Chapter 21
Pricing and Hedging in Jump Models

This chapter considers the pricing and hedging of financial derivatives using discontinuous processes that can model sharp movements in asset prices. Unlike in the case of continuous asset price modeling, the uniqueness of risk-neutral probability measures can be lost and, as a consequence, the computation of perfect replicating hedging strategies may not be possible in general.

21.1 Fitting the Distribution of Market Returns

The modeling of risky asset by stochastic processes with continuous paths, based on Brownian motions, suffers from several defects. First, the path continuity assumption does not seem reasonable in view of the possibility of sudden price variations (jumps) resulting of market crashes, gaps or opening jumps, see *e.g.* Chapter 1 of Cont and Tankov (2004). Secondly, the modeling of risky asset prices by Brownian motion relies on the use of the Gaussian distribution which tends to underestimate the probabilities of extreme events.

The following scripts allow us to fetch DJI and STI index data using the R package quantmod. The command diff(log(stock)) computes log-returns

$$d \log S_t \simeq \log S_{t+dt} - \log S_t = \log \frac{S_{t+dt}}{S_t}, \qquad t \geqslant 0,$$

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with $dt = 1/365$, which are modeled by the stochastic differential equation

$$d \log S_t = \sigma dB_t + r dt - \frac{\sigma^2}{2} dt$$

satisfied by geometric Brownian motion $S_t = S_0 \, \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \geqslant 0$.

```
install.packages("quantmod");library(quantmod)
getSymbols("^STI",from="1990-01-03",to="2015-01-03",src="yahoo");stock=Ad(`STI`);
getSymbols("^DJI",from="1990-01-03",to=Sys.Date(),src="yahoo");stock=Ad(`DJI`);
stock.rtn=diff(log(stock));returns <- as.vector(stock.rtn)
m=mean(returns,na.rm=TRUE);s=sd(returns,na.rm=TRUE);times=index(stock.rtn)
n = sum(is.na(returns))+sum(!is.na(returns));x=seq(1,n);y=rnorm(n,mean=m,sd=s)
plot(times,returns,pch=19,xaxs="i",cex=0.03,col="blue", ylab="", xlab="", main = '')
segments(x0 = times, x1 = times, y0 = 0, y1 = returns,col="blue")
points(times,y,pch=19,cex=0.3,col="red")
abline(h = m+3*s, col="black", lwd =1);abline(h = m, col="black", lwd =1);abline(h = m-3*s,
      col="black", lwd =1)
length(returns[abs(returns-m)>3*s])/length(stock.rtn)
length(y[abs(y-m)>3*s])/length(y);2*(1-pnorm(3*s,0,s))
```

The next Figures 21.1-21.6 illustrate the mismatch between the distributional properties of market log-returns *vs.* standardized Gaussian returns, which tend to underestimate the probabilities of extreme events. Note that when $X \simeq \mathcal{N}(0, \sigma^2)$, 99.73% of samples of X are falling within the interval $[-3\sigma, +3\sigma]$, *i.e.* $\mathbb{P}(|X| \leqslant 3\sigma) = 0.9973002$.

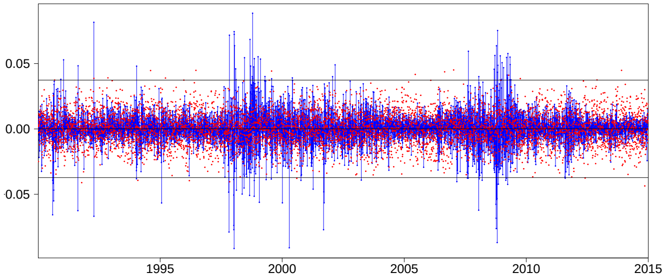

Fig. 21.1: Market returns *vs.* normalized Gaussian returns.

```
stock.ecdf=ecdf(as.vector(stock.rtn));x <- seq(-0.25, 0.25, length=100);px <- pnorm((x-m)/s)
plot(stock.ecdf, xlab = 'Sample Quantiles', lwd=3, col="blue",ylab = '', main = '')
lines(x, px, type="l", lty=2, lwd=3, col="red",xlab="x value",ylab="Probability", main="")
legend("topleft", legend=c("Empirical CDF", "Gaussian CDF"),col=c("blue", "red"), lty=1:2,
      cex=0.8)
```

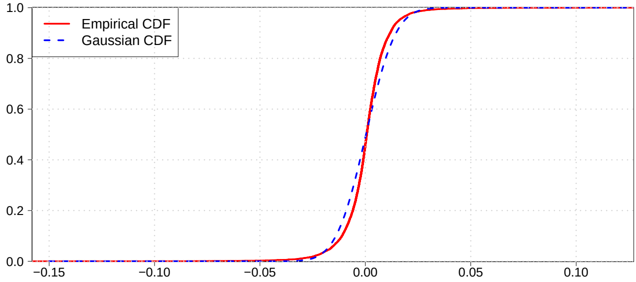

Fig. 21.2: Empirical *vs.* Gaussian CDF.

The following Quantile-Quantile graph is plotting the normalized empirical quantiles against the standard Gaussian quantiles, and is obtained with the `qqnorm(returns)` command.

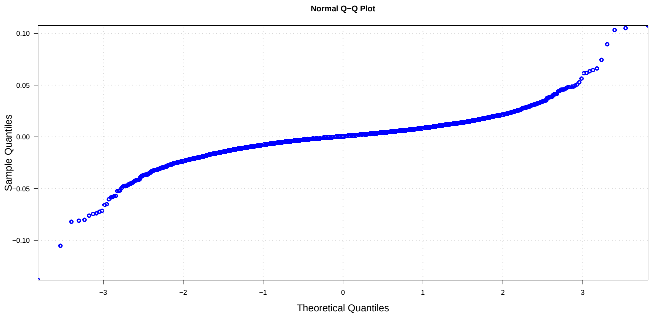

Fig. 21.3: Quantile-Quantile plot.

```
ks.test(y,"pnorm",mean=m,sd=s)
ks.test(returns,"pnorm",mean=m,sd=s)
```

The Kolmogorov-Smirnov test clearly rejects the null (normality) hypothesis of market returns.

```
One-sample Kolmogorov-Smirnov test

data: returns
D = 0.075577, p-value < 2.2e-16
```

```
alternative hypothesis: two-sided
```

This mismatch can be further illustrated by the empirical probability density plot in Figure 21.4, which is obtained from the following R code.

```
x <- seq(-0.1, 0.1, length=100);qx <- dnorm(x,mean=m,sd=s)
returns.dens=density(stock.rtn,na.rm=TRUE)
dev.new(width=10, height=5)
plot(returns.dens, xlim=c(-0.1,0.1),xlab = 'x', lwd=3, col="red",ylab = '', main = '',panel.first
      = abline(h = 0, col='grey', lwd =0.2), las=1, cex.axis=1.2, cex.lab=1.3)
lines(x, qx, type="l", lty=2, lwd=3, col="blue",xlab="x value",ylab="Density", main="")
legend("topleft", legend=c("Empirical density", "Gaussian density"),col=c("red", "blue"),
lty=1:2, cex=1.2)
```

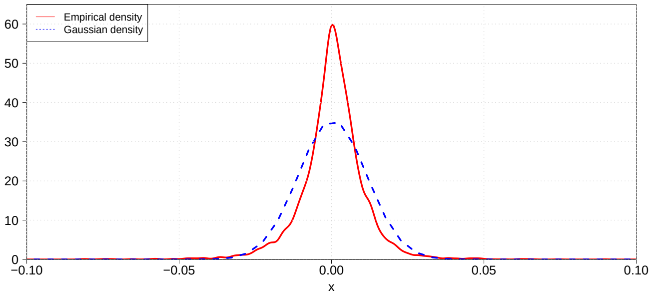

Fig. 21.4: Empirical density *vs.* normalized Gaussian density.

