# Chapter 3
# Pricing and Hedging in Discrete Time

We consider the pricing and hedging of financial derivatives in the $N$-step Cox-Ross-Rubinstein (CRR) model with $N+1$ time instants $t = 0, 1, \ldots, N$. Vanilla options are priced and hedged using backward induction, and exotic options with arbitrary claim payoffs are dealt with using the Clark-Ocone formula in discrete time.



## 3.1 Pricing Contingent Claims

Let us consider an attainable contingent claim with (random) claim payoff $C \geqslant 0$ and maturity $N \geqslant 1$. Recall that by the Definition 2.16 of attainability there exists a (predictable) self-financing portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ that *hedges* the claim with payoff $C$, in the sense that

$$\bar{\xi}_N \cdot \overline{S}_N = \sum_{k=0}^{d} \xi_N^{(k)} S_N^{(k)} = C \tag{3.1}$$

at time $N$. If (3.1) holds, then, investing the amount

$$V_0 = \bar{\xi}_1 \cdot \overline{S}_0 = \sum_{k=0}^{d} \xi_1^{(k)} S_0^{(k)} \tag{3.2}$$

at time $t = 0$, and the amount

$$V_t = \bar{\xi}_t \bullet \overline{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \tag{3.3}$$

at times $t = 1, 2, \ldots, N$ into a self-financing hedging portfolio $\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N}$ will allow one to hedge the option and to reach the perfect replication equality (3.1) at time $t = N$.

**Definition 3.1.** *The value* (3.2)-(3.3) *at time* $t$ *of a portfolio strategy* $(\xi_t)_{t=1,2,\ldots,N}$ *will be called an* arbitrage-free price *of* $C$*, and denoted by* $\pi_t(C)$*,* $t = 0, 1, \ldots, N$*, provided that this portfolio strategy:*

*a) is predictable and self-financing, and*

*b) hedges the claim* $C$ *at maturity time* $N$*, i.e. Relation* (3.1) *is satisfied.*

Recall that arbitrage-free prices can be used to ensure that financial derivatives are “marked” at their fair value (mark to market).

Next we develop a second approach to the pricing of contingent claims, based on conditional expectations and martingale arguments. We will need the following lemma, in which $\widetilde{V}_t := V_t/(1+r)^t$ denotes the discounted portfolio value, $t = 0, 1, \ldots, N$.

Relation (3.4) in the following lemma has a natural interpretation by saying that when a portfolio is self-financing the value $\widetilde{V}_t$ of the (discounted) portfolio at time $t$ is given by summing up the (discounted) trading profits or losses registered over all trading time periods from time 0 to time $t$. Note that in (3.4), the use of the vector of discounted asset prices

$$\overline{X}_t := \big(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\big), \qquad t = 0, 1, \ldots, N,$$

allows us to add up the discounted trading profits or losses $\bar{\xi}_t \bullet \big(\overline{X}_t - \overline{X}_{t-1}\big)$ since they are expressed in units of currency “at time 0”. Indeed, in general, \$1 at time $t = 0$ cannot be added to \$1 at time $t = 1$ without proper discounting.

**Lemma 3.2.** *The following statements are equivalent:*

*(i) The portfolio strategy* $\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N}$ *is self-financing, i.e.*

$$\bar{\xi}_t \bullet \overline{S}_t = \bar{\xi}_{t+1} \bullet \overline{S}_t, \qquad t = 1, 2, \ldots, N-1.$$

*(ii) Under discounting, we have* $\bar{\xi}_t \bullet \overline{X}_t = \bar{\xi}_{t+1} \bullet \overline{X}_t$ *for all* $t = 1, 2, \ldots, N-1$.

*(iii) The discounted portfolio value* $\widetilde{V}_t$ *can be written as the stochastic summation*

$$\widetilde{V}_t = \widetilde{V}_0 + \underbrace{\sum_{k=1}^{t} \bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)}_{\text{Sum of profits or losses}}, \qquad t = 0, 1, \ldots, N, \tag{3.4}$$

*of discounted trading profits or losses.*

*Proof.* First, the self-financing condition $(i)$, *i.e.*

$$\bar{\xi}_t \cdot \overline{S}_t = \bar{\xi}_{t+1} \cdot \overline{S}_t, \qquad t = 1, 2, \ldots, N-1,$$

is clearly equivalent to $(ii)$ by division of both sides by $(1+r)^{t-1}$.

