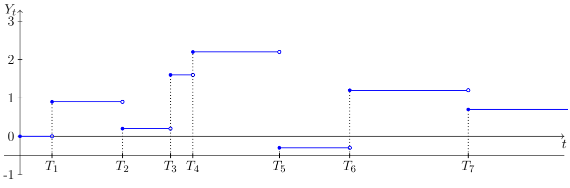

Fig. 20.6: Sample path of a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$.

Example. Assume that the jump sizes Z are Gaussian distributed with mean δ and variance η^2, with

$$\nu(dy) = \frac{1}{\sqrt{2\pi\eta^2}}\,\mathrm{e}^{-(y-\delta)^2/(2\eta^2)}dy.$$

```
N<-50;Tk<-cumsum(rexp(N,rate=0.5)); Zk<-rexp(N,rate=0.5); Yk<-cumsum(c(0,Zk))
plot(stepfun(Tk,Yk),xlim = c(0,10),lwd=2,do.points = F,main="L=0.5",col="blue")
Zk<-rnorm(N,mean=0,sd=1); Yk<-cumsum(c(0,Zk))
plot(stepfun(Tk,Yk),xlim = c(0,10),lwd=2,do.points = F,main="L=0.5",col="blue")
```

Given that $\{N_T = n\}$, the n jump sizes of $(Y_t)_{t\in\mathbb{R}_+}$ on $[0,T]$ are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. Based on this fact, the next proposition allows us to compute the *Moment Generating Function* (MGF) of the increment $Y_T - Y_t$.

Proposition 20.6. *For any $t \in [0,T]$ and $\alpha \in \mathbb{R}$ we have*

$$\mathbb{E}\big[\mathrm{e}^{(Y_T-Y_t)\alpha}\big] = \exp\big((T-t)\lambda\big(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1\big)\big). \tag{20.10}$$

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Z_k)_{k\geqslant 1}$, for all $\alpha \in \mathbb{R}$ we have, by conditioning on the value of $N_T - N_t = n$,

$$\begin{aligned}
\mathbb{E}\big[\mathrm{e}^{(Y_T-Y_t)\alpha}\big] &= \mathbb{E}\left[\exp\left(\alpha\sum_{k=N_t+1}^{N_T} Z_k\right)\right] = \mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^{N_T-N_t} Z_{k+N_t}\right)\right] \\
&= \mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^{N_T-N_t} Z_k\right)\right] \\
&= \sum_{n\geqslant 0}\mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^{N_T-N_t} Z_k\right)\Big| N_T-N_t=n\right]\mathbb{P}(N_T-N_t=n) \\
&= \sum_{n\geqslant 0}\mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^{n} Z_k\right)\right]\mathbb{P}(N_T-N_t=n) \\
&= \mathrm{e}^{-(T-t)\lambda}\sum_{n\geqslant 0}\frac{\lambda^n}{n!}(T-t)^n\mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^{n} Z_k\right)\right] \\
&= \mathrm{e}^{-(T-t)\lambda}\sum_{n\geqslant 0}\frac{\lambda^n}{n!}(T-t)^n\prod_{k=1}^{n}\mathbb{E}\big[\mathrm{e}^{\alpha Z_k}\big] \\
&= \mathrm{e}^{-(T-t)\lambda}\sum_{n\geqslant 0}\frac{\lambda^n}{n!}(T-t)^n\big(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]\big)^n \\
&= \exp\Big((T-t)\lambda\big(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1\big)\Big).
\end{aligned}$$

□

As a consequence of Proposition 20.6, we can derive the following version of the Lévy-Khintchine formula, after approximating $f : [0,T] \longrightarrow \mathbb{R}$ a bounded deterministic function of time by indicator functions:

$$\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] = \exp\left(\lambda\int_0^T\int_{-\infty}^{\infty}\big(\mathrm{e}^{yf(t)}-1\big)\nu(dy)dt\right). \tag{20.11}$$

We note that we can also write

$$\mathbb{E}\big[\mathrm{e}^{(Y_T-Y_t)\alpha}\big] = \exp\left((T-t)\lambda\int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y}-1)\nu(dy)\right)$$

$$= \exp\left((T-t)\lambda \int_{-\infty}^{\infty} \mathrm{e}^{\alpha y}\nu(dy) - (T-t)\lambda \int_{-\infty}^{\infty} \nu(dy)\right),$$

since the probability distribution $\nu(dy)$ of Z satisfies

$$\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big] = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha y}\nu(dy) \quad \text{and} \quad \int_{-\infty}^{\infty} \nu(dy) = 1.$$

From the moment generating function (20.10) we can compute the expectation and variance of Y_t for fixed t. Note that the proofs of those identities require to exchange the differentiation and expectation operators, which is possible when the moment generating function (20.10) takes finite values for all α in a certain neighborhood $(-\varepsilon, \varepsilon)$ of 0.

Proposition 20.7. *i) The expectation of Y_t is given as the product of the mean number of jump times $\mathbb{E}[N_t] = \lambda t$ and the mean jump size $\mathbb{E}[Z]$, i.e.,*

$$\boxed{\mathbb{E}[Y_t] = \mathbb{E}[N_t]\mathbb{E}[Z] = \lambda t \mathbb{E}[Z]. \qquad (20.12)}$$

ii) Regarding the variance, we have

$$\boxed{\mathrm{Var}\,[Y_t] = \mathbb{E}[N_t]\mathbb{E}\big[|Z|^2\big] = \lambda t \mathbb{E}\big[|Z|^2\big]. \qquad (20.13)}$$

Proof. (i) We use the relation

$$\mathbb{E}[Y_t] = \frac{\partial}{\partial\alpha}\mathbb{E}\big[\mathrm{e}^{\alpha Y_t}\big]_{|\alpha=0} = \lambda t \int_{-\infty}^{\infty} y\nu(dy) = \lambda t \mathbb{E}[Z].$$

(ii) By (20.10), we have

$$\begin{aligned}
\mathbb{E}\big[Y_t^2\big] &= \frac{\partial^2}{\partial\alpha^2}\mathbb{E}\big[\mathrm{e}^{\alpha Y_t}\big]_{|\alpha=0} \\
&= \frac{\partial^2}{\partial\alpha^2}\exp\big(\lambda t(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1)\big)_{|\alpha=0} \\
&= \frac{\partial}{\partial\alpha}\left(\lambda t \mathbb{E}\big[Z\,\mathrm{e}^{\alpha Z}\big]\exp\big(\lambda t(\mathbb{E}\big[\mathrm{e}^{\alpha Z}\big]-1)\big)\right)_{|\alpha=0} \\
&= \lambda t \mathbb{E}\big[Z^2\big] + (\lambda t \mathbb{E}[Z])^2 \\
&= \lambda t \int_{-\infty}^{\infty} y^2\nu(dy) + (\lambda t)^2\left(\int_{-\infty}^{\infty} y\nu(dy)\right)^2 \\
&= \lambda t \mathbb{E}[Z^2] + (\lambda t \mathbb{E}[Z])^2.
\end{aligned}$$

□

Relation (20.12) can be directly recovered using series summations, as

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\sum_{k=1}^{N_t} Z_k\right] \\
&= \sum_{n\geqslant 1} \mathbb{E}\left[\sum_{k=1}^{N_t} Z_k \,\middle|\, N_t = n\right] \mathbb{P}(N_t = n) \\
&= \mathrm{e}^{-\lambda t} \sum_{n\geqslant 1} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\sum_{k=1}^{n} Z_k \,\middle|\, N_t = n\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n\geqslant 1} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\sum_{k=1}^{n} Z_k\right] \\
&= \lambda t \,\mathrm{e}^{-\lambda t} \mathbb{E}[Z] \sum_{n\geqslant 1} \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= \lambda t \mathbb{E}[Z] \\
&= \mathbb{E}[N_t]\mathbb{E}[Z].
\end{aligned}
$$

As a consequence, the *dispersion index* of the compound Poisson process

$$
\frac{\mathrm{Var}\,[Y_t]}{\mathbb{E}[Y_t]} = \frac{\mathbb{E}\big[|Z|^2\big]}{\mathbb{E}[Z]}, \qquad t \geqslant 0.
$$

coincides with the dispersion index of the random jump size Z. By a multivariate version of Theorem A.14, Proposition 20.6 can be used to show the next result.

Proposition 20.8. (i) *The compound Poisson process*

$$
Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \geqslant 0,
$$

has independent increments, i.e. for any finite sequence of times $t_0 < t_1 < \cdots < t_n$*, the increments*

$$
Y_{t_1} - Y_{t_0},\ Y_{t_2} - Y_{t_1}, \ldots,\ Y_{t_n} - Y_{t_{n-1}}
$$

are mutually independent random variables.

(ii) *In addition, the increment* $Y_t - Y_s$ *is stationary,* $0 \leqslant s \leqslant t$*, i.e. the distribution of* $Y_{t+h} - Y_{s+h}$ *does not depend of* $h \geqslant 0$.

Proof. This result relies on the fact that the result of Proposition 20.6 can be extended to sequences $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, as

$$
\mathbb{E}\left[\prod_{k=1}^{n} \mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right] = \mathbb{E}\left[\exp\left(i \sum_{k=1}^{n} \alpha_k (Y_{t_k} - Y_{t_{k-1}})\right)\right]
$$

$$= \exp\left(\lambda \sum_{k=1}^{n}(t_k - t_{k-1}) \int_{-\infty}^{\infty}(\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \qquad (20.14)$$

$$= \prod_{k=1}^{n} \exp\left((t_k - t_{k-1})\lambda \int_{-\infty}^{\infty}(\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right)$$

$$= \prod_{k=1}^{n} \mathbb{E}\left[\mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right],$$

which also shows the stationarity in distribution of $Y_{t+h} - Y_{s+h}$ in $h \geqslant 0$, for $0 \leqslant s \leqslant t$. □

Since the compensated compound Poisson process also has independent and centered increments by (20.7) we have the following counterpart of Proposition 20.4, cf. also Example 2 page 272.