The next code and graph present a comparison of market prices to a calibrated lognormal distribution.

```
x <- seq(0, max(stock), length=100);qx <- dlnorm(x,mean=mean(log(stock)), sd=sd(log(stock)))
stock.dens=density(stock,na.rm=TRUE);dev.new(width=10, height=5)
plot(stock.dens, xlab = 'x', lwd=3, col="red",ylab = '', main = '',panel.first = abline(h = 0,
      col='grey', lwd =0.2), las=1, cex.axis=1, cex.lab=1, xaxs='i', yaxs='i')
lines(x, qx, type="l", lty=2, lwd=3, col="blue",xlab="x value",ylab="Density", main="")
legend("topright", legend=c("Empirical density", "Lognormal density"),col=c("red", "blue"),
      lty=1:2, cex=1.2)
```

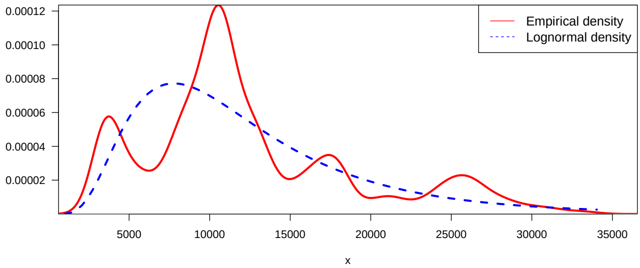

Fig. 21.5: Empirical density *vs.* normalized lognormal density.

Power tail distributions

We note that the empirical density has significantly higher kurtosis (leptokurtic distribution) and non zero skewness in comparison with the Gaussian probability density. On the other hand, power tail probability densities of the form $\varphi(x) \simeq C_\alpha / x^\alpha$, $x \to \infty$, can provide a better fit of empirical probability density functions, as shown in Figure 21.6.

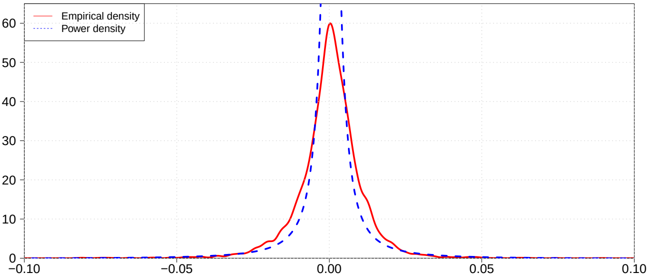

Fig. 21.6: Empirical density *vs.* power density.

The above fitting of empirical probability density function is using a power probability density function defined by a rational fraction obtained by the following R script.

```
install.packages("pracma")
library(pracma); x <- seq(-0.25, 0.25, length=1000)
returns.dens=density(returns,na.rm=TRUE, from = -0.1, to = 0.1, n = 1000)
a<-rationalfit(returns.dens$x, returns.dens$y, d1=2, d2=2)
dev.new(width=10, height=5)
plot(returns.dens$x, returns.dens$y, lwd=3, type = "l",xlab = 'x', col="red",ylab = '', main =
       '', panel.first = abline(h = 0, col='grey', lwd =0.2),las=1, cex.axis=1, cex.lab=1,
       xaxs='i', yaxs='i')
lines(x,(a$p1[3]+a$p1[2]*x+a$p1[1]*x^2)/(a$p2[3]+a$p2[2]*x+a$p2[1]*x^2), type="l", lty=2,
       lwd=3, col="blue",xlab="x value",ylab="Density", main="")
legend("topright", legend=c("Empirical density", "Power density"),col=c("red", "blue"), lty=1:2,
       cex=1.2)
```

The output of the **rationalfit** command is

```
$p1
[1] -0.184717249 -0.001591433  0.001385017

$p2
[1]  1.000000e+00 -6.460948e-04  1.314672e-05
```

which yields a rational fraction of the form

$$x \mapsto \frac{0.001385017 - 0.001591433 \times x - 0.184717249 \times x^2}{1.314672\ 10^{-5} - 6.460948\ 10^{-4} \times x + x^2}$$
$$\simeq -0.184717249 - \frac{0.001591433}{x} + \frac{0.001385017}{x^2},$$

which approximates the empirical probability density function of DJI returns in the least squares sense.

A solution to this tail problem is to use stochastic processes with jumps, that will account for sudden variations of the asset prices. On the other hand, such jump models are generally based on the Poisson distribution which has a slower tail decay than the Gaussian distribution. This allows one to assign higher probabilities to extreme events, resulting in a more realistic modeling of asset prices. *Stable distributions* with parameter $\alpha \in (0, 2)$ provide typical examples of probability laws with power tails, as their probability density functions behave asymptotically as $x \mapsto C_\alpha / |x|^{1+\alpha}$ when $x \to \pm\infty$, see Figure 20.12 for stable processes.

Edgeworth and Gram-Charlier expansions

Let

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \qquad x \in \mathbb{R},$$

denote the standard normal density function, and let

$$\Phi(x) := \int_{-\infty}^{x} \varphi(y) dy, \qquad x \in \mathbb{R},$$

denote the standard normal cumulative distribution function. Let also

$$H_n(x) := \frac{(-1)^n}{\varphi(x)} \frac{\partial^n \varphi}{\partial x^n}(x), \qquad x \in \mathbb{R},$$

denote the Hermite polynomial of degree n, with $H_0(x) = 1$.

Given X a random variable, the sequence $(\kappa_n^X)_{n \geqslant 1}$ of cumulants of X has been introduced in Thiele (1899). In what follows we will use the Moment Generating Function (MGF) of the random variable X, defined as

$$\mathcal{M}_X(t) := \mathbb{E}\big[e^{tX} \big] = 1 + \sum_{n \geqslant 1} \frac{t^n}{n!} \mathbb{E}[X^n], \qquad t \in \mathbb{R}. \tag{21.1}$$

Definition 21.1. *The* cumulants *of a random variable X are defined to be the coefficients $(\kappa_n^X)_{n \geqslant 1}$ appearing in the series expansion*

$$\log \left(\mathbb{E}\big[e^{tX} \big] \right) = \log \left(1 + \sum_{n \geqslant 1} \frac{t^n}{n!} \mathbb{E}[X^n] \right) = \sum_{n \geqslant 1} \kappa_n^X \frac{t^n}{n!}, \quad t \in \mathbb{R}, \tag{21.2}$$

of the logarithmic moment generating function (log-MGF) of X.