Assuming now that $(ii)$ holds, by (2.8) we have

$$V_0 = \bar{\xi}_1 \cdot \overline{S}_0 \quad \text{and} \quad V_t = \bar{\xi}_t \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t = 1, 2, \ldots, N.$$

which shows that (3.4) is satisfied for $t = 0, 1$. Next, for $t = 1, 2, \ldots, N$ we have the telescoping identity

$$\begin{aligned}
\widetilde{V}_t &= \widetilde{V}_0 + \sum_{k=1}^{t} \big(\widetilde{V}_k - \widetilde{V}_{k-1}\big) \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \big(\bar{\xi}_k \cdot \overline{X}_k - \bar{\xi}_{k-1} \cdot \overline{X}_{k-1}\big) \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \big(\bar{\xi}_k \cdot \overline{X}_k - \bar{\xi}_k \cdot \overline{X}_{k-1}\big) \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big), \qquad t = 1, 2, \ldots, N.
\end{aligned}$$

Finally, assuming that $(iii)$ holds, we get

$$\widetilde{V}_t - \widetilde{V}_{t-1} = \bar{\xi}_t \cdot \big(\overline{X}_t - \overline{X}_{t-1}\big), \qquad t = 1, 2, \ldots, N,$$

which rewrites as

$$\bar{\xi}_t \cdot \overline{X}_t - \bar{\xi}_{t-1} \cdot \overline{X}_{t-1} = \bar{\xi}_t \cdot \big(\overline{X}_t - \overline{X}_{t-1}\big), \qquad t = 2, 3, \ldots, N,$$

or

$$\bar{\xi}_{t-1} \cdot \overline{X}_{t-1} = \bar{\xi}_t \cdot \overline{X}_{t-1}, \qquad t = 2, 3, \ldots, N,$$

which implies $(ii)$. □

In Relation (3.4), the term $\bar{\xi}_t \cdot \big(\overline{X}_t - \overline{X}_{t-1}\big)$ represents the (discounted) trading profit or loss

$$\widetilde{V}_t - \widetilde{V}_{t-1} = \bar{\xi}_t \cdot \big(\overline{X}_t - \overline{X}_{t-1}\big),$$

of the self-financing portfolio strategy $\big(\bar{\xi}_j\big)_{j=1,2,\ldots,N}$ over the time interval $(t-1,t]$, computed by multiplication of the portfolio allocation $\bar{\xi}_t$ with the change of price $\overline{X}_t - \overline{X}_{t-1}$, $t = 1,2,\ldots,N$.

**Remark 3.3.** *As a consequence of Lemma 3.2, if a contingent claim with payoff $C$ is attainable by a (predictable) self-financing portfolio strategy $\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N}$, then the discounted claim payoff*

$$\widetilde{C} := \frac{C}{(1+r)^N}$$

*rewrites as the sum of discounted trading profits or losses*

$$\widetilde{C} = \widetilde{V}_N = \bar{\xi}_N \cdot \overline{X}_N = \widetilde{V}_0 + \sum_{t=1}^{N} \bar{\xi}_t \cdot \big(\overline{X}_t - \overline{X}_{t-1}\big). \tag{3.5}$$

The sum (3.4) is also referred to as a discrete-time *stochastic integral* of the portfolio strategy $\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N}$ with respect to the random process $\big(\overline{X}_t\big)_{t=0,1,\ldots,N}$.

**Remark 3.4.** *By Proposition 2.13, the process $\big(\overline{X}_t\big)_{t=0,1,\ldots,N}$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, hence by the martingale transform Theorem 2.11 and Lemma 3.2, $(\widetilde{V}_t)_{t=0,1,\ldots,N}$ in (3.4) is also martingale under $\mathbb{P}^*$, provided that $\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N}$ is a self-financing and predictable process.*

The above remarks will be used in the proof of Theorem 3.5.