Proposition 20.9. *The compensated compound Poisson process*

$$M_t := Y_t - \lambda t \mathbb{E}[Z], \qquad t \geqslant 0,$$

is a martingale.

```
lambda = 0.6;T=10;Tn=c();S=0;n=0;
while (S<T) {S=S+rexp(1,rate=lambda); Tn=c(Tn,S); n=n+1}
Z<-cumsum(c(0,rep(1,n))); Zn<-cumsum(c(0,rexp(n,rate=2)));
Y <- function(t) {return(stepfun(Tn,Zn)(t))};t <- seq(0,10,0.01)
par(oma=c(0,0.1,0,0))
plot(t,Y(t)-0.5*lambda*t,xlim = c(0,10),ylim =
c(-2,2),xlab="t",ylab=expression(paste('Y'[t],'-t')),type="l",lwd=2,col="blue",main="", xaxs =
      "i", yaxs = "i", xaxs = "i", yaxs = "i", las = 1, cex.axis=1.2, cex.lab=1.4)
abline(h = 0, col="black", lwd =2)
points(Tn,Y(Tn)-0.5*lambda*Tn,pch=1,cex=0.8,col="blue",lwd=2);grid()
```

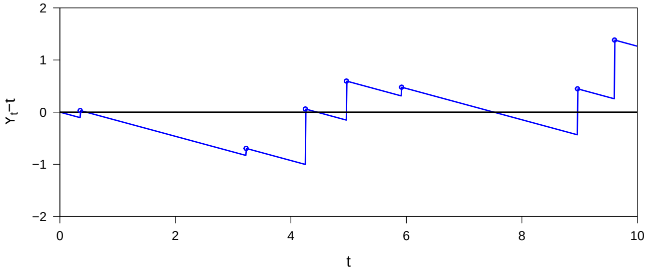

Fig. 20.7: Sample path of a compensated compound Poisson process $(Y_t - \lambda t \mathbb{E}[Z])_{t \in \mathbb{R}_+}$.

20.3 Stochastic Integrals and Itô Formula with Jumps

Definition 20.10. *Based on the relation*

$$\Delta Y_t = Z_{N_t} \Delta N_t,$$

we define the stochastic integral of a stochastic process $(\phi_t)_{t\in[0,T]}$ *with respect to* $(Y_t)_{t\in[0,T]}$ *by*

$$\boxed{\int_0^T \phi_t dY_t = \int_0^T \phi_t Z_{N_t} dN_t := \sum_{k=1}^{N_T} \phi_{T_k} Z_k. \qquad (20.15)}$$

In particular, the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ in Definition 20.5 admits the stochastic integral representation

$$Y_t = Y_0 + \sum_{k=1}^{N_t} Z_k = Y_0 + \int_0^t Z_{N_s} dN_s.$$

Note that the expression (20.15) of $\int_0^T \phi_t dY_t$ has a natural financial interpretation as the value at time T of a portfolio containing a (possibly fractional) quantity ϕ_t of a risky asset at time t, whose price evolves according to random returns Z_k, generating profits/losses $\phi_{T_k} Z_k$ at random times T_k.

The next result is also called the smoothing lemma, cf. Theorem 9.2.1 in Brémaud (1999).

Proposition 20.11. *Let* $(\phi_t)_{t\in\mathbb{R}_+}$ *be a stochastic process* adapted *to the filtration generated by* $(Y_t)_{t\in\mathbb{R}_+}$*, admitting left limits, and such that*

$$\mathbb{E}\left[\int_0^T |\phi_t| dt\right] < \infty, \qquad T > 0.$$

The expected value of the compound Poisson stochastic integral can be expressed as

$$\boxed{\mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] = \mathbb{E}\left[\int_0^T \phi_{t^-} Z_{N_t} dN_t\right] = \lambda \mathbb{E}[Z] \mathbb{E}\left[\int_0^T \phi_{t^-} dt\right], \qquad (20.16)}$$

where ϕ_{t^-} *denotes the left limit*

$$\phi_{t^-} := \lim_{s \nearrow t} \phi_s, \qquad t > 0.$$

Proof. By Proposition 20.9 the compensated compound Poisson process $(Y_t - \lambda t \mathbb{E}[Z])_{t\in\mathbb{R}_+}$ is a *martingale*, and the adaptedness of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, makes $(\phi_{t^-})_{t>0}$ *predictable*, *i.e.* adapted with respect to the filtration

$$\mathcal{F}_{t^-} := \sigma(Y_s \ : \ s \in [0,t)), \qquad t > 0.$$

Hence, by an argument similar to the first part of the proof of Proposition 7.1 and concluded by dominated convergence as in the proof of Theorem 9.2.1 in Brémaud (1999), the stochastic integral process

$$t \longmapsto \int_0^t \phi_{s^-} d\big(Y_s - \lambda\mathbb{E}[Z]ds\big) = \int_0^t \phi_{s^-}\big(Z_{N_s} dN_s - \lambda\mathbb{E}[Z]ds\big)$$

is also a martingale. We can then use the fact that the expectation of a martingale remains constant over time, *i.e.*,

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_0^T \phi_{t^-}\big(dY_t - \lambda\mathbb{E}[Z]dt\big)\right] \\ &= \mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] - \lambda\mathbb{E}[Z]\mathbb{E}\left[\int_0^T \phi_{t^-} dt\right]. \end{aligned}$$

□

For example, taking $\phi_t = Y_t := N_t$ we have

$$\int_0^T N_{t^-} dN_t = \sum_{k=1}^{N_T}(k-1) = \frac{1}{2}N_T(N_T-1),$$

hence

$$\begin{aligned} \mathbb{E}\left[\int_0^T N_{t^-} dN_t\right] &= \frac{1}{2}\left(\mathbb{E}[N_T^2] - \mathbb{E}[N_T]\right) \\ &= \frac{(\lambda T)^2}{2} \\ &= \lambda\int_0^T \lambda t dt \\ &= \lambda\int_0^T \mathbb{E}[N_t]dt, \end{aligned}$$

as in (20.16). Note however that while the identity in expectations (20.16) holds for the left limit ϕ_{t^-}, it need not hold for ϕ_t itself. Indeed, taking $\phi_t = Y_t := N_t$ we have

$$\int_0^T N_t dN_t = \sum_{k=1}^{N_T} k = \frac{1}{2}N_T(N_T+1),$$

hence

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T N_t dN_t\right] &= \frac{1}{2}\left(\mathbb{E}\left[N_T^2\right]+\mathbb{E}[N_T]\right)\\
&= \frac{1}{2}\left((\lambda T)^2+2\lambda T\right)\\
&= \frac{(\lambda T)^2}{2}+\lambda T\\
&\neq \lambda\mathbb{E}\left[\int_0^T N_t dt\right].
\end{aligned}
$$

Under similar conditions, the compound Poisson compensated stochastic integral can be shown to satisfy the Itô isometry (20.17) in the next proposition.

Proposition 20.12. *Let $(\phi_t)_{t\in\mathbb{R}_+}$ be a stochastic process* adapted *to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, admitting left limits, and such that*

$$
\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right] < \infty, \qquad T > 0.
$$

The expected value of the squared compound Poisson compensated stochastic integral can be computed as

$$
\boxed{\mathbb{E}\left[\left(\int_0^T \phi_{t^-}(dY_t - \lambda\mathbb{E}[Z]dt)\right)^2\right] = \lambda\mathbb{E}\left[|Z|^2\right]\mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 dt\right],} \tag{20.17}
$$

Note that in (20.17), the generic jump size Z is squared but λ is not.

Proof. From the stochastic Fubini-type theorem, we have

$$
\begin{aligned}
&\left(\int_0^T \phi_{t^-}(dY_t - \lambda\mathbb{E}[Z]dt)\right)^2 && (20.18)\\
&\quad = 2\int_0^T \phi_{t^-}\int_0^{t^-}\phi_{s^-}(dY_s - \lambda\mathbb{E}[Z]ds)(dY_t - \lambda\mathbb{E}[Z]dt) && (20.19)\\
&\qquad + \int_0^T |\phi_{t^-}|^2 |Z_{N_t}|^2 dN_t, && (20.20)
\end{aligned}
$$

where integration over the diagonal $\{s = t\}$ has been excluded in (20.19) as the inner integral has an upper limit t^- rather than t. Next, taking expectation on both sides of (20.18)-(20.20), we find

$$
\mathbb{E}\left[\left(\int_0^T \phi_{t^-}(dY_t - \lambda\mathbb{E}[Z]dt)\right)^2\right] = \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 |Z_{N_t}|^2 dN_t\right]
$$

$$= \lambda \mathbb{E}\big[|Z|^2\big] \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 dt \right],$$

where we used the vanishing of the expectation of the double stochastic integral:

$$\mathbb{E}\left[\int_0^T \phi_{t^-} \int_0^{t^-} \phi_{s^-} (dY_s - \lambda \mathbb{E}[Z] ds)(dY_t - \lambda \mathbb{E}[Z] dt) \right] = 0,$$

and the martingale property of the compensated compound Poisson process

$$t \longmapsto \left(\sum_{k=1}^{N_t} |Z_k|^2 \right) - \lambda t \mathbb{E}\big[Z^2\big], \qquad t \geqslant 0,$$

as in the proof of Proposition 20.11. The isometry relation (20.17) can also be proved using simple predictable processes, similarly to the proof of Proposition 4.21. □

Extensions

a) Take $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion independent of $(Y_t)_{t\in\mathbb{R}_+}$ and $(X_t)_{t\in\mathbb{R}_+}$ a jump-diffusion process of the form

$$X_t := \int_0^t u_s dB_s + \int_0^t v_s ds + Y_t, \qquad t \geqslant 0,$$

where $(u_t)_{t\in\mathbb{R}_+}$ is a stochastic process which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$, and such that

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 |u_t|^2 dt \right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T |\phi_t v_t| dt \right] < \infty, \quad T > 0.$$

In this case, the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ can be defined by

$$\begin{aligned} \int_0^T \phi_t dX_t &:= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \int_0^T \phi_t dY_t \\ &= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \sum_{k=1}^{N_T} \phi_{T_k} Z_k, \qquad T > 0. \end{aligned}$$

For the mixed continuous-jump martingale

$$X_t := \int_0^t u_s dB_s + Y_t - \lambda t \mathbb{E}[Z], \qquad t \geqslant 0,$$

we then have the isometry:

$$\mathbb{E}\left[\left(\int_0^T \phi_{t^-} dX_t\right)^2\right] = \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 |u_t|^2 dt\right] + \lambda \mathbb{E}\left[|Z|^2\right] \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 dt\right]. \tag{20.21}$$

provided that $(\phi_t)_{t\in\mathbb{R}_+}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$. The isometry formula (20.21) will be used in Section 21.6 for mean-variance hedging in jump-diffusion models.

b) When $(X_t)_{t\in\mathbb{R}_+}$ takes the form

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dY_s, \qquad t \geqslant 0,$$

the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ can be defined as

$$\begin{aligned}\int_0^T \phi_t dX_t &:= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \int_0^T \eta_t \phi_t dY_t \\ &= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \sum_{k=1}^{N_T} \phi_{T_k} \eta_{T_k} Z_k, \qquad T > 0.\end{aligned}$$

Itô Formula with Jumps

The next proposition gives the simplest instance of the Itô formula with jumps, in the case of a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity λ.