The cumulants of X were originally called "semi-invariants" due to the property $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$, $n \geqslant 1$, when X and Y are independent random variables. Indeed, in this case we have

$$\begin{aligned}
\sum_{n \geqslant 1} \kappa_n^{X+Y} \frac{t^n}{n!} &= \log \left(\mathbb{E}\big[e^{t(X+Y)} \big] \right) \\
&= \log \left(\mathbb{E}\big[e^{tX} \big] \mathbb{E}\big[e^{tY} \big] \right) \\
&= \log \mathbb{E}\big[e^{tX} \big] + \log \mathbb{E}\big[e^{tY} \big] \\
&= \sum_{n \geqslant 1} \kappa_n^X \frac{t^n}{n!} + \sum_{n \geqslant 1} \kappa_n^Y \frac{t^n}{n!} \\
&= \sum_{n \geqslant 1} \left(\kappa_n^X + \kappa_n^Y \right) \frac{t^n}{n!}, \qquad t \in \mathbb{R},
\end{aligned}$$

showing that $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$, $n \geqslant 1$.

a) First moment and cumulant. Taking $n = 1$, we find $\kappa_1^X = \mathbb{E}[X]$.
b) Variance and second cumulant. We have

$$\kappa_2^X = \mathbb{E}\big[X^2 \big] - (\mathbb{E}[X])^2 = \mathbb{E}\big[(X - \mathbb{E}[X])^2 \big],$$

and $\sqrt{\kappa_2^X}$ is the *standard deviation* of X.
c) The third cumulant of X is given as the third central moment

$$\kappa_3^X = \mathbb{E}[(X - \mathbb{E}[X])^3],$$

and the coefficient

$$\frac{\kappa_3^X}{(\kappa_2^X)^{3/2}} = \frac{\mathbb{E}\big[(X-\mathbb{E}[X])^3\big]}{(\mathbb{E}[(X-\mathbb{E}[X])^2])^{3/2}}$$

is the *skewness* of X.

d) Similarly, we have

$$\begin{aligned}\kappa_4^X &= \mathbb{E}\big[(X-\mathbb{E}[X])^4\big] - 3(\kappa_2^X)^2 \\ &= \mathbb{E}\big[(X-\mathbb{E}[X])^4\big] - 3\big(\mathbb{E}\big[(X-\mathbb{E}[X])^2\big]\big)^2,\end{aligned}$$

and the *excess kurtosis* of X is defined as

$$\frac{\kappa_4^X}{(\kappa_2^X)^2} = \frac{\mathbb{E}[(X-\mathbb{E}[X])^4]}{(\mathbb{E}[(X-\mathbb{E}[X])^2])^2} - 3.$$

Example: Gaussian moments and cumulants

When X is centered we have $\kappa_1^X = 0$ and $\kappa_2^X = \mathbb{E}\big[X^2\big] = \mathrm{Var}\,[X]$, and X becomes Gaussian if and only if $\kappa_n^X = 0$, $n \geqslant 3$, *i.e.*

$$\kappa_n^X = \mathbb{1}_{\{n=2\}}\sigma^2, \qquad n \geqslant 1,$$

or

$$\big(\kappa_1^X, \kappa_2^X, \kappa_3^X, \kappa_4^X, \ldots\big) = (0, \sigma^2, 0, 0, \ldots).$$

Example: Poisson moments and cumulants

In the particular case of a Poisson random variable $Z \simeq \mathcal{P}(\lambda)$ with intensity $\lambda > 0$, we have

$$\begin{aligned}\mathbb{E}_\lambda\big[\mathrm{e}^{tZ}\big] &= \sum_{n\geqslant 0} \mathrm{e}^{nt}\mathbb{P}(Z=n) \\ &= \mathrm{e}^{-\lambda}\sum_{n\geqslant 0}\frac{(\lambda\,\mathrm{e}^t)^n}{n!} \\ &= \mathrm{e}^{\lambda(\mathrm{e}^t-1)}, \qquad t \in \mathbb{R}_+,\end{aligned} \tag{21.3}$$

hence $\kappa_n^Z = \lambda$, $n \geqslant 1$, or

$$\big(\kappa_1^Z, \kappa_2^Z, \kappa_3^Z, \kappa_4^Z, \ldots\big) = (\lambda, \lambda, \lambda, \lambda, \ldots),$$

The next proposition summarizes the Gram-Charlier expansion method to obtain series expansion of a probability density function, see Gram (1883), Charlier (1914) and § 17.6 of Cramér (1946).

Proposition 21.2. *(Proposition 2.1 in Tanaka et al. (2010)) The Gram-Charlier expansion of the continuous probability density function $\phi_X(x)$ of a random variable X is given by*

$$\phi_X(x) = \frac{1}{\sqrt{\kappa_2^X}}\varphi\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right) + \frac{1}{\sqrt{\kappa_2^X}}\sum_{n=3}^{\infty} c_n H_n\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right)\varphi\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right),$$

where $c_0 = 1$, $c_1 = c_2 = 0$, and the sequence $(c_n)_{n\geqslant 3}$ is given from the cumulants $(\kappa_n^X)_{n\geqslant 1}$ of X as

$$c_n = \frac{1}{(\kappa_2^X)^{n/2}}\sum_{m=1}^{[n/3]}\sum_{\substack{l_1+\cdots+l_m=n\\ l_1,\ldots,l_m\geqslant 3}}\frac{\kappa_{l_1}^X\cdots\kappa_{l_m}^X}{m!l_1!\cdots l_m!}, \qquad n\geqslant 3.$$

The coefficients c_3 and c_4 can be expressed from the skewness $\kappa_3^X/(\kappa_2^X)^{3/2}$ and the excess kurtosis $\kappa_4^X/(\kappa_2^X)^2$ as

$$c_3 = \frac{\kappa_3^X}{3!(\kappa_2^X)^{3/2}} \quad\text{and}\quad c_4 = \frac{\kappa_4^X}{4!(\kappa_2^X)^2}.$$

a) The first-order expansion

$$\phi_X^{(1)}(x) = \frac{1}{\sqrt{\kappa_2^X}}\varphi\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right)$$

corresponds to normal moment matching approximation.

b) The third-order expansion is given by

$$\phi_X^{(3)}(x) = \frac{1}{\sqrt{\kappa_2^X}}\varphi\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right)\left(1 + c_3 H_3\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right)\right)$$

c) The fourth-order expansion is given by

$$\phi_X^{(4)}(x) = \frac{1}{\sqrt{\kappa_2^X}}\varphi\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right)\left(1 + c_3 H_3\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right) + c_4 H_4\left(\frac{x-\kappa_1^X}{\sqrt{\kappa_2^X}}\right)\right).$$