**Theorem 3.5.** *The arbitrage-free price $\pi_t(C)$ of any (integrable) attainable contingent with claim payoff $C$ is given by*

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0,1,\ldots,N, \tag{3.6}$$

*where $\mathbb{P}^*$ denotes any risk-neutral probability measure.*

*Proof. i) Short proof.* Since the claim payoff $C$ is attainable, there exists a self-financing portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ such that $C = V_N$, *i.e.* $\widetilde{C} = \widetilde{V}_N$. In addition, by Theorem 2.11 Lemma 3.2 the process $(\widetilde{V}_t)_{t=0,1,\ldots,N}$ is a martingale under $\mathbb{P}^*$, hence we have

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{V}_N \,\big|\, \mathcal{F}_t\big] = \mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathcal{F}_t\big], \qquad t = 0,1,\ldots,N, \tag{3.7}$$

which shows (3.8). To conclude, we note that by Definition 3.1 the arbitrage-free price $\pi_t(C)$ of the claim at time $t$ is equal to the value $V_t$ of the self-financing hedging $C$.

*ii) Long proof.* For completeness, we include a self-contained, step-by-step derivation of (3.7) by following the argument of Theorem 2.11, as follows. By Remark 3.3 we have

$$\begin{aligned}
\mathbb{E}^*\big[\widetilde{C}\,\big|\,\mathcal{F}_t\big] &= \mathbb{E}^*\big[\widetilde{V}_N\,\big|\,\mathcal{F}_t\big] \\
&= \mathbb{E}^*\left[\widetilde{V}_0 + \sum_{k=1}^{N} \bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^*\big[\widetilde{V}_0\,\big|\,\mathcal{F}_t\big] + \sum_{k=1}^{N} \mathbb{E}^*\big[\bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\big|\,\mathcal{F}_t\big] \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \mathbb{E}^*\big[\bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\big|\,\mathcal{F}_t\big] + \sum_{k=t+1}^{N} \mathbb{E}^*\big[\bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\big|\,\mathcal{F}_t\big] \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big) + \sum_{k=t+1}^{N} \mathbb{E}^*\big[\bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\big|\,\mathcal{F}_t\big] \\
&= \widetilde{V}_t + \sum_{k=t+1}^{N} \mathbb{E}^*\big[\bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\big|\,\mathcal{F}_t\big],
\end{aligned}$$

where we used Relation (3.4) of Lemma 3.2. In order to obtain (3.8) we need to show that

$$\sum_{k=t+1}^{N} \mathbb{E}^*\big[\bar{\xi}_k \cdot \big(\overline{X}_k - \overline{X}_{k-1}\big)\,\big|\,\mathcal{F}_t\big] = 0,$$

or

$$\mathbb{E}^*\big[\bar{\xi}_j \cdot \big(\overline{X}_j - \overline{X}_{j-1}\big)\,\big|\,\mathcal{F}_t\big] = 0,$$

for all $j = t+1, \ldots, N$. Since $0 \leqslant t \leqslant j-1$ we have $\mathcal{F}_t \subset \mathcal{F}_{j-1}$, hence by the tower property (A.33) of conditional expectations, we get

$$\mathbb{E}^*\big[\bar{\xi}_j \cdot \big(\overline{X}_j - \overline{X}_{j-1}\big)\,\big|\,\mathcal{F}_t\big] = \mathbb{E}^*\big[\mathbb{E}^*\big[\bar{\xi}_j \cdot \big(\overline{X}_j - \overline{X}_{j-1}\big)\,\big|\,\mathcal{F}_{j-1}\big]\,\big|\,\mathcal{F}_t\big],$$

therefore it suffices to show that

$$\mathbb{E}^*\big[\bar{\xi}_j \cdot \big(\overline{X}_j - \overline{X}_{j-1}\big)\,\big|\,\mathcal{F}_{j-1}\big] = 0, \quad j = 1, 2, \ldots, N.$$

We note that the portfolio allocation $\bar{\xi}_j$ over the time period $[j-1, j]$ is predictable, *i.e.* it is decided at time $j-1$, and it thus depends only on the information $\mathcal{F}_{j-1}$ known up to time $j-1$, hence

$$\mathbb{E}^*\big[\bar{\xi}_j \cdot \big(\overline{X}_j - \overline{X}_{j-1}\big)\,\big|\,\mathcal{F}_{j-1}\big] = \bar{\xi}_j \cdot \mathbb{E}^*\big[\overline{X}_j - \overline{X}_{j-1}\,\big|\,\mathcal{F}_{j-1}\big].$$