Proposition 20.13. *Itô formula for the standard Poisson process. We have*

$$f(N_t) = f(0) + \int_0^t (f(N_s) - f(N_{s^-}))dN_s, \qquad t \geqslant 0,$$

where N_{s^-} denotes the left limit $N_{s^-} = \lim_{h\searrow 0} N_{s-h}$.

Proof. We note that

$$N_s = N_{s^-} + 1 \ \text{ if } \ dN_s = 1 \ \text{ and } \ k = N_{T_k} = 1 + N_{T_k^-}, \qquad k \geqslant 1.$$

Hence we have the *telescoping sum*

$$\begin{aligned}f(N_t) &= f(0) + \sum_{k=1}^{N_t} (f(k) - f(k-1)) \\ &= f(0) + \sum_{k=1}^{N_t} (f(N_{T_k}) - f(N_{T_k^-}))\end{aligned}$$

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$$
\begin{aligned}
&= f(0) + \sum_{k=1}^{N_t} (f(1+N_{T_k^-}) - f(N_{T_k^-})) \\
&= f(0) + \int_0^t (f(1+N_{s^-}) - f(N_{s^-}))dN_s \\
&= f(0) + \int_0^t (f(N_s) - f(N_s - 1))dN_s \\
&= f(0) + \int_0^t (f(N_s) - f(N_{s^-}))dN_s,
\end{aligned}
$$

where N_{s^-} denotes the left limit $N_{s^-} = \lim_{h \searrow 0} N_{s-h}$. □

The next result deals with the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ in (20.5) via a similar argument.

Proposition 20.14. *Itô formula for the compound Poisson process* $(Y_t)_{t\in\mathbb{R}_+}$. *We have the* pathwise Itô formula

$$
f(Y_t) = f(0) + \int_0^t (f(Y_s) - f(Y_{s^-}))dN_s, \qquad t \geqslant 0. \tag{20.22}
$$

Proof. We have

$$
\begin{aligned}
f(Y_t) &= f(0) + \sum_{k=1}^{N_t} \big(f\big(Y_{T_k}\big) - f\big(Y_{T_k^-}\big)\big) \\
&= f(0) + \sum_{k=1}^{N_t} \big(f\big(Y_{T_k^-} + Z_k\big) - f\big(Y_{T_k^-}\big)\big) \\
&= f(0) + \int_0^t (f(Y_{s^-} + Z_{N_s}) - f(Y_{s^-}))dN_s \\
&= f(0) + \int_0^t (f(Y_s) - f(Y_{s^-}))dN_s, \qquad t \geqslant 0.
\end{aligned}
$$

□

From the expression

$$
Y_t = Y_0 + \sum_{k=1}^{N_t} Z_k = Y_0 + \int_0^t Z_{N_s} dN_s,
$$

the Itô formula (20.22) can be decomposed using a compensated Poisson stochastic integral as

$$
\begin{aligned}
df(Y_t) = {} & (f(Y_t) - f(Y_{t^-}))dN_t - \mathbb{E}[(f(y+Z) - f(y)]_{y=Y_{t^-}} dt \\
& + \mathbb{E}[(f(y+Z) - f(y)]_{y=Y_{t^-}} dt,
\end{aligned} \tag{20.23}
$$

where

$$
(f(Y_t) - f(Y_{t^-}))dN_t - \mathbb{E}[(f(y+Z_{N_t}) - f(y)]_{y=Y_{t^-}} dt
$$

is the differential of a martingale by the smoothing lemma Proposition 20.11.

More generally, we have the following result.

Proposition 20.15. *For an Itô process of the form*

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s + \int_0^t \eta_s dY_s, \qquad t \geqslant 0,$$

and f a $\mathcal{C}^2(\mathbb{R})$ function, we have the Itô formula

$$\boxed{\begin{aligned} f(X_t) = f(X_0) + \int_0^t v_s f'(X_s) ds + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds \\ + \int_0^t (f(X_s) - f(X_{s^-})) dN_s, \qquad t \geqslant 0. \end{aligned}} \tag{20.24}$$

Proof. By combining the Itô formula for Brownian motion with the Itô formula for the compound Poisson process of Proposition 20.14, we find

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds + \int_0^t v_s f'(X_s) ds \\ &\quad + \sum_{k=1}^{N_T} \big(f\big(X_{T_k^-} + \eta_{T_k} Z_k\big) - f\big(X_{T_k^-}\big)\big) \\ &= f(X_0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds + \int_0^t v_s f'(X_s) ds \\ &\quad + \int_0^t (f(X_{s^-} + \eta_s Z_{N_s}) - f(X_{s^-})) dN_s, \quad t \geqslant 0, \end{aligned}$$

which yields (20.24). □

The integral Itô formula (20.24) can be rewritten in differential notation as

$$\boxed{df(X_t) = v_t f'(X_t) dt + u_t f'(X_t) dB_t + \frac{|u_t|^2}{2} f''(X_t) dt + (f(X_t) - f(X_{t^-})) dN_t,} \tag{20.25}$$

$t \geqslant 0$. For a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ given by

$$X_t = \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dN_s, \qquad t \geqslant 0,$$

the Itô formula with jumps reads

$$f(X_t) = f(0) + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t |u_s|^2 f''(X_s) dB_s$$

$$
\begin{aligned}
&+\int_0^t v_s f'(X_s)ds + \int_0^t (f(X_{s^-}+\eta_s) - f(X_{s^-}))dN_s \\
&= f(0) + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t |u_s|^2 f''(X_s)dB_s \\
&+\int_0^t v_s f'(X_s)ds + \int_0^t (f(X_s) - f(X_{s^-}))dN_s.
\end{aligned}
$$

Itô multiplication table with jumps

Given two Itô processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ written in differential notation as

$$dX_t = u_t dB_t + v_t dt + \eta_t dN_t, \qquad t \geqslant 0,$$

and

$$dY_t = a_t dB_t + b_t dt + c_t dN_t, \qquad t \geqslant 0,$$

the Itô formula for jump processes can also be written as

$$d(X_tY_t) = X_{t^-}dY_t + Y_{t^-}dX_t + dX_t \bullet dY_t$$

where the product $dX_t \bullet dY_t$ is computed according to the following extension of the Itô multiplication Table 4.1. The relation $dB_t \bullet dN_t = 0$ is due to the fact that $(N_t)_{t\in\mathbb{R}_+}$ has finite variation on any finite interval.

$\bullet$	dt	dB_t	dN_t
dt	0	0	0
dB_t	0	dt	0
dN_t	0	0	dN_t

Table 20.1: Itô multiplication table with jumps.

In other words, we have

$$
\begin{aligned}
dX_t \bullet dY_t &= (v_t dt + u_t dB_t + \eta_t dN_t)(b_t dt + a_t dB_t + c_t dN_t) \\
&= v_t b_t dt \bullet dt + u_t b_t dB_t \bullet dt + \eta_t b_t dN_t \bullet dt \\
&\quad v_t a_t dt \bullet dB_t + u_t a_t dB_t \bullet dB_t + \eta_t a_t dN_t \bullet dB_t \\
&\quad + v_t c_t dt \bullet dN_t + u_t c_t dB_t \bullet dN_t + \eta_t c_t dN_t \bullet dN_t \\
&= + u_t a_t dB_t \bullet dB_t + \eta_t c_t dN_t \bullet dN_t \\
&= u_t a_t dt + \eta_t c_t dN_t,
\end{aligned}
$$

since

$$dN_t \bullet dN_t = (dN_t)^2 = dN_t,$$

as $\Delta N_t \in \{0,1\}$. In particular, we have

$$(dX_t)^2 = (v_t dt + u_t dB_t + \eta_t dN_t)^2 = u_t^2 dt + \eta_t^2 dN_t.$$

Jump processes with infinite activity

Given $\eta(s)$, $s \in \mathbb{R}_+$, a deterministic function of time and $(X_t)_{t\in\mathbb{R}_+}$ an Itô process of the form

$$X_t := X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s + \int_0^t \eta(s) dY_t, \qquad t \geqslant 0,$$

the Itô formula with jumps (20.24) can be rewritten as

$$f(X_t) = f(X_0) + \int_0^t v_s f'(X_s) ds + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds$$
$$+ \int_0^t (f(X_{s^-} + \eta(s)\Delta Y_s) - f(X_{s^-}))\, dN_s - \lambda \int_0^t \mathbb{E}\big[f(x+\eta(s)Z) - f(x)\big]_{|x=X_{s^-}} ds$$
$$+ \lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_{s^-}))\, \nu(dy) ds, \qquad t \geqslant 0,$$

using the compensated martingale

$$\begin{aligned} &\int_0^t (f(X_s) - f(X_{s^-}))\, dN_s - \lambda \int_0^t \mathbb{E}\big[f(x+\eta(s)Z) - f(x)\big]_{|x=X_{s^-}} ds \\ &\quad = \int_0^t (f(X_{s^-} + \eta(s)\Delta Y_s) - f(X_{s^-}))\, dN_s \\ &\qquad -\lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_s))\, \nu(dy) ds, \end{aligned} \tag{20.26}$$

with the relation $dX_s = \eta_s \Delta Y_s$. We note that from the relation

$$\mathbb{E}[Z] = \int_{-\infty}^{\infty} y \nu(dy),$$

the above compensator term (20.26) rewrites as

$$\begin{aligned} &\lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_{s^-}))\, \nu(dy) ds \\ &\quad = \lambda \int_0^t \int_{-\infty}^{\infty} \big(f(X_{s^-} + \eta(s)y) - f(X_{s^-}) - \eta(s) y f'(X_{s^-})\big)\, \nu(dy) ds \\ &\qquad +\lambda \mathbb{E}[Z] \int_0^t \eta(s) f'(X_{s^-}) ds. \end{aligned} \tag{20.27}$$