The next R code presents a fit of first to fourth order Gram-Charlier density approximations to the empirical distribution of asset returns.

```
install.packages("SimMultiCorrData");install.packages("PDQutils")
library(SimMultiCorrData);library(PDQutils)
x <- seq(-0.25, 0.25, length=1000);dev.new(width=10, height=5)
plot(returns.dens$x, returns.dens$y, xlim=c(-0.1,0.1), xlab = 'x', type = 'l', lwd=3,
      col="red",ylab = '', main = '',panel.first = abline(h = 0, col='grey', lwd =0.2),las=1,
      cex.axis=1, cex.lab=1,xaxs='i', yaxs='i')
lines(x, qx, type="l", lty=2, lwd=3, col="blue")
m<-calc_moments(returns[!is.na(returns)])
cumulants<-c(m[1],m[2]**2);d2 <- dapx_edgeworth(x, cumulants)
lines(x, d2, type="l", lty=2, lwd=3, col="blue")
cumulants<-c(m[1],m[2]**2,m[3]*m[2]**3);d3 <- dapx_edgeworth(x, cumulants)
lines(x, d3, type="l", lty=2, lwd=3, col="green")
cumulants<-c(m[1],m[2]**2,0.5*m[3]*m[2]**3,0.2*m[4]*m[2]**4)
d4 <- dapx_edgeworth(x, cumulants);lines(x, d4, type="l", lty=2, lwd=3, col="purple")
legend("topleft", legend=c("Empirical density", "Gaussian density", "Third order
      Gram-Charlier", "Fourth order Gram-Charlier"),col=c("red", "blue", "green", "purple"),
      lty=1:2,cex=1.2)
grid()
```

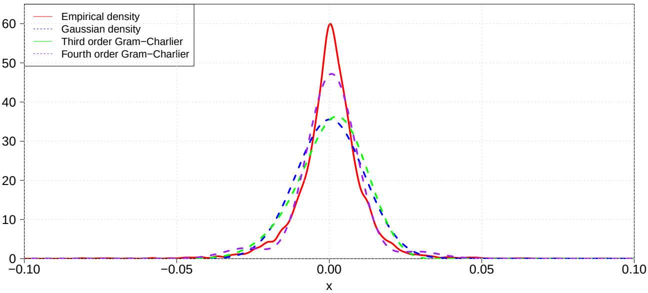

Fig. 21.7: Gram-Charlier expansions

21.2 Risk-Neutral Probability Measures

Consider an asset price process modeled by the equation,

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t^-} dY_t, \tag{21.4}$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is the compound Poisson process defined in Section 20.2, with jump size distribution $\nu(dx)$ under $\mathbb{P}_\nu$. The equation (21.4) has for solution

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{\sigma^2}{2} t\right) \prod_{k=1}^{N_t} (1 + Z_k), \tag{21.5}$$

$t \geqslant 0$. An important issue for non-arbitrage pricing is to determine a risk-neutral probability measure (or martingale measure) $\mathbb{P}^*$ under which the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} := (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and this goal can be achieved using the Girsanov Theorem for jump processes, cf. Section 20.5. Similarly to Lemma 5.13, we have the following result.

Lemma 21.3. Discounting lemma. *The discounted asset price process*

$$\widetilde{S}_t := \mathrm{e}^{-rt}S_t, \qquad t \geqslant 0,$$

satisfies the equation

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma\widetilde{S}_t dB_t + \widetilde{S}_{t^-} dY_t. \tag{21.6}$$

In addition, Equation 21.6 can be rewritten as

$$d\widetilde{S}_t = (\mu - r + \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z] - \sigma u)\widetilde{S}_t dt + \sigma\widetilde{S}_t(dB_t + udt) + \widetilde{S}_{t^-}(dY_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt),$$

for any $u \in \mathbb{R}$. When the drift parameter u, the intensity $\tilde{\lambda} > 0$ and the jump size distribution $\tilde{\nu}$ are chosen to satisfy the condition

$$\mu - r + \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z] - \sigma u = 0 \tag{21.7}$$

with $\sigma u + r - \mu > 0$, then

$$\tilde{\lambda} = \frac{\sigma u + r - \mu}{\mathbb{E}_{\tilde{\nu}}[Z]} > 0,$$

and the Girsanov Theorem 20.21 for jump processes shows that

$$dB_t + udt + dY_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ defined in Theorem 20.21. As a consequence, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ becomes a martingale is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$.

In this setting, the non-uniqueness of the risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ is apparent since additional degrees of freedom are involved in the choices of u, λ and the measure $\tilde{\nu}$, whereas in the continuous case the choice of $u = (\mu - r)/\sigma$ in (7.14) was unique.

21.3 Pricing in Jump Models

Recall that a market is without arbitrage if and only it admits at least one risk-neutral probability measure.

Consider the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ constructed in Theorem 20.21, under which the discounted asset price process

$$d\widetilde{S}_t = \sigma \widetilde{S}_t d\widehat{B}_t + \widetilde{S}_{t^-}(dY_t - \tilde{\lambda}\mathbb{E}_\nu[Z]dt),$$

is a martingale, and $\widehat{B}_t = B_t + ut$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. Then, the arbitrage-free price of a claim with payoff C is given by

$$\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[C \mid \mathcal{F}_t] \tag{21.8}$$

under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$.

Clearly the price (21.8) of C is no longer unique in the presence of jumps due to an infinity of possible choices of parameters $u, \tilde{\lambda}, \tilde{\nu}$ satisfying the martingale condition (21.7), and such a market is not complete, except if either $\tilde{\lambda} = \lambda = 0$, or ($\sigma = 0$ and $\tilde{\nu} = \nu = \delta_1$).

Various techniques can be used for the selection of a risk-neutral probability measure, such as the determination of a minimal entropy risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ that minimizes the Kullback-Leibler relative entropy

$$\mathbb{Q} \mapsto I(\mathbb{Q},\mathbb{P}) := \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]$$

among the probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$.

Pricing vanilla options

The price of a vanilla option with payoff of the form $\phi(S_T)$ on the underlying asset S_T can be written from (21.8) as

$$\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t], \tag{21.9}$$

where the expectation can be computed as

$$\begin{aligned}
&\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi(S_T) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(S_0 \exp\left(\mu T + \sigma B_T - \frac{\sigma^2}{2}T\right)\prod_{k=1}^{N_T}(1+Z_k)\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(S_t \exp\left((T-t)\mu + (B_T - B_t)\sigma - \frac{\sigma^2}{2}(T-t)\right)\prod_{k=N_t+1}^{N_T}(1+Z_k)\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x \exp\left((T-t)\mu + (B_T - B_t)\sigma - \frac{\sigma^2}{2}(T-t)\right)\prod_{k=N_t+1}^{N_T}(1+Z_k)\right)\right]_{x=S_t} \\
&= \sum_{n\geqslant 0} \mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}(N_T - N_t = n)
\end{aligned}$$

$$\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma-(T-t)\sigma^2/2}\prod_{k=N_t+1}^{N_T}(1+Z_k)\right)\,\Big|\,N_T-N_t=n\right]_{x=S_t}$$