Finally we note that

$$\begin{aligned}\mathbb{E}^*\big[\overline{X}_j - \overline{X}_{j-1} \,\big|\, \mathcal{F}_{j-1}\big] &= \mathbb{E}^*\big[\overline{X}_j \,\big|\, \mathcal{F}_{j-1}\big] - \mathbb{E}^*\big[\overline{X}_{j-1} \,\big|\, \mathcal{F}_{j-1}\big] \\ &= \mathbb{E}^*\big[\overline{X}_j \,\big|\, \mathcal{F}_{j-1}\big] - \overline{X}_{j-1} \\ &= 0, \qquad j = 1, 2, \ldots, N,\end{aligned}$$

because $\big(\overline{X}_t\big)_{t=0,1,\ldots,N}$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, and this concludes the proof of (3.7). Let

$$\widetilde{C} = \frac{C}{(1+r)^N}$$

denote the discounted payoff of the claim $C$. We will show that under any risk-neutral probability measure $\mathbb{P}^*$ the discounted value of any self-financing portfolio hedging $C$ is given by

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathcal{F}_t\big], \qquad t = 0, 1, \ldots, N, \tag{3.8}$$

which shows that

$$V_t = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t]$$

after multiplication of both sides by $(1+r)^t$. Next, we note that (3.8) follows from the martingale transform result of Theorem 2.11. □

Note that (3.6) admits an interpretation in an insurance framework, in which $\pi_t(C)$ represents an insurance premium and $C$ represents the random value of an insurance claim made by a subscriber. In this context, the premium of the insurance contract reads as the average of the values (3.6) of the random claims after discounting for the time value of money.

**Remark 3.6.** *By Remark 3.4 the self-financing discounted portfolio value process*

$$\big(\widetilde{V}_t\big)_{t=0,1,\ldots,N} = \big((1+r)^{-t}\pi_t(C)\big)_{t=0,1,\ldots,N}$$

*hedging the claim $C$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$. This fact can be recovered from Theorem 3.5 as in Remark 3.3, since from the tower property (A.33) of conditional expectation we have*

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathcal{F}_t\big] \\ &= \mathbb{E}^*\big[\mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathcal{F}_{t+1}\big] \,\big|\, \mathcal{F}_t\big] \\ &= \mathbb{E}^*\big[\widetilde{V}_{t+1} \,\big|\, \mathcal{F}_t\big], \qquad t = 0, 1, \ldots, N-1.\end{aligned} \tag{3.9}$$

*This will also allow us to compute $V_t$ by backward induction on $t = 0, 1, \ldots, N-1$, starting from $V_N = C$, see (3.13) below.*



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In particular, for $t = 0$ we obtain the price at time 0 of the contingent claim with payoff $C$, *i.e.*

$$\pi_0(C) = \mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathcal{F}_0\big] = \mathbb{E}^*\big[\widetilde{C}\big] = \frac{1}{(1+r)^N}\mathbb{E}^*[C].$$

## 3.2 Pricing Vanilla Options in the CRR Model

In this section we consider the pricing of contingent claims in the discrete-time Cox-Ross-Rubinstein (Cox et al. (1979)) model of Section 2.6, with $d = 1$ and

$$S_t^{(0)} = S_0^{(0)}(1+r)^t, \qquad t = 0, 1, \ldots, N,$$

and

$$S_t^{(1)} = S_0^{(1)} \prod_{k=1}^{t}(1+R_k) = \left\{ \begin{array}{ll} (1+b)S_{t-1}^{(1)} & \text{if } R_t = b \\ \\ (1+a)S_{t-1}^{(1)} & \text{if } R_t = a \end{array} \right\} = (1+R_t)S_{t-1}^{(1)},$$

$t = 1, \ldots, N$. More precisely we are concerned with vanilla options whose payoffs depend on the terminal value of the underlying asset, as opposed to exotic options whose payoffs may depend on the whole path of the underlying asset price until expiration time.

Recall that the portfolio value process $(V_t)_{t=1,2,\ldots N}$ and the discounted portfolio value process $(\widetilde{V}_t)_{t=1,2,\ldots N}$ respectively satisfy

$$V_0 = \widetilde{V}_0 = \bar{\xi}_1 \cdot \bar{S}_0$$

and

$$V_t = \bar{\xi}_t \cdot \bar{S}_t \ \text{ and } \ \widetilde{V}_t = \frac{1}{(1+r)^t}V_t = \frac{1}{(1+r)^t}\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_t \cdot \overline{X}_t, \quad t = 1, 2, \ldots, N.$$