The expression (20.27) above is at the basis of the extension of Itô's formula to Lévy processes with an infinite number of jumps on any interval under the conditions

$$\int_{|y|\leqslant 1} y^2 \nu(dy) < \infty \quad \text{and} \quad \nu([-1,1]^c) < \infty,$$

using the bound

$$|f(x+y) - f(x) - y f'(x)| \leqslant C y^2, \qquad y \in [-1,1],$$

that follows from Taylor's theorem for f a $\mathcal{C}^2(\mathbb{R})$ function. This yields

$$
\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t v_s f'(X_s)ds + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds \\
&+ \int_0^t \left(f(X_{s^-} + \eta(s)\Delta Y_s) - f(X_{s^-})\right) dN_s - \lambda \int_0^t \mathbb{E}\left[f(x+\eta(s)Z) - f(x)\right]_{|x=X_{s^-}} ds \\
&+ \lambda \int_0^t \int_{-\infty}^{\infty} \left(f(X_{s^-} + \eta(s)y) - f(X_{s^-}) - \eta(s)yf'(X_{s^-})\right) \nu(dy)ds \\
&+ \lambda \mathbb{E}[Z] \int_0^t \eta(s) f'(X_{s^-})ds, \qquad t \geqslant 0,
\end{aligned}
$$

see *e.g.* Theorem 1.16 in Øksendal and Sulem (2005) and Theorem 4.4.7 in Applebaum (2009) in the setting of Poisson random measures.

By construction, compound Poisson processes only have a *finite* number of jumps on any interval. They belong to the family of *Lévy processes* which may have an infinite number of jumps on any finite time interval, see *e.g.* § 4.4.1 of Cont and Tankov (2004). Such processes, also called "infinite activity Lévy processes" are also useful in financial modeling, cf. Cont and Tankov (2004), and include the gamma process, stable processes, variance gamma processes, inverse Gaussian processes, etc, as in the following illustrations.

1. Gamma process.

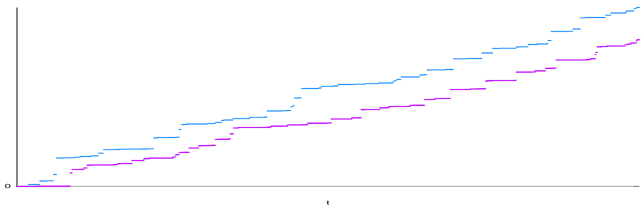

Fig. 20.8: Sample trajectories of a gamma process.

The next R code can be used to generate the gamma process paths of Figure 20.8.

```
N=2000; t <- 0:N; dt <- 1.0/N; nsim <- 6; alpha=20.0
X = matrix(0, nsim, N)
for (i in 1:nsim){X[i,]=rgamma(N,alpha*dt);}
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
plot(t, X[1, ], xlab = "time", type = "l", ylim = c(0, 2*N*alpha*dt), col = 0)
for (i in 1:nsim){points(t, X[i, ], xlab = "time", type = "p", pch=20, cex =0.02, col = i)}
```

2. Variance gamma process.

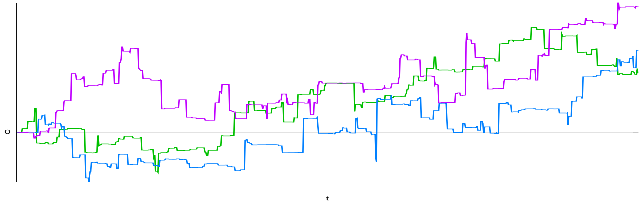

Fig. 20.9: Sample trajectories of a variance gamma process.

3. Inverse Gaussian process.

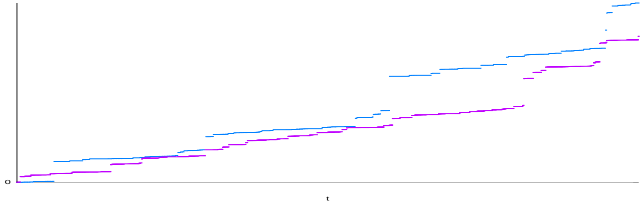

Fig. 20.10: Sample trajectories of an inverse Gaussian process.

4. Negative Inverse Gaussian process.

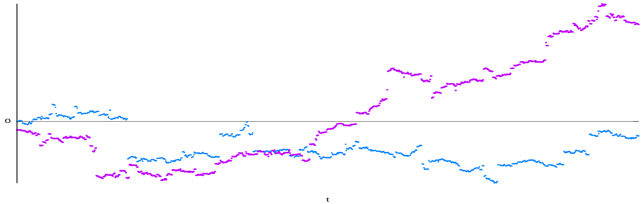

Fig. 20.11: Sample trajectories of a negative inverse Gaussian process.

5. Stable process.

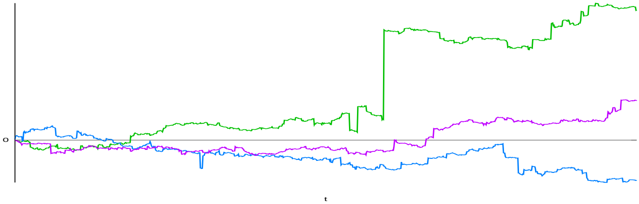

Fig. 20.12: Sample trajectories of a stable process.

The above sample paths of a stable process can be compared to the USD/CNY exchange rate over the year 2015, according to the date retrieved using the following code.

```
library(quantmod);myPars <- chart_pars();myPars$cex<-1.5
getSymbols("USDCNY=X",from="2015-01-01",to="2015-12-06",src="yahoo")
rate=Ad(`USDCNY=X`);myTheme <- chart_theme();myTheme$col$line.col <- "blue"
myTheme$rylab <- FALSE;chart_Series(rate, pars=myPars, theme = myTheme,
       name="USDCNY=X")
getSymbols("EURCHF=X",from="2013-12-30",to="2016-01-01",src="yahoo")
rate=Ad(`EURCHF=X`);chart_Series(rate, pars=myPars, theme = myTheme)
```

The adjusted close price Ad() is the closing price after adjustments for applicable splits and dividend distributions.

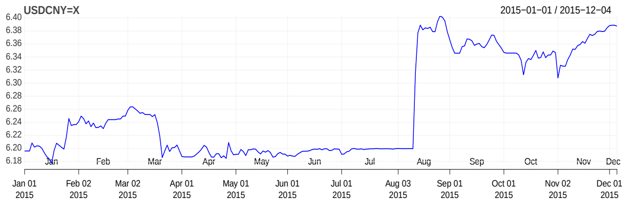

Fig. 20.13: USD/CNY Exchange rate data.

Cumulants of stochastic integrals with jumps

Using the stochastic integral of a deterministic function $f(t)$ with respect to $(Y_t)_{t\in\mathbb{R}_+}$ defined as

$$\int_0^T f(t)dY_t = \sum_{k=1}^{N_T} Z_k f(T_k),$$

Relation (20.11) can be used to show that, more generally, the moment generating function of $\int_0^T f(t)dY_t$ is given by

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] &= \exp\left(\lambda\int_0^T\int_{-\infty}^{\infty}\left(\mathrm{e}^{yf(t)}-1\right)\nu(dy)dt\right)\\ &= \exp\left(\lambda\int_0^T\left(\mathbb{E}\left[\mathrm{e}^{f(t)Z}\right]-1\right)dt\right).\end{aligned}$$

We also have

$$\begin{aligned}\log\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] &= \lambda\int_0^T\int_{\mathbb{R}}\left(\mathrm{e}^{yf(t)}-1\right)\nu(dy)dt\\ &= \lambda\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^T\int_{\mathbb{R}}y^n f^n(t)\nu(dy)dt\\ &= \lambda\sum_{n=1}^{\infty}\frac{1}{n!}\mathbb{E}[Z^n]\int_0^T f^n(t)dt,\end{aligned}$$

hence the *cumulant* of order $n \geqslant 1$ of $\displaystyle\int_0^T f(t)dY_t$, see Definition 21.1, is given by

$$\kappa_n = \lambda\mathbb{E}[Z^n]\int_0^T f^n(t)dt,$$

which recovers (20.12) and (20.13) by taking $f(t) := \mathbb{1}_{[0,T]}(t)$ when $n = 1, 2$.

20.4 Stochastic Differential Equations with Jumps

In the continuous asset price model, the returns of the riskless asset price process $(A_t)_{t\in\mathbb{R}_+}$ and of the risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ are modeled as

$$\frac{dA_t}{A_t} = rdt \quad\text{and}\quad \frac{dS_t}{S_t} = \mu dt + \sigma dB_t.$$

In this section we are interested in using jump processes in order to model an asset price process $(S_t)_{t\in\mathbb{R}_+}$.

i) Constant market return $\eta > -1$.