$$= \mathrm{e}^{-(T-t)\tilde{\lambda}}\sum_{n\geqslant 0}\frac{((T-t)\tilde{\lambda})^n}{n!}$$

$$\times\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma-(T-t)\sigma^2/2}\prod_{k=1}^{n}(1+Z_k)\right)\right]_{x=S_t}$$

$$= \mathrm{e}^{-\tilde{\lambda}(T-t)}\sum_{n\geqslant 0}\frac{(\tilde{\lambda}(T-t))^n}{n!}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{\text{n times}}$$

$$\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma-(T-t)\sigma^2/2}\prod_{k=1}^{n}(1+z_k)\right)\right]_{x=S_t}\tilde{\nu}(dz_1)\cdots\tilde{\nu}(dz_n),$$

hence the price of the vanilla option with payoff $\phi(S_T)$ is given by

$$\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T)\mid\mathcal{F}_t]$$

$$= \frac{1}{\sqrt{2(T-t)\pi}}\,\mathrm{e}^{-(r+\tilde{\lambda})(T-t)}\sum_{n\geqslant 0}\frac{(\tilde{\lambda}(T-t))^n}{n!}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{\text{n+1 times}}$$

$$\phi\left(S_t\,\mathrm{e}^{(T-t)\mu+\sigma x-(T-t)\sigma^2/2}\prod_{k=1}^{n}(1+z_k)\right)\,\mathrm{e}^{-x^2/(2(T-t))}\tilde{\nu}(dz_1)\cdots\tilde{\nu}(dz_n)dx.$$

21.4 Exponential Lévy Models

Instead of modeling the asset price $(S_t)_{t\in\mathbb{R}_+}$ through a stochastic exponential (21.5) solution of the stochastic differential equation with jumps of the form (21.4), we may consider an exponential price process of the form

$$\begin{aligned}
S_t &:= S_0\,\mathrm{e}^{\mu t+\sigma B_t+Y_t}\\
&= S_0\exp\left(\mu t+\sigma B_t+\sum_{k=1}^{N_t}Z_k\right)\\
&= S_0\,\mathrm{e}^{\mu t+\sigma B_t}\prod_{k=1}^{N_t}\mathrm{e}^{Z_k}\\
&= S_0\,\mathrm{e}^{\mu t+\sigma B_t}\prod_{0\leqslant s\leqslant t}\mathrm{e}^{\Delta Y_t},\qquad t\geqslant 0,
\end{aligned}$$

from Relation (20.9), *i.e.* $\Delta Y_t = Z_{N_t}\Delta N_t$. The process $(S_t)_{t\in\mathbb{R}_+}$ is equivalently given by the log-return dynamics

$$d\log S_t = \mu dt + \sigma dB_t + dY_t, \qquad t \geqslant 0.$$

In the exponential Lévy model we also have

$$S_t = S_0 \,\mathrm{e}^{(\mu+\sigma^2/2)t+\sigma B_t - \sigma^2 t/2 + Y_t}$$

and the process S_t satisfies the stochastic differential equation

$$\begin{aligned} dS_t &= \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dB_t + S_{t^-}(\,\mathrm{e}^{\Delta Y_t} - 1) dN_t \\ &= \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dB_t + S_{t^-}(\,\mathrm{e}^{Z_{N_t}} - 1) dN_t, \end{aligned}$$

hence the process S_t has jumps of size $S_{T_k^-}(\,\mathrm{e}^{Z_k} - 1)$, $k \geqslant 1$, and (21.7) reads

$$\mu + \frac{\sigma^2}{2} - r = \sigma u - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[\,\mathrm{e}^{Z} - 1].$$

Under this condition we can choose a risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ under which $(\,\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and the expected value

$$\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t]$$

represents a (non-unique) arbitrage-free price at time $t \in [0, T]$ for the contingent claim with payoff $\phi(S_T)$.

This arbitrage-free price can be expressed as

$$\begin{aligned} &\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\big[\phi(S_T) \,\big|\, \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\big[\phi(S_0\,\mathrm{e}^{\mu T+\sigma B_T+Y_T}) \,\big|\, \mathcal{F}_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\big[\phi(S_t\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma+Y_T-Y_t}) \,\big|\, \mathcal{F}_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\big[\phi(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma+Y_T-Y_t})\big]_{x=S_t} \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\exp\left((T-t)\mu + (B_T-B_t)\sigma + \sum_{k=N_t+1}^{N_T} Z_k\right)\right)\right]_{x=S_t} \\ &= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \\ &\quad \times \sum_{n\geqslant 0}\frac{(\tilde{\lambda}(T-t))^n}{n!}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma}\exp\left(\sum_{k=1}^{n} Z_k\right)\right)\right]_{x=S_t}. \end{aligned}$$

Merton (1976) model

We assume that $(Z_k)_{k\geqslant 1}$ is a family of independent identically distributed Gaussian $\mathcal{N}(\delta, \eta^2)$ random variables under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ with

$$\mu + \frac{\sigma^2}{2} - r = \sigma u - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[\,\mathrm{e}^Z - 1] = \sigma u - \tilde{\lambda}(\,\mathrm{e}^{\delta+\eta^2/2} - 1),$$

as in (21.7), hence by the Girsanov Theorem 20.21 for jump processes, $B_t + ut + Y_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[\,\mathrm{e}^Z - 1]t$ is a martingale and $B_t + ut$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. For simplicity we choose $u = 0$, which yields

$$\mu = r - \frac{\sigma^2}{2} - \tilde{\lambda}(\,\mathrm{e}^{\delta+\eta^2/2} - 1).$$

Proposition 21.4. *The price of the European call option in the Merton model is given by*

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[(S_T - K)^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-\tilde{\lambda}\,\mathrm{e}^{\delta+\eta^2/2}(T-t)} \sum_{n\geqslant 0} \frac{(\tilde{\lambda}\,\mathrm{e}^{\delta+n\eta^2/2}(T-t))^n}{n!} \\
&\qquad \times \mathrm{Bl}\left(S_t, K, \sigma^2 + n\eta^2/(T-t), r + n\frac{\delta+\eta^2/2}{T-t} - \tilde{\lambda}(\,\mathrm{e}^{\delta+\eta^2/2} - 1), T-t\right),
\end{aligned}$$

$0 \leqslant t \leqslant T$.