Here we will be concerned with the pricing of vanilla options with payoffs of the form

$$C = h\big(S_N^{(1)}\big),$$

*e.g.* $h(x) = (x - K)^+$ in the case of a European call option. Equivalently, the discounted claim payoff

$$\widetilde{C} = \frac{C}{(1+r)^N}$$

satisfies $\widetilde{C} = \widetilde{h}\big(S_N^{(1)}\big)$ with $\widetilde{h}(x) = h(x)/(1+r)^N$. For example in the case of the European call option with strike price $K$ we have

$$\widetilde{h}(x) = \frac{1}{(1+r)^N}(x-K)^+.$$

From Theorem 3.5, the discounted value of a portfolio hedging the attainable (discounted) claim payoff $\widetilde{C}$ is given by

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{h}\big(S_N^{(1)}\big) \bigm| \mathcal{F}_t\big], \qquad t = 0, 1, \ldots, N,$$

under the risk-neutral probability measure $\mathbb{P}^*$. As a consequence of Theorem 3.5, we have the following proposition.

**Proposition 3.7.** *The arbitrage-free price $\pi_t(C)$ at time $t = 0, 1, \ldots, N$ of the contingent claim with payoff $C = h\big(S_N^{(1)}\big)$ is given by*

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*\big[h\big(S_N^{(1)}\big) \bigm| \mathcal{F}_t\big], \qquad t = 0, 1, \ldots, N. \tag{3.10}$$

In the next proposition we implement the calculation of (3.10) in the CRR model.*

**Proposition 3.8.** *The price $\pi_t(C)$ of the contingent claim with payoff $C = h\big(S_N^{(1)}\big)$ satisfies*

$$\pi_t(C) = v\big(t, S_t^{(1)}\big), \qquad t = 0, 1, \ldots, N,$$

*where the function $v(t, x)$ is given by*

$$\begin{aligned} v(t,x) &= \frac{1}{(1+r)^{N-t}}\mathbb{E}^*\left[h\left(x\prod_{j=t+1}^{N}(1+R_j)\right)\right] \qquad (3.11)\\ &= \frac{1}{(1+r)^{N-t}}\sum_{k=0}^{N-t}\binom{N-t}{k}(p^*)^k(q^*)^{N-t-k}h\big(x(1+b)^k(1+a)^{N-t-k}\big), \end{aligned}$$

*where the risk-neutral probabilities $p^*, q^*$ are defined as*

$$p^* := \frac{r-a}{b-a} \qquad \text{and} \qquad q^* := 1 - p^* = \frac{b-r}{b-a}. \tag{3.12}$$

*Proof.* From the relations

$$S_N^{(1)} = S_t^{(1)}\prod_{j=t+1}^{N}(1+R_j),$$

* Download the corresponding (non-recursive) **IPython notebook** that can be run **here** or **here**.

and (3.10) we have, using Property (v) of the conditional expectation, see page 67, and the independence of the asset returns $\{R_1, \ldots, R_t\}$ and $\{R_{t+1}, \ldots, R_N\}$,

$$\begin{aligned}\pi_t(C) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[h(S_N^{(1)}) \,\big|\, \mathcal{F}_t\big] \\ &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\left[h\left(S_t^{(1)} \prod_{j=t+1}^N (1+R_j)\right) \,\Bigg|\, S_t^{(1)}\right] \\ &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\left[h\left(x \prod_{j=t+1}^N (1+R_j)\right)\right]_{x=S_t^{(1)}},\end{aligned}$$

where we used Property (v) of the conditional expectation, see page 67, and the independence of asset returns. Next, we note that the number of times $R_j$ is equal to $b$ for $j \in \{t+1, \ldots, N\}$, has a binomial distribution with parameter $(N-t, p^*)$ since the set of paths from time $t+1$ to time $N$ containing $j$ times "$(1+b)$" has cardinality $\binom{N-t}{j}$ and each such path has probability

$$(p^*)^j (q^*)^{N-t-j}, \qquad j = 0, \ldots, N-t.$$

In Figure 3.1 we enumerate the $120 = \binom{10}{7} = \binom{10}{3}$ possible paths corresponding to $n = 5$ and $k = 2$.