   In the case of discontinuous asset prices, let us start with the simplest example of a constant market return η written as

$$\eta := \frac{S_t - S_{t^-}}{S_{t^-}}, \qquad (20.28)$$

   assuming the presence of a jump at time $t > 0$, *i.e.*, $\Delta N_t = 1$. Using the identity $\Delta S_t = S_t - S_{t^-}$, Relation (20.28) rewrites as

$$\eta\Delta N_t = \frac{S_t - S_{t^-}}{S_{t^-}} = \frac{\Delta S_t}{S_{t^-}}, \tag{20.29}$$

or

$$dS_t = \eta S_{t^-} dN_t, \tag{20.30}$$

which is a stochastic differential equation with respect to the standard Poisson process, with constant volatility $\eta \in \mathbb{R}$. Note that the left limit S_{t^-} in (20.30) occurs naturally from the definition (20.29) of market returns when dividing by the previous index value S_{t^-}.

In the presence of a jump at time t, *i.e.* when $dN_t = 1$, the equation (20.29) also reads

$$S_t = (1+\eta)S_{t^-}, \qquad dN_t = 1,$$

which can be applied by induction at the successive jump times $T_1, T_2, \ldots, T_{N_t}$ until time t, to derive the solution

$$S_t = S_0(1+\eta)^{N_t}, \qquad t \geqslant 0,$$

of (20.30).

The use of the left limit S_{t^-} turns out to be necessary when computing pathwise solutions by solving for S_t from S_{t^-}.

ii) Time-dependent market returns $\eta_t > -1$, $t \geqslant 0$.

Next, consider the case where η_t is time-dependent, *i.e.*,

$$dS_t = \eta_t S_{t^-} dN_t. \tag{20.31}$$

At each jump time T_k, Relation (20.31) reads

$$dS_{T_k} = S_{T_k} - S_{T_k^-} = \eta_{T_k} S_{T_k^-},$$

i.e.,

$$S_{T_k} = (1+\eta_{T_k})S_{T_k^-},$$

and repeating this argument for all $k = 1, 2, \ldots, N_t$ yields the product solution

$$\begin{aligned} S_t &= S_0 \prod_{k=1}^{N_t} (1+\eta_{T_k}) \\ &= S_0 \prod_{\substack{\Delta N_s = 1 \\ 0 \leqslant s \leqslant t}} (1+\eta_s) \\ &= S_0 \prod_{0 \leqslant s \leqslant t} (1+\eta_s \Delta N_s), \quad t \geqslant 0. \end{aligned}$$

By a similar argument, we obtain the following proposition.

Proposition 20.16. *The stochastic differential equation with jumps*

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt), \tag{20.32}$$

admits the solution

$$\boxed{S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \int_0^t \eta_s ds\right) \prod_{k=1}^{N_t} (1 + \eta_{T_k}), \qquad t \geqslant 0.}$$

Note that the equations

$$dS_t = \mu_t S_{t^-} dt + \eta_t S_{t^-}(dN_t - \lambda dt)$$

and

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt)$$

are equivalent because $S_{t^-} dt = S_t dt$ as the set $\{T_k\}_{k \geqslant 1}$ of jump times has zero measure of length.

A random simulation of the numerical solution of the above equation (20.32) is given in Figure 20.14 for $\eta = 1.29$ and constant $\mu = \mu_t$, $t \geqslant 0$.

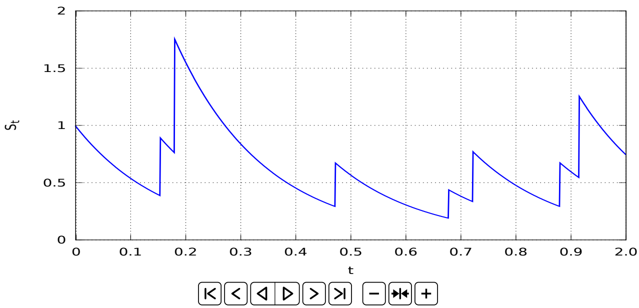

Fig. 20.14: Geometric Poisson process.*

The above simulation can be compared to the real sales ranking data of Figure 20.15.

* The animation works in Acrobat Reader on the entire pdf file.

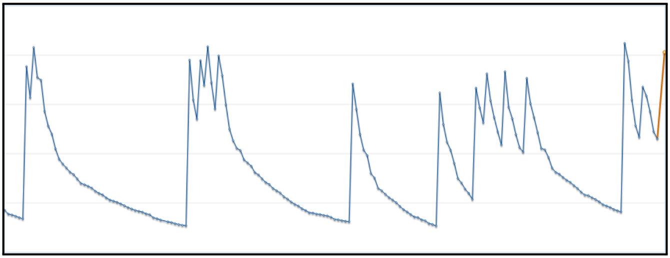

Fig. 20.15: Ranking data.

Next, consider the equation

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda \mathbb{E}[Z]dt)$$

driven by the compensated compound Poisson process $(Y_t - \lambda \mathbb{E}[Z]t)_{t\in\mathbb{R}_+}$, also written as

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(Z_{N_t} dN_t - \lambda \mathbb{E}[Z]dt),$$

with solution

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \mathbb{E}[Z] \int_0^t \eta_s ds\right) \prod_{k=1}^{N_t} (1 + \eta_{T_k} Z_k) \qquad t \geqslant 0. \quad (20.33)$$

A random simulation of the geometric compound Poisson process (20.33) is given in Figure 20.16.

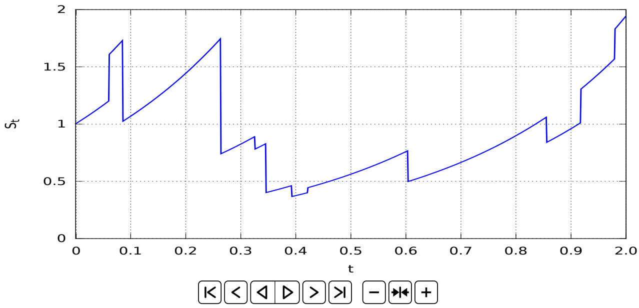

Fig. 20.16: Geometric compound Poisson process.*

In the case of a jump-diffusion stochastic differential equation of the form

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda \mathbb{E}[Z]dt) + \sigma_t S_t dB_t,$$

we get

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \mathbb{E}[Z] \int_0^t \eta_s ds + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t |\sigma_s|^2 ds\right)$$
$$\times \prod_{k=1}^{N_t} (1 + \eta_{T_k} Z_k), \qquad t \geqslant 0.$$

A random simulation of the geometric Brownian motion with compound Poisson jumps is given in Figure 20.17.

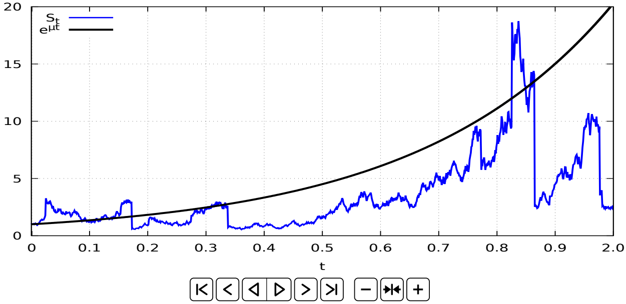

Fig. 20.17: Geometric Brownian motion with compound Poisson jumps.*

By rewriting S_t as

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \eta_s (dY_s - \lambda \mathbb{E}[Z]ds) + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t |\sigma_s|^2 ds\right)$$
$$\times \prod_{k=1}^{N_t} \left((1 + \eta_{T_k} Z_k)\, \mathrm{e}^{-\eta_{T_k} Z_k}\right),$$

$t \geqslant 0$, one can extend this jump model to processes with an infinite number of jumps on any finite time interval, cf. Cont and Tankov (2004). The next Figure 20.18 shows a number of downward and upward jumps occurring in the SMRT historical share price data, with a typical geometric Brownian behavior in between jumps.

* The animation works in Acrobat Reader on the entire pdf file.

* The animation works in Acrobat Reader on the entire pdf file.

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https://personal.ntu.edu.sg/nprivault/indext.html

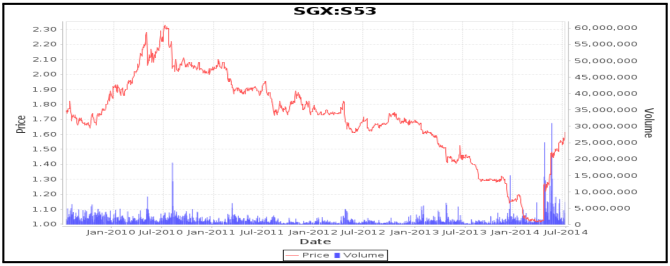

Fig. 20.18: SMRT Share price.

20.5 Girsanov Theorem for Jump Processes

Recall that in its simplest form, cf. Section 7.2, the Girsanov Theorem 7.3 for Brownian motion states the following.

Let $\mu \in \mathbb{R}$. Under the probability measure $\widetilde{\mathbb{P}}_{-\mu}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{-\mu}}{d\mathbb{P}} := e^{-\mu B_T - \mu^2 T/2},$$

the random variable $B_T + \mu T$ has the centered Gaussian distribution $\mathcal{N}(0, T)$.

This fact follows from the calculation

$$\begin{aligned}
\widetilde{\mathbb{E}}_{-\mu}[f(B_T + \mu T)] &= \mathbb{E}[f(B_T + \mu T)\, e^{-\mu B_T - \mu^2 T/2}] \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x + \mu T)\, e^{-\mu x - \mu^2 T/2}\, e^{-x^2/(2T)} dx \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(x + \mu T)\, e^{-(x+\mu T)^2/(2T)} dx \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(y)\, e^{-y^2/(2T)} dy \\
&= \mathbb{E}[f(B_T)], \qquad (20.34)
\end{aligned}$$

for any bounded measurable function f on $\mathbb{R}$, which shows that $B_T + \mu T$ is a centered Gaussian random variable under $\widetilde{\mathbb{P}}_{-\mu}$.