Proof. We have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\qquad \times \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma}\exp\left(\sum_{k=1}^n Z_k\right)\right)\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \mathbb{E}\left[\phi(x\,\mathrm{e}^{(T-t)\mu+n\delta+X_n})\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \int_{-\infty}^{\infty} \phi\big(S_t\,\mathrm{e}^{(T-t)\mu+n\delta+y}\big) \frac{\mathrm{e}^{-y^2/(2((T-t)\sigma^2+n\eta^2))}}{\sqrt{4((T-t)\sigma^2+n\eta^2)\pi}} dy,
\end{aligned}$$

where

$$X_n := (B_T - B_t)\sigma + \sum_{k=1}^n (Z_k - \delta) \simeq \mathcal{N}(0, (T-t)\sigma^2 + n\eta^2), \qquad n \geqslant 0,$$

is a centered Gaussian random variable with variance

$$v_n^2 := (T-t)\sigma^2 + \sum_{k=1}^n \operatorname{Var} Z_k = (T-t)\sigma^2 + n\eta^2.$$

Hence when $\phi(x) = (x-K)^+$ is the payoff function of a European call option, using the relation

$$\mathrm{Bl}\big(x, K, v_n^2/\tau, r, \tau\big) = \mathrm{e}^{-r\tau}\mathbb{E}\big[\big(x\,\mathrm{e}^{X_n - v_n^2/2 + r\tau} - K\big)^+\big]$$

we get

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}}\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[(S_T-K)^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \mathbb{E}\big[\big(x\,\mathrm{e}^{(T-t)\mu + n\delta + X_n} - K\big)^+\big]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathbb{E}\big[\big(x\,\mathrm{e}^{(r-\sigma^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1))(T-t)+n\delta+X_n} - K\big)^+\big]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathbb{E}\big[\big(x\,\mathrm{e}^{n\delta+n\eta^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)+X_n-v_n^2/2+(T-t)r} - K\big)^+\big]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathrm{Bl}\big(S_t\,\mathrm{e}^{n\delta+n\eta^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)}, K, \sigma^2 + n\eta^2/(T-t), r, T-t\big).
\end{aligned}$$

We may also write

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}}\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[(S_T-K)^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{\big((T-t)\tilde{\lambda}\big)^n}{n!}\, \mathrm{e}^{n\delta+n\eta^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)} \\
&\quad \times \mathrm{Bl}\left(S_t, K\,\mathrm{e}^{-n\delta-n\eta^2/2+\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)}, \sigma^2 + n\eta^2/(T-t), r, T-t\right) \\
&= \mathrm{e}^{-\tilde{\lambda}\,\mathrm{e}^{\delta+\eta^2/2}(T-t)} \sum_{n\geqslant 0} \frac{(\tilde{\lambda}\,\mathrm{e}^{\delta+n\eta^2/2}(T-t))^n}{n!}
\end{aligned}$$

$$\times \mathrm{Bl}\left(S_t, K, \sigma^2 + n\eta^2/(T-t), r + n\frac{\delta+\eta^2/2}{T-t} - \tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1), T-t\right).$$

$\square$

21.5 Black-Scholes PDE with Jumps

In this section, we consider the asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled by the equation (21.4), *i.e.*

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t^-} dY_t, \tag{21.10}$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process with jump size distribution $\nu(dx)$. Recall that by the Markov property of $(S_t)_{t\in\mathbb{R}_+}$, the price (21.9) at time t of the option with payoff $\phi(S_T)$ can be written as a function $f(t, S_t)$ of t and S_t, *i.e.*

$$f(t, S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathcal{F}_t] = \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid S_t], \tag{21.11}$$

with the terminal condition $f(T,x) = \phi(x)$. In addition, the process

$$t \mapsto \mathrm{e}^{(T-t)r} f(t, S_t)$$

is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ by the same argument as in (7.1).

In the next proposition we derive a Partial Integro-Differential Equation (PIDE) for the function $(t,x) \mapsto f(t,x)$.

Proposition 21.5. *The price $f(t, S_t)$ of the vanilla option with payoff function ϕ in the model (21.10) satisfies the Partial* Integro-Differential *Equation (PIDE)*

$$\boxed{\begin{aligned} rf(t,x) &= \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x) \\ &\quad + \tilde{\lambda}\int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - yx\frac{\partial f}{\partial x}(t,x)\right)\tilde{\nu}(dy), \end{aligned}} \tag{21.12}$$

under the terminal condition $f(T,x) = \phi(x)$.

Proof. We have

$$dS_t = rS_t dt + \sigma S_t d\widehat{B}_t + S_{t^-}(dY_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt), \tag{21.13}$$

where $\widehat{B}_t = B_t + ut$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. Next, by the Itô formula with jumps (20.24), we have

$$
\begin{aligned}
& df(t,S_t) \\
&= \frac{\partial f}{\partial t}(t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,S_t)dt + \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)dt \\
&\quad - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)dt + (f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t \\
&= \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t + (f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t \\
&\quad - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt \\
&\quad + \left(\frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)\right)dt \\
&\quad + \left(\tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t} - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)\right)dt.
\end{aligned}
$$

Based on the discounted portfolio value differential

$$
\begin{aligned}
& d(\mathrm{e}^{-rt}f(t,S_t)) \\
&= \mathrm{e}^{-rt}\sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t \\
&+ \mathrm{e}^{-rt}\left(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt\right) \\
&+ \mathrm{e}^{-rt}\left(-rf(t,S_t) + \frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)\right)dt \qquad (21.14) \\
&+ \mathrm{e}^{-rt}\left(\tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t} - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)\right)dt, \qquad (21.15)
\end{aligned}
$$

obtained from the Itô Table 20.1 with jumps, and the facts that

- the Brownian motion $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$,
- by the smoothing lemma Proposition 20.11, the process given by the differential

  $$(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt,$$

  is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$, see also (20.23),
- the discounted portfolio value process $t \mapsto \mathrm{e}^{-rt}f(t,S_t)$, is also a martingale under the risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$,

we conclude to the vanishing of the terms (21.14)-(21.15) above, *i.e.*

$$
\begin{aligned}
& -rf(t,S_t) + \frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t) \\
&\quad + \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t} - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t) = 0,
\end{aligned}
$$

or

$$\frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x)$$
$$+\tilde{\lambda}\int_{-\infty}^{\infty}(f(t,x(1+y)) - f(t,x))\tilde{\nu}(dy) - \tilde{\lambda}x\frac{\partial f}{\partial x}(t,x)\int_{-\infty}^{\infty} y\tilde{\nu}(dy) = rf(t,x),$$

which leads to the Partial *Integro-Differential* Equation (21.12). □

A major technical difficulty when solving the PIDE (21.12) numerically is that the operator

$$f \mapsto \int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - yx\frac{\partial f}{\partial x}(t,x)\right)\tilde{\nu}(dy)$$

is *nonlocal*, therefore adding significant difficulties to the application of standard discretization schemes, cf. *e.g.* Section 22.4.