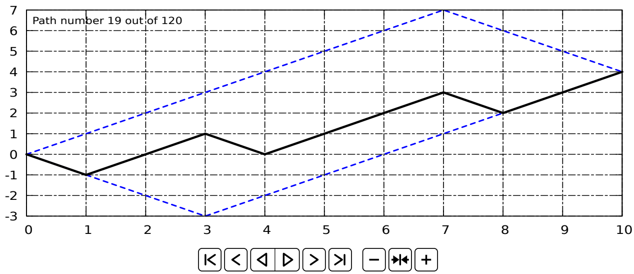


Fig. 3.1: Graph of $120 = \binom{10}{7} = \binom{10}{3}$ paths with $n = 5$ and $k = 2$.*

Hence we have

* Animated figure (works in Acrobat Reader).

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[h\big(S_N^{(1)}\big) \,\big|\, \mathcal{F}_t\big]$$
$$= \frac{1}{(1+r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (q^*)^{N-t-k} h\big(S_t^{(1)}(1+b)^k(1+a)^{N-t-k}\big).$$

□

In the above proof we have also shown that $\pi_t(C)$ is given by the conditional expected value

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[h\big(S_N^{(1)}\big) \,\big|\, \mathcal{F}_t\big] = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[h\big(S_N^{(1)}\big) \,\big|\, S_t^{(1)}\big]$$

given the value of $S_t^{(1)}$ at time $t = 0, 1, \ldots, N$, due to the Markov property of $\big(S_t^{(1)}\big)_{t=0,1,\ldots,N}$. In particular, the price of the claim with payoff $C$ is written as the average (path integral) of the values of the contingent claim over all possible paths starting from $S_t^{(1)}$.

### Market terms and data

Intrinsic value. The *intrinsic value* at time $t = 0, 1, \ldots, N$ of the option with payoff $C = h\big(S_N^{(1)}\big)$ is given by the immediate exercise payoff $h\big(S_t^{(1)}\big)$. The *extrinsic value* at time $t = 0, 1, \ldots, N$ of the option is the remaining difference $\pi_t(C) - h\big(S_t^{(1)}\big)$ between the option price $\pi_t(C)$ and the immediate exercise payoff $h\big(S_t^{(1)}\big)$. In general, the option price $\pi_t(C)$ decomposes as

$$\pi_t(C) = \underbrace{h\big(S_t^{(1)}\big)}_{\text{Intrinsic value}} + \underbrace{\pi_t(C) - h\big(S_t^{(1)}\big)}_{\text{Extrinsic value}}, \qquad t = 0, 1, \ldots, N.$$

Moneyness. The moneyness is the ratio of the intrinsic value of the option *vs.* the current price of the underlying asset, *i.e.*

$$M_t(C) := \frac{h\big(S_t^{(1)}\big)}{S_t^{(1)}}, \qquad t = 0, 1, \ldots, N.$$

The option is said to be "*in the money*" (ITM) when $M_t > 0$, "*at the money*" (ATM) when $M_t = 0$, and "*out of the money*" (OTM) when $M_t < 0$.

Gearing. The *gearing* at time $t = 0, 1, \ldots, N$ of the option with payoff $C = h\big(S_N^{(1)}\big)$ is defined as the ratio

$$\mathrm{G}_t := \frac{S_t^{(1)}}{\pi_t(C)} = \frac{S_t^{(1)}}{v\big(t, S_t^{(1)}\big)},$$

telling how many time the option price $v\big(t, S_t^{(1)}\big)$ is “contained” in the stock price $S_t^{(1)}$ at time $t = 0, 1, \ldots, N$.

Break-even price. The *break-even* price $\mathrm{BEP}_t$ of the underlying asset at time $t = 0, 1, \ldots, N$, see also Exercise 1.12, is the value of $S$ for which the intrinsic option value $h\big(S_t^{(1)}\big)$ equals the option price $\pi_t(C)$. In other words, $\mathrm{BEP}_t$ represents the price of the underlying asset for which we would break even if the option was exercised immediately. For European call options with payoff function $h(x) = (x - K)^+$, it is given by

$$\mathrm{BEP}_t := K + \pi_t(C) = K + v\big(t, S_t^{(1)}\big), \qquad t = 0, 1, \ldots, N,$$

whereas for European put options with payoff function $h(x) = (K - x)^+$, it is given by

$$\mathrm{BEP}_t := K - \pi_t(C) = K - v\big(t, S_t^{(1)}\big), \qquad t = 0, 1, \ldots, N.$$