More generally, the Girsanov Theorem states that $(B_t + \mu t)_{t \in [0,T]}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{-\mu}$.

When Brownian motion is replaced with a standard Poisson process $(N_t)_{t\in[0,T]}$, a spatial shift of the type

$$B_t \longmapsto B_t + \mu t$$

can no longer be used because $N_t + \mu t$ cannot be a Poisson process, whatever the change of probability applied, since by construction, the paths of the standard Poisson process has jumps of unit size and remain constant between jump times.

The correct way to extend the Girsanov Theorem to the Poisson case is to replace the space shift with a shift of the intensity of the Poisson process as in the following statement.

Proposition 20.17. *Consider a random variable N_T having the Poisson distribution $\mathcal{P}(\lambda T)$ with parameter λT under $\mathbb{P}_\lambda$. Under the probability measure $\widetilde{\mathbb{P}}_{\tilde\lambda}$ defined by the Radon-Nikodym density*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\tilde\lambda}}{\mathrm{d}\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde\lambda-\lambda)T}\left(\frac{\widetilde\lambda}{\lambda}\right)^{N_T},$$

the random variable N_T has a Poisson distribution with intensity $\widetilde\lambda T$. As a consequence, the compensated process $\big(N_t - \widetilde\lambda t\big)_{t\in[0,T]}$ is a martingale under $\widetilde{\mathbb{P}}_{\tilde\lambda}$.

Proof. This follows from the relation

$$\begin{aligned}\widetilde{\mathbb{P}}_{\tilde\lambda}(N_T = k) &= \mathrm{e}^{-(\tilde\lambda-\lambda)T}\left(\frac{\widetilde\lambda}{\lambda}\right)^{k}\mathbb{P}_\lambda(N_T = k)\\ &= \mathrm{e}^{-(\tilde\lambda-\lambda)T}\left(\frac{\widetilde\lambda}{\lambda}\right)^{k}\mathrm{e}^{-\lambda T}\frac{(\lambda T)^k}{k!}\\ &= \mathrm{e}^{-\tilde\lambda T}\frac{(\widetilde\lambda T)^k}{k!}, \qquad k \geqslant 0.\end{aligned}$$

□

Assume now that $(N_t)_{t\in[0,T]}$ is a standard Poisson process with intensity λ under a probability measure $\mathbb{P}_\lambda$. In order to extend (20.34) to the Poisson case we can replace the space shift with a *time contraction* (or dilation)

$$N_t \longmapsto N_{(1+c)t}$$

by a factor $1 + c$, where

$$c := -1 + \frac{\widetilde\lambda}{\lambda} > -1,$$

or $\widetilde\lambda = (1 + c)\lambda$. We note that

$$\begin{aligned}\mathbb{P}_\lambda\big(N_{(1+c)T}=k\big) &= \frac{(\lambda(1+c)T)^k}{k!}\,\mathrm{e}^{-\lambda(1+c)T}\\ &= (1+c)^k\,\mathrm{e}^{-\lambda cT}\mathbb{P}_\lambda(N_T=k)\\ &= \widetilde{\mathbb{P}}_{\tilde\lambda}(N_T=k), \qquad k\geqslant 0,\end{aligned}$$

hence

$$\frac{d\widetilde{\mathbb{P}}_{\tilde\lambda}}{d\mathbb{P}_\lambda} := (1+c)^{N_T}\,\mathrm{e}^{-\lambda cT},$$

and by analogy with (20.34) we have

$$\begin{aligned}\mathbb{E}_\lambda\big[f(N_{(1+c)T})\big] &= \sum_{k\geqslant 0} f(k)\mathbb{P}_\lambda\big(N_{(1+c)T}=k\big) \\ &= \mathrm{e}^{-\lambda cT}\sum_{k\geqslant 0} f(k)(1+c)^k\mathbb{P}_\lambda(N_T=k)\\ &= \mathrm{e}^{-\lambda cT}\mathbb{E}_\lambda\big[f(N_T)(1+c)^{N_T}\big]\\ &= \mathbb{E}_\lambda\left[f(N_T)\frac{d\widetilde{\mathbb{P}}_{\tilde\lambda}}{d\mathbb{P}_\lambda}\right]\\ &= \widetilde{\mathbb{E}}_{\tilde\lambda}[f(N_T)],\end{aligned} \tag{20.35}$$

for any bounded function f on $\mathbb{N}$. In other words, taking $f(x) := \mathbb{1}_{\{x\leqslant n\}}$, we have

$$\mathbb{P}_\lambda\big(N_{(1+c)T}\leqslant n\big) = \widetilde{\mathbb{P}}_{\tilde\lambda}(N_T\leqslant n), \qquad n\geqslant 0,$$

or

$$\widetilde{\mathbb{P}}_{\tilde\lambda}(N_{T/(1+c)}\leqslant n) = \mathbb{P}_\lambda\big(N_T\leqslant n\big), \qquad n\geqslant 0.$$

As a consequence, we have the following proposition.

Proposition 20.18. *Let $\lambda, \widetilde\lambda > 0$, and set*

$$c := -1 + \frac{\widetilde\lambda}{\lambda} > -1.$$

The process $\big(N_{t/(1+c)}\big)_{t\in[0,T]}$ is a standard Poisson process with intensity λ under the probability measure $\widetilde{\mathbb{P}}_{\tilde\lambda}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde\lambda}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde\lambda-\lambda)T}\left(\frac{\widetilde\lambda}{\lambda}\right)^{N_T} = \mathrm{e}^{-c\lambda T}(1+c)^{N_T}.$$

In particular, the compensated Poisson processes

$$N_{t/(1+c)} - \lambda t \quad \textit{and} \quad N_t - \widetilde\lambda t, \qquad 0\leqslant t\leqslant T,$$

are martingales under $\widetilde{\mathbb{P}}_{\tilde\lambda}$.

Proof. As in (20.35), we have

$$\mathbb{E}_\lambda[f(N_T)] = \widetilde{\mathbb{E}}_{\tilde{\lambda}}\big[f\big(N_{T/(1+c)}\big)\big],$$

i.e., under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ the distribution of $N_{T/(1+c)}$ is that of a standard Poisson random variable with parameter λT. Since $\big(N_{t/(1+c)}\big)_{t\in[0,T]}$ has independent increments, $\big(N_{t/(1+c)}\big)_{t\in[0,T]}$ is a standard Poisson process with intensity λ under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, and the compensated process $(N_{t/(1+c)} - \lambda t)_{t\in[0,T]}$ is a martingale under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ by (7.2). Similarly, the compensated process

$$(N_t - (1+c)\lambda t)_{t\in[0,T]} = \big(N_t - \tilde{\lambda} t\big)_{t\in[0,T]}$$

has independent increments and is a martingale under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$. □

We also have

$$N_{t/(1+c)} = \sum_{n\geqslant 1} \mathbb{1}_{[T_n,\infty)}\left(\frac{t}{1+c}\right) = \sum_{n\geqslant 1} \mathbb{1}_{[(1+c)T_n,\infty)}(t), \quad t \geqslant 0,$$

which shows that the jump times $((1+c)T_n)_{n\geqslant 1}$ of $\big(N_{t/(1+c)}\big)_{t\in[0,T]}$ are distributed under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ as the jump times of a Poisson process with intensity λ.

The next R code shows that the compensated Poisson process $(N_{t/(1+c)} - \lambda t)_{t\in[0,T]}$, remains a martingale after the Poisson process interjump times $(\tau_k)_{k\geqslant 1}$ have been generated using exponential random variables with parameter $\tilde{\lambda} > 0$.

```
lambda = 0.5;lambdat=2;c=-1+lambdat/lambda;n = 20;Z<-cumsum(c(0,rep(1,n)))
for (k in 1:n){tau_k <- rexp(n,rate=lambdat); Tn <- cumsum(tau_k)}
N <- function(t) {return(stepfun(Tn,Z)(t))};t <- seq(0,10,0.01)
plot(t,N(t/(1+c))-lambda*t,xlim = c(0,10),ylim =
      c(-2,2),xlab="t",ylab="Nt-t",type="l",lwd=2,col="blue",main="", xaxs = "i", yaxs = "i",
      xaxs = "i", yaxs = "i");abline(h = 0, col="black", lwd =2)
points(Tn*(1+c),N(Tn)-lambda*Tn*(1+c),pch=1,cex=0.8,col="blue",lwd=2)
```

When $\mu \neq r$, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ written as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = (\mu - r)dt + \sigma(dN_t - \lambda dt) \tag{20.36}$$

is not a martingale under $\mathbb{P}_\lambda$. However, we can rewrite (20.36) as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma\left(dN_t - \left(\lambda - \frac{\mu - r}{\sigma}\right)dt\right)$$

and letting

$$\widetilde{\lambda} := \lambda - \frac{\mu - r}{\sigma} = (1 + c)\lambda$$

with

$$c := -\frac{\mu - r}{\sigma\lambda},$$

we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma\big(dN_t - \widetilde{\lambda}dt\big)$$

hence the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by the Radon-Nikodym density

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{\mathrm{d}\mathbb{P}_{\lambda}} := \mathrm{e}^{-\lambda cT}(1+c)^{N_T} = \mathrm{e}^{(\mu-r)/\sigma}\left(1 - \frac{\mu - r}{\sigma\lambda}\right)^{N_T}.$$

We note that if

$$\mu - r \leqslant \sigma\lambda$$

then the risk-neutral probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ exists and is unique, therefore by Theorems 5.7 and 5.11 the market is without arbitrage and complete. If $\mu - r > \sigma\lambda$ then the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is always increasing, and arbitrage becomes possible by borrowing from the savings account and investing on the risky underlying asset.

Girsanov Theorem for compound Poisson processes

In the case of compound Poisson processes, the Girsanov Theorem can be extended to variations in jump sizes in addition to time variations, and we have the following more general result.