In addition, we have shown that the change $df(t,S_t)$ in the portfolio value (21.11) is given by

$$df(t,S_t) = \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t + rf(t,S_t)dt \tag{21.16}$$
$$+(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt.$$

Fixed jump size

In the case of Poisson jumps with fixed size a, *i.e.* when $Y_t = aN_t$ and $\nu(dx) = \delta_a(dx)$, the PIDE (21.12) reads

$$rf(t,x) = \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x)$$
$$+\tilde{\lambda}\left(f(t,x(1+a)) - f(t,x) - ax\frac{\partial f}{\partial x}(t,x)\right),$$

and we have

$$df(t,S_t) = \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t + rf(t,S_t)dt$$
$$+(f(t,S_{t^-}(1+a)) - f(t,S_{t^-}))dN_t - \tilde{\lambda}(f(t,S_t(1+a)) - f(t,S_t))dt.$$

21.6 Mean-Variance Hedging with Jumps

Consider a portfolio valued

$$V_t := \eta_t A_t + \xi_t S_t = \eta_t \,\mathrm{e}^{rt} + \xi_t S_t$$

at time $t \in \mathbb{R}_+$, and satisfying the self-financing condition (5.3), *i.e.*

$$dV_t = \eta_t dA_t + \xi_t dS_t = r\eta_t \mathrm{e}^{rt} dt + \xi_t dS_t.$$

Assuming that the portfolio value takes the form $V_t = f(t, S_t)$ at all times $t \in [0, T]$, by (21.13) we have

$$\begin{aligned} dV_t &= df(t, S_t) \\ &= r\eta_t \mathrm{e}^{rt} dt + \xi_t dS_t \\ &= r\eta_t \mathrm{e}^{rt} dt + \xi_t (rS_t dt + \sigma S_t d\widehat{B}_t + S_{t^-}(dY_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt)) \\ &= rV_t dt + \sigma \xi_t S_t d\widehat{B}_t + \xi_t S_{t^-}(dY_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt) \\ &= rf(t, S_t)dt + \sigma \xi_t S_t d\widehat{B}_t + \xi_t S_{t^-}(dY_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt), \end{aligned} \tag{21.17}$$

has to match

$$\begin{aligned} & df(t, S_t) = rf(t, S_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t) d\widehat{B}_t \\ & + (f(t, S_{t^-}(1 + Z_{N_t})) - f(t, S_{t^-}))dN_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[(f(t, x(1+Z)) - f(t, x))]_{x=S_t} dt, \end{aligned} \tag{21.18}$$

which is obtained from (21.16).

In such a situation we say that the claim payoff C can be exactly replicated.

Exact replication is possible in essentially only two situations:

(i) *Continuous market*, $\lambda = \tilde{\lambda} = 0$. In this case we find the usual Black-Scholes Delta:

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t). \tag{21.19}$$

(ii) *Poisson jump market*, $\sigma = 0$ and $Y_t = aN_t$, $\nu(dx) = \delta_a(dx)$. In this case, by matching (21.17) to (21.18) we find

$$\xi_t = \frac{1}{aS_{t^-}}(f(t, S_{t^-}(1+a)) - f(t, S_{t^-})). \tag{21.20}$$

Note that in the limit $a \to 0$ this expression recovers the Black-Scholes Delta formula (21.19).

When Conditions (i) or (ii) above are not satisfied, exact replication is not possible, and this results into an hedging error given from (21.17) and (21.18) by

$$\begin{aligned} V_T - \phi(S_T) &= V_T - f(T, S_T) \\ &= V_0 + \int_0^T dV_t - f(0, S_0) - \int_0^T df(t, S_t) \\ &= V_0 - f(0, S_0) + \sigma \int_0^T S_t \left(\xi_t - \frac{\partial f}{\partial x}(t, S_t) \right) d\widehat{B}_t \\ &\quad + \int_0^T \xi_t S_{t^-}(Z_{N_t} dN_t - \tilde{\lambda}\mathbb{E}_{\tilde{\nu}}[Z]dt) \end{aligned}$$

$$
\begin{aligned}
&-\int_0^T (f(t, S_{t^-}(1+Z_{N_t})) - f(t, S_{t^-}))dN_t \\
&+\tilde{\lambda}\int_0^T \mathbb{E}_{\tilde{\nu}}[(f(t, x(1+Z)) - f(t, x))]_{x=S_t} dt.
\end{aligned}
$$

Fixed jump size

Proposition 21.6. *Assume that $Y_t = aN_t$, i.e. $\nu(dx) = \delta_a(dx)$. The mean-square hedging error is minimized by*

$$
V_0 = f(0, S_0) = \mathrm{e}^{-rT}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T)],
$$

and

$$
\boxed{\xi_t = \frac{\sigma^2}{\sigma^2 + a^2\tilde{\lambda}} \frac{\partial f}{\partial x}(t, S_{t^-}) + \frac{a^2\tilde{\lambda}}{\sigma^2 + a^2\tilde{\lambda}} \times \frac{f(t, S_{t^-}(1+a)) - f(t, S_{t^-})}{aS_{t^-}},} \quad (21.21)
$$

$t \in [0, T]$.

Proof. We have

$$
\begin{aligned}
V_T - f(T, S_T) = {} & V_0 - f(0, S_0) + \sigma \int_0^T S_{t^-}\left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right) d\widehat{B}_t \\
& - \int_0^T (f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})(dN_t - \tilde{\lambda}dt),
\end{aligned}
$$

hence the mean-square hedging error is given by

$$
\begin{aligned}
&\mathbb{E}_{u,\tilde{\lambda}}\big[\big(V_T - f(T, S_T)\big)^2\big] \\
&= (V_0 - f(0, S_0))^2 + \sigma^2\mathbb{E}_{u,\tilde{\lambda}}\left[\left(\int_0^T S_{t^-}\left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right) d\widehat{B}_t\right)^2\right] \\
&\quad + \mathbb{E}_{u,\tilde{\lambda}}\left[\left(\int_0^T (f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})(dN_t - \tilde{\lambda}dt)\right)^2\right] \\
&= (V_0 - f(0, S_0))^2 + \sigma^2\mathbb{E}_{u,\tilde{\lambda}}\left[\int_0^T S_{t^-}^2\left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right)^2 dt\right] \\
&\quad + \tilde{\lambda}\mathbb{E}_{u,\tilde{\lambda}}\left[\int_0^T \big((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\big)^2 dt\right],
\end{aligned}
$$

where we applied the Itô isometry (20.21). Clearly, the initial portfolio value V_0 minimizing the above quantity is

$$V_0 = f(0, S_0) = \mathrm{e}^{-rT}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T)].$$

Next, let us find the optimal portfolio strategy $(\xi_t)_{t\in[0,T]}$ minimizing the remaining hedging error

$$\mathbb{E}_{u,\tilde{\lambda}}\left[\int_0^T \left(\sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right)^2 + \tilde{\lambda}\big((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\big)^2\right) dt\right].$$

For all $t \in (0, T]$, the almost-sure minimum of

$$\xi_t \mapsto \sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right)^2 + \tilde{\lambda}\big((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\big)^2$$

is given by differentiation with respect to ξ_t, as the solution of

$$2\sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-})\right) - 2a\tilde{\lambda} S_{t^-} \left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\right) = 0,$$

i.e.

$$\xi_t = \frac{\sigma^2}{\sigma^2 + a^2\tilde{\lambda}} \frac{\partial f}{\partial x}(t, S_{t^-}) + \frac{a^2\tilde{\lambda}}{\sigma^2 + a^2\tilde{\lambda}} \times \frac{f(t, S_{t^-}(1+a)) - f(t, S_{t^-})}{aS_{t^-}},$$

$t \in (0, T]$. □

When hedging only the risk generated by the Brownian part, we let

$$\xi_t = \frac{\partial f}{\partial x}(t, S_{t^-})$$

as in the Black-Scholes model, and in this case the hedging error due to the presence of jumps becomes

$$\mathbb{E}_{u,\tilde{\lambda}}\big[(V_T - f(T, S_T))^2\big] = \tilde{\lambda}\mathbb{E}_{u,\tilde{\lambda}}\left[\int_0^T \big((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-})\big)^2 dt\right],$$

$t \in (0, T]$. We note that the optimal strategy (21.21) is a weighted average of the Brownian and jump hedging strategies (21.19) and (21.20) according to the respective variance parameters σ^2 and $a^2\tilde{\lambda}$ of the continuous and jump components.

Clearly, if $a\tilde{\lambda} = 0$ we get

$$\xi_t = \frac{\partial f}{\partial x}(t, S_{t^-}), \qquad t \in (0, T],$$

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which is the Black-Scholes perfect replication strategy, and when $\sigma = 0$ we recover

$$\xi_t = \frac{f(t,(1+a)S_{t^-}) - f(t,S_{t^-})}{aS_{t^-}}, \qquad t \in (0,T].$$

which is (21.20). See § 10.4.2 of Cont and Tankov (2004) for mean-variance hedging in exponential Lévy model, and § 12.6 of Di Nunno et al. (2009) for mean-variance hedging by the Malliavin calculus.

Note that the fact that perfect replication is not possible in a jump-diffusion model can be interpreted as a more realistic feature of the model, as perfect replication is not possible in the real world.

See Jeanblanc and Privault (2002) for an example of a complete market model with jumps, in which continuous and jump noise are mutually excluding each other over time.

In Table 21.1 we summarize the properties of geometric Brownian motion *vs.* jump-diffusion models in terms of asset price and market behaviors.

Model / Properties	Geometric Brownian motion	Jump-diffusion model	Real world
Discontinuous asset prices	✗	✓	✓
Fat tailed market returns	✗	✓	✓
Complete market	✓	✗	✗
Unique prices and risk-neutral measure	✓	✗	✗

Table 21.1: Market models and their properties.

Exercises

Exercise 21.1 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + \eta S_{t^-}(dN_t - \alpha dt),$$

where $\eta > 0$.

a) Find the value of $\alpha \in \mathbb{R}$ such that the discounted process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Compute the price at time $t \in [0,T]$ of a power option with payoff $|S_T|^2$ at maturity T.

Exercise 21.2 Consider a long forward contract with payoff $S_T - K$ on a jump diffusion risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t^-} dY_t.$$

a) Show that the forward claim admits a unique arbitrage-free price to be computed in a market with risk-free rate $r > 0$.
b) Show that the forward claim admits an exact replicating portfolio strategy based on the two assets S_t and e^{rt}.
c) Recover portfolio strategy of Question (b) using the optimal portfolio strategy formula (21.21).

Exercise 21.3 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$, independent of $(B_t)_{t\in\mathbb{R}_+}$, under a probability measure $\mathbb{P}^*$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + \eta S_{t^-} dN_t + \sigma S_t dB_t. \tag{21.22}$$

a) Solve the equation (21.22).
b) We assume that μ, η and the risk-free rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$. What relation does this impose on μ, η, λ and r?
c) Under the relation of Question (b), compute the price at time $t \in [0, T]$ of a European call option on S_T with strike price K and maturity T, using a series expansion of Black-Scholes functions.

Exercise 21.4 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + Y_{N_t} S_{t^-} dN_t,$$

where $(Y_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of uniformly distributed random variables on $[-1, 1]$.

a) Show that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Compute the price at time 0 of a European call option on S_T with strike price K and maturity T, using a series of multiple integrals.

Exercise 21.5 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + Y_{N_t} S_{t^-} (dN_t - \alpha dt),$$

where $(Y_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of uniformly distributed random variables on $[0, 1]$.

a) Find the value of $\alpha \in \mathbb{R}$ such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.

b) Compute the price at time $t \in [0,T]$ of the long forward contract with maturity T and payoff $S_T - K$.

Exercise 21.6 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a risk-neutral probability measure $\mathbb{P}^*$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + \alpha S_{t^-}(dN_t - \lambda dt), \tag{21.23}$$

where $\alpha > 0$. Consider a portfolio with value

$$V_t = \eta_t \mathrm{e}^{rt} + \xi_t S_t$$

at time $t \in [0,T]$, and satisfying the self-financing condition

$$dV_t = r\eta_t \mathrm{e}^{rt} dt + \xi_t dS_t.$$

We assume that the portfolio hedges a claim payoff $C = \phi(S_T)$, and that its value can be written as a function $V_t = f(t, S_t)$ of t and S_t for all times $t \in [0,T]$.

a) Solve the stochastic differential equation (21.23).
b) Price the claim $C = \phi(S_T)$ at time $t \in [0,T]$ using a series expansion.
c) Show that under self-financing, the variation dV_t of the portfolio value V_t satisfies

$$dV_t = rf(t, S_t)dt + \alpha\xi_t S_{t^-}(dN_t - \lambda dt). \tag{21.24}$$

d) Show that the claim payoff $C = \phi(S_T)$ can be exactly replicated by the delta hedging strategy

$$\xi_t = \frac{1}{\alpha S_{t^-}}\big(f(t, S_{t^-}(1+\alpha)) - f(t, S_{t^-})\big).$$

Exercise 21.7 Pricing by the Esscher transform (Gerber and Shiu (1994)). Consider a compound Poisson process $(Y_t)_{t\in[0,T]}$ with $\mathbb{E}\big[\mathrm{e}^{\theta(Y_t - Y_s)}\big] = \mathrm{e}^{(t-s)m(\theta)}$, $0 \leqslant s \leqslant t$, with $m(\theta)$ a function of $\theta \in \mathbb{R}$, and the asset price process $S_t := \mathrm{e}^{rt+Y_t}$, $t \in [0,T]$. Given $\theta \in \mathbb{R}$, let

$$N_t := \frac{\mathrm{e}^{\theta Y_t}}{\mathbb{E}\big[\mathrm{e}^{\theta Y_t}\big]} = \mathrm{e}^{\theta Y_t - tm(\theta)} = S_t^\theta \,\mathrm{e}^{-r\theta t - tm(\theta)},$$

and consider the probability measure $\mathbb{P}^\theta$ defined as

$$\frac{\mathrm{d}\mathbb{P}^\theta_{|\mathcal{F}_t}}{\mathrm{d}\mathbb{P}_{|\mathcal{F}_t}} := \frac{N_T}{N_t} = \mathrm{e}^{(Y_T - Y_t)\theta - (T-t)m(\theta)}, \qquad 0 \leqslant t \leqslant T.$$

a) Check that $(N_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Find a condition on θ such that the discounted price process $(\,\mathrm{e}^{-rt}S_t)_{t\in[0,T]} = \left(\,\mathrm{e}^{Y_t}\right)_{t\in[0,T]}$ is a martingale under $\mathbb{P}^\theta$.
c) Price the European call option with payoff $(S_T - K)^+$ by taking $\mathbb{P}^\theta$ as risk-neutral probability measure.