Premium. The option *premium* $\mathrm{OP}_t$ can be defined as the variation required from the underlying asset price in order to reach the break-even price for which the intrinsic option payoff equals the current option price, *i.e.* we have

$$\mathrm{OP}_t := \frac{\mathrm{BEP}_t - S_t^{(1)}}{S_t^{(1)}} = \frac{K + v\big(t, S_t^{(1)}\big) - S_t^{(1)}}{S_t^{(1)}}, \qquad t = 0, 1, \ldots, N,$$

for European call options, and

$$\mathrm{OP}_t := \frac{S_t^{(1)} - \mathrm{BEP}_t}{S_t^{(1)}} = \frac{S_t^{(1)} + v\big(t, S_t^{(1)}\big) - K}{S_t^{(1)}}, \qquad t = 0, 1, \ldots, N,$$

for European put options. The term “premium” is sometimes also used to denote the arbitrage-free price $v\big(t, S_t^{(1)}\big)$ of the option.

### Pricing by backward induction in the CRR model

In the Cox-Ross-Rubinstein (Cox et al. (1979)) model of Section 2.6, the discounted portfolio value $\widetilde{V}_t$ can be computed by solving the *backward induction* relation (3.9), using the martingale property of the discounted portfolio value process $(\widetilde{V}_t)_{t=0,1,\ldots,N}$ under the risk-neutral probability measure $\mathbb{P}^*$.

**Proposition 3.9.** *The function $v(t,x)$ defined from the arbitrage-free prices of the contingent claim with payoff $C = h\big(S_N^{(1)}\big)$ at times $t = 0,1,\ldots,N$ by*

$$v\big(t, S_t^{(1)}\big) = V_t = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*\big[h\big(S_N^{(1)}\big) \,\big|\, \mathcal{F}_t\big]$$

*satisfies the backward recursion**

$$v(t,x) = \frac{q^*}{1+r}v\big(t+1, x(1+a)\big) + \frac{p^*}{1+r}v\big(t+1, x(1+b)\big), \qquad (3.13)$$

*with the terminal condition*

$$v(N,x) = h(x), \qquad x > 0.$$

*Proof.* By the tower property of conditional expectations, letting

$$\widetilde{v}\big(t, S_t^{(1)}\big) := \frac{1}{(1+r)^t}v\big(t, S_t^{(1)}\big), \qquad t = 0,1,\ldots,N,$$

we have, using the Markov property of $\big(S_t^{(1)}\big)_{t=0,1,\ldots,N}$,

$$\begin{aligned}
&\widetilde{v}\big(t, S_t^{(1)}\big) = \widetilde{V}_t \\
&= \mathbb{E}^*\big[\widetilde{h}\big(S_N^{(1)}\big) \,\big|\, \mathcal{F}_t\big] \\
&= \mathbb{E}^*\big[\mathbb{E}^*\big[\widetilde{h}\big(S_N^{(1)}\big) \,\big|\, \mathcal{F}_{t+1}\big] \,\big|\, \mathcal{F}_t\big] \\
&= \mathbb{E}^*\big[\widetilde{V}_{t+1} \,\big|\, \mathcal{F}_t\big] \\
&= \mathbb{E}^*\big[\widetilde{v}\big(t+1, S_{t+1}^{(1)}\big) \,\big|\, S_t^{(1)}\big] \\
&= \widetilde{v}\big(t+1, (1+a)S_t^{(1)}\big)\mathbb{P}^*(R_{t+1} = a) + \widetilde{v}\big(t+1, (1+b)S_t^{(1)}\big)\mathbb{P}^*(R_{t+1} = b) \\
&= q^*\widetilde{v}\big(t+1, (1+a)S_t^{(1)}\big) + p^*\widetilde{v}\big(t+1, (1+b)S_t^{(1)}\big),
\end{aligned}$$

which shows that $\widetilde{v}(t,x)$ satisfies

$$\widetilde{v}(t,x) = q^*\widetilde{v}\big(t+1, x(1+a)\big) + p^*\widetilde{v}\big(t+1, x(1+b)\big), \qquad (3.14)$$

while the terminal condition $\widetilde{V}_N = \widetilde{h}\big(S_N^{(1)}\big)$ implies

$$\widetilde{v}(N,x) = \widetilde{h}(x), \qquad x > 0.$$

□

* Download the corresponding (backward recursive) **IPython notebook** that can be run **here** or **