Theorem 20.19. *Let $(Y_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\nu(dx)$. Consider another intensity parameter $\widetilde{\lambda} > 0$ and jump size distribution $\widetilde{\nu}(dx)$, and let*

$$\psi(x) := \frac{\widetilde{\lambda}}{\lambda}\frac{\widetilde{\nu}(dx)}{\nu(dx)} - 1, \qquad x \in \mathbb{R}. \tag{20.37}$$

Then,

under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ *defined by the Radon-Nikodym density*

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}}{d\widetilde{\mathbb{P}}_{\lambda,\nu}} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\prod_{k=1}^{N_T}(1+\psi(Z_k)),$$

the process

$$Y_t := \sum_{k=1}^{N_t} Z_k, \qquad t \geqslant 0,$$

is a compound Poisson process with

- *modified intensity* $\widetilde{\lambda} > 0$, *and*
- *modified jump size distribution* $\widetilde{\nu}(dx)$.

Proof. For any bounded measurable function f on $\mathbb{R}$, we extend (20.35) to the following change of variable

$$\begin{aligned}
\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[f(Y_T)] &= \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\mathbb{E}_{\lambda,\nu}\left[f(Y_T)\prod_{i=1}^{N_T}(1+\psi(Z_i))\right] \\
&= \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\sum_{k\geqslant 0}\mathbb{E}_{\lambda,\nu}\left[f\left(\sum_{i=1}^{k}Z_i\right)\prod_{i=1}^{k}(1+\psi(Z_i))\ \Big|\ N_T=k\right]\mathbb{P}_{\lambda}(N_T=k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\lambda T)^k}{k!}\mathbb{E}_{\lambda,\nu}\left[f\left(\sum_{i=1}^{k}Z_i\right)\prod_{i=1}^{k}(1+\psi(Z_i))\right] \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\lambda T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\prod_{i=1}^{k}(1+\psi(z_i))\nu(dz_1)\cdots\nu(dz_k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\widetilde{\lambda}T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\left(\prod_{i=1}^{k}\frac{\widetilde{\nu}(dz_i)}{\nu(dz_i)}\right)\nu(dz_1)\cdots\nu(dz_k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\widetilde{\lambda}T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\widetilde{\nu}(dz_1)\cdots\widetilde{\nu}(dz_k).
\end{aligned}$$

This shows that under $\mathbb{P}_{\tilde{\lambda},\tilde{\nu}}$, Y_T has the distribution of a compound Poisson process with intensity $\widetilde{\lambda}$ and jump size distribution $\widetilde{\nu}$. We refer to Proposition 9.6 of Cont and Tankov (2004) for the independence of increments of $(Y_t)_{t\in\mathbb{R}_+}$ under $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$. $\square$

Example. In case $\nu \simeq \mathcal{N}(\alpha,\sigma^2)$ and $\widetilde{\nu} \simeq \mathcal{N}(\beta,\eta^2)$, we have

$$\nu(dx) = \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\alpha)^2\right), \quad \widetilde{\nu}(dx) = \frac{dx}{\sqrt{2\pi\eta^2}} \exp\left(-\frac{1}{2\eta^2}(x-\beta)^2\right),$$

$x \in \mathbb{R}$, hence

$$\frac{\widetilde{\nu}(dx)}{\nu(dx)} = \frac{\eta}{\sigma} \exp\left(\frac{1}{2\eta^2}(x-\beta)^2 - \frac{1}{2\sigma^2}(x-\alpha)^2\right),$$

and $\psi(x)$ in (20.37) is given by

$$1+\psi(x) = \frac{\widetilde{\lambda}}{\lambda}\frac{\widetilde{\nu}(dx)}{\nu(dx)} = \frac{\widetilde{\lambda}\eta}{\lambda\sigma} \exp\left(\frac{1}{2\eta^2}(x-\beta)^2 - \frac{1}{2\sigma^2}(x-\alpha)^2\right), \quad x \in \mathbb{R}.$$

Note that the compound Poisson process with intensity $\widetilde{\lambda} > 0$ and jump size distribution $\widetilde{\nu}$ can be built as

$$X_t := \sum_{k=1}^{N_{\tilde{\lambda}t/\lambda}} h(Z_k),$$

provided that $\widetilde{\nu}$ is the *pushforward* measure of ν by the function $h : \mathbb{R} \to \mathbb{R}$, *i.e.*,

$$\mathbb{P}(h(Z_k) \in A) = \mathbb{P}(Z_k \in h^{-1}(A)) = \nu(h^{-1}(A)) = \widetilde{\nu}(A),$$

for all (measurable) subsets A of $\mathbb{R}$. As a consequence of Theorem 20.19 we have the following proposition.

Proposition 20.20. *The compensated process*

$$Y_t - \widetilde{\lambda} t \mathbb{E}_{\tilde{\nu}}[Z]$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}}{d\widetilde{\mathbb{P}}_{\lambda,\nu}} = \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \prod_{k=1}^{N_T} (1+\psi(Z_k)).$$

Finally, the Girsanov Theorem can be extended to the linear combination of a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ independent of $(B_t)_{t\in\mathbb{R}_+}$, as in the following result which is a particular case of Theorem 33.2 of Sato (1999).

Theorem 20.21. *Let $(Y_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\nu(dx)$. Consider another jump size distribution $\widetilde{\nu}(dx)$ and intensity parameter $\widetilde{\lambda} > 0$, and let*

$$\psi(x) := \frac{\widetilde{\lambda}}{\lambda}\frac{d\widetilde{\nu}}{d\nu}(x) - 1, \qquad x \in \mathbb{R},$$

and let $(u_t)_{t\in\mathbb{R}_+}$ *be a bounded adapted process. Then, the process*

$$\left(B_t + \int_0^t u_s ds + Y_t - \widetilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]t\right)_{t\in\mathbb{R}_+}$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ *defined by the Radon-Nikodym density*

$$\frac{d\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}}{d\widetilde{\mathbb{P}}_{\lambda,\nu}} = \exp\left(-(\widetilde{\lambda}-\lambda)T - \int_0^T u_s dB_s - \frac{1}{2}\int_0^T |u_s|^2 ds\right)\prod_{k=1}^{N_T}(1+\psi(Z_k)). \tag{20.38}$$

As a consequence of Theorem 20.21, if

$$B_t + \int_0^t v_s ds + Y_t \tag{20.39}$$

is not a martingale under $\widetilde{\mathbb{P}}_{\lambda,\nu}$, it will become a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ provided that $u, \tilde{\lambda}$ and $\widetilde{\nu}$ are chosen in such a way that

$$v_s = u_s - \widetilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z], \qquad s \in \mathbb{R}, \tag{20.40}$$

in which case (20.39) can be rewritten into the martingale decomposition

$$dB_t + u_t dt + dY_t - \widetilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt,$$

in which both $\left(B_t + \int_0^t u_s ds\right)_{t\in\mathbb{R}_+}$ and $\left(Y_t - \widetilde{\lambda}t\mathbb{E}_{\tilde{\nu}}[Z]\right)_{t\in\mathbb{R}_+}$ are martingales under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$

The following remarks will be of importance for arbitrage-free pricing in jump models in Chapter 21.

a) When $\widetilde{\lambda} = \lambda = 0$, Theorem 20.21 coincides with the usual Girsanov Theorem for Brownian motion, in which case (20.40) admits only one solution given by $u = v$ and there is uniqueness of $\widetilde{\mathbb{P}}_{u,0,0}$.
b) Uniqueness also occurs when $u = 0$ in the absence of Brownian motion, and with Poisson jumps of fixed size a (*i.e.*, $\widetilde{\nu}(dx) = \nu(dx) = \delta_a(dx)$) since in this case (20.40) also admits only one solution $\widetilde{\lambda} = v$ and there is uniqueness of $\widetilde{\mathbb{P}}_{0,\tilde{\lambda},\delta_a}$.

When $\mu \neq r$, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ defined by

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = (\mu - r)dt + \sigma dB_t + \eta(dY_t - \lambda t\mathbb{E}_\nu[Z])$$

is not martingale under $\mathbb{P}_{\lambda,\nu}$, however we can rewrite the equation as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma(udt + dB_t) + \eta\left(dY_t - \left(\frac{u\sigma}{\eta} + \lambda \mathbb{E}_\nu[Z] - \frac{\mu - r}{\eta}\right)dt\right)$$

and choosing u, $\widetilde{\nu}$, and $\widetilde{\lambda}$ such that

$$\widetilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z] = \frac{u\sigma}{\eta} + \lambda \mathbb{E}_\nu[Z] - \frac{\mu - r}{\eta}, \tag{20.41}$$

we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma(udt + dB_t) + \eta\big(dY_t - \widetilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt\big).$$

Hence the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$, and the market is without arbitrage by Theorem 5.7 and the existence of a risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. However, the market is not complete due to the non uniqueness of solutions $\big(u,\widetilde{\nu},\widetilde{\lambda}\big)$ to (20.41), and Theorem 5.11 does not apply in this situation.

Exercises

Exercise 20.1 Analysis of user login activity to the DBX digibank app showed that the times elapsed between two logons are independent and exponentially distributed with mean $1/\lambda$. Find the CDF of the time $T - T_{N_T}$ elapsed since the last logon before time T, given that the user has logged on at least once.

Hint: The number of logins until time $t > 0$ can be modeled by a standard Poisson process $(N_t)_{t\in[0,T]}$ with intensity λ.

Exercise 20.2 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$, started at $N_0 = 0$.

a) Solve the stochastic differential equation

$$dS_t = \eta S_{t^-}dN_t - \eta\lambda S_t dt = \eta S_{t^-}(dN_t - \lambda dt).$$

b) Using the first Poisson jump time T_1, solve the stochastic differential equation

$$dS_t = -\lambda\eta S_t dt + dN_t, \qquad t \in (0, T_2).$$

Exercise 20.3 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$, and the stochastic differential equation

$$dX_t = \alpha X_t dt + \sigma dB_t + \eta dN_t.$$

a) Write down the Itô formula for $df(X_t)$.

b) Write down the Itô formula for $d(X_t^2)$.

Exercise 20.4 Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dB_t + \eta S_{t^-} dY_t$, *i.e.*

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s + \eta \int_0^t S_{s^-} dY_s, \qquad t \geqslant 0, \qquad (20.42)$$

where $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma \geqslant 0$, $\eta \geqslant 0$ are constants, and $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process with intensity $\lambda \geqslant 0$ and *i.i.d.* jump sizes Z_k, $k \geqslant 1$.

a) Write a differential equation satisfied by $u(t) := \mathbb{E}[S_t]$, $t \geqslant 0$.
   Hint: Use the smoothing lemma Proposition 6.9.
b) Find the value of $\mathbb{E}[S_t]$, $t \geqslant 0$, in terms of S_0, μ, η, λ and $\mathbb{E}[Z]$.

Exercise 20.5 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = \alpha X_t dt + \sigma dN_t$ over the time intervals $[0, T_1)$, $[T_1, T_2)$, $[T_2, T_3)$, $[T_3, T_4)$, where $X_0 = 1$.
b) Write a differential equation for $f(t) := \mathbb{E}[X_t]$, and solve it for $t \in \mathbb{R}_+$.

Exercise 20.6 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = \sigma X_{t^-} dN_t$ for $(X_t)_{t\in\mathbb{R}_+}$, where $\sigma > 0$ and $X_0 = 1$.
b) Show that the solution $(S_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dS_t = r dt + \sigma S_{t^-} dN_t,$$

   is given by $S_t = S_0 X_t + r X_t \int_0^t X_s^{-1} ds$.
c) Compute $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t / X_s]$, $0 \leqslant s \leqslant t$.
d) Compute $\mathbb{E}[S_t]$, $t \geqslant 0$.

Exercise 20.7 Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with intensity $\lambda > 0$, started at $N_0 = 0$.

a) Is the process $t \mapsto N_t - 2\lambda t$ a *sub*martingale, a martingale, or a *super*martingale?
b) Let $r > 0$. Solve the stochastic differential equation

$$dS_t = r S_t dt + \sigma S_{t^-} (dN_t - \lambda dt).$$

c) Is the process $t \mapsto S_t$ of Question (b) a *sub*martingale, a martingale, or a *super*martingale?

d) Compute the price at time 0 of the European call option with strike price $K = S_0 \mathrm{e}^{(r-\lambda\sigma)T}$, where $\sigma > 0$.

Exercise 20.8 Affine stochastic differential equation with jumps. Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = adN_t + \sigma X_{t^-} dN_t$, where $\sigma > 0$, and $a \in \mathbb{R}$.
b) Compute $\mathbb{E}[X_t]$ for $t \in \mathbb{R}_+$.

Exercise 20.9 Consider the compound Poisson process $Y_t := \sum_{k=1}^{N_t} Z_k$, where $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity $\lambda > 0$, and $(Z_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of $\mathcal{N}(0,1)$ Gaussian random variables. Solve the stochastic differential equation

$$dS_t = rS_t dt + \eta S_{t^-} dY_t,$$

where $\eta, r \in \mathbb{R}$.

Exercise 20.10 Show, by direct computation or using the moment generating function (20.10), that the variance of the compound Poisson process Y_t with intensity $\lambda > 0$ satisfies

$$\mathrm{Var}\,[Y_t] = \lambda t \mathbb{E}\left[|Z|^2\right] = \lambda t \int_{-\infty}^{\infty} x^2 \nu(dx).$$

Exercise 20.11 Consider an exponential compound Poisson process of the form

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B_t + Y_t}, \qquad t \geqslant 0,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process of the form (20.8).

a) Derive the stochastic differential equation with jumps satisfied by $(S_t)_{t\in\mathbb{R}_+}$.
b) Let $r > 0$. Find a family $\left(\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}\right)$ of probability measures under which the discounted asset price $\mathrm{e}^{-rt} S_t$ is a martingale.

Exercise 20.12 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + Z_{N_t} S_{t^-} dN_t, \tag{20.43}$$

where $(Z_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of random variables of the form

$$Z_k = \mathrm{e}^{X_k} - 1, \quad \text{where} \quad X_k \simeq \mathcal{N}(0, \sigma^2), \quad k \geqslant 1.$$

a) Solve the equation (20.43).
b) We assume that μ and the risk-free interest rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$. What relation does this impose on μ and r?
c) Under the relation of Question (b), compute the price at time t of the European call option on S_T with strike price κ and maturity $T > 0$, using a series expansion of Black-Scholes functions.

Exercise 20.13 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be the mean-reverting process defined by the stochastic differential equation

$$dS_t = -\alpha S_t dt + \sigma(dN_t - \beta dt), \tag{20.44}$$

where $S_0 > 0$ and $\alpha, \beta > 0$.

a) Solve the equation (20.44) for S_t.
b) Compute $f(t) := \mathbb{E}[S_t]$ for all $t \in \mathbb{R}_+$.
c) Under which condition on α, β, σ and λ does the process S_t become a *sub*martingale?
d) Propose a method for the calculation of expectations of the form $\mathbb{E}[\phi(S_T)]$ where ϕ is a payoff function.

Exercise 20.14 Let $(N_t)_{t\in[0,T]}$ be a standard Poisson process started at $N_0 = 0$, with intensity $\lambda > 0$ under the probability measure $\mathbb{P}_\lambda$, and consider the compound Poisson process $(Y_t)_{t\in[0,T]}$ with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$.

a) Under the probability measure $\mathbb{P}_\lambda$, the process $t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z])$ is a:

submartingale |	martingale |	supermartingale |

b) Consider the process $(S_t)_{t\in[0,T]}$ given by

$$dS_t = \mu S_t dt + \sigma S_{t^-} dY_t.$$

Find $\widetilde{\lambda}$ such that the discounted process $(\widetilde{S}_t)_{t\in[0,T]} := (e^{-rt}S_t)_{t\in[0,T]}$ is a martingale under the probability measure $\mathbb{P}_{\tilde{\lambda}}$ defined by the Radon-Nikodym density

$$\frac{d\mathbb{P}_{\tilde{\lambda}}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_T}.$$

with respect to $\mathbb{P}_\lambda$.
c) Price the forward contract with payoff $S_T - \kappa$.

Exercise 20.15 Consider $(Y_t)_{t\in\mathbb{R}_+}$ a compound Poisson process written as

$$Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,$$

where $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ and $(Z_k)_{k\geqslant 1}$ is an *i.i.d* family of random variables with probability distribution $\nu(dx)$ on $\mathbb{R}$, under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + S_{t^-} dY_t. \tag{20.45}$$

a) Solve the equation (20.45).
b) We assume that μ, $\nu(dx)$ and the risk-free interest rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$. What relation does this impose on μ, $\nu(dx)$ and r?
c) Under the relation of Question (b), compute the price at time t of the European call option on S_T with strike price κ and maturity $T > 0$, using a series expansion of integrals.

Exercise 20.16 Consider a standard Poisson process $(N_t)_{t\in[0,T]}$ with intensity $\lambda > 0$ and a standard Brownian motion $(B_t)_{t\in[0,T]}$ independent of $(N_t)_{t\in[0,T]}$ under the probability measure $\mathbb{P}_\lambda$. Let also $(Y_t)_{t\in[0,T]}$ be a compound Poisson process with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$ under $\mathbb{P}_\lambda$, and consider the jump process $(S_t)_{t\in[0,T]}$ solution of

$$dS_t = rS_t dt + \sigma S_t dB_t + \eta S_{t^-} \big(dY_t - \widetilde{\lambda}\mathbb{E}[Z_1]dt\big).$$

with $r, \sigma, \eta, \lambda, \widetilde{\lambda} > 0$.

a) Assume that $\widetilde{\lambda} = \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	martingale |	supermartingale |

b) Assume $\widetilde{\lambda} > \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	martingale |	supermartingale |

c) Assume $\widetilde{\lambda} < \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

submartingale |	martingale |	supermartingale |

d) Consider the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by its Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{d\mathbb{P}_{\lambda}} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_T}.$$

with respect to $\mathbb{P}_{\lambda}$. Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, the discounted price process $(\mathrm{e}^{-rt}S_t)_{t\in[0,T]}$ is a:

submartingale		martingale		supermartingale	

Exercise 20.17 Let $(N_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ be a standard Poisson process with intensity $\lambda > 0$ and an independent standard Brownian motion under a probability measure $\mathbb{P}$. Let also $(Y_t)_{t\in[0,T]}$ be a compound Poisson process with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$ under $\mathbb{P}$, and let $\mu, \sigma > 0$. Let also $\widetilde{\mathbb{P}}$ denote the probability measure defined by the density

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := e^{-(\widetilde{\lambda}-\lambda)T-\mu B_T/\sigma-\mu^2T/(2\sigma^2)}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_T}$$

with respect to $\mathbb{P}$, where $\widetilde{\lambda} > \lambda > 0$. Which of the following processes are martingales under $\widetilde{\mathbb{P}}$?

a) B_t,
b) $\mu t/\sigma + B_t$,
c) $\mu t/\sigma - B_t$,
d) $-\mu t/\sigma + B_t$,
e) $Y_t - \widetilde{\lambda}\mathbb{E}[Z_1]t$,
f) $Y_t - \lambda\mathbb{E}[Z_1]t$,
g) $\mu t/\sigma + B_t + Y_t - \widetilde{\lambda}\mathbb{E}[Z_1]t$,
h) $\mu t/\sigma + B_t - (Y_t - \widetilde{\lambda}\mathbb{E}[Z_1]t)$,
i) $-\mu t/\sigma + B_t + Y_t - \widetilde{\lambda}\mathbb{E}[Z_1]t$,
j) $\mu t/\sigma + B_t + Y_t - \lambda\mathbb{E}[Z_1]t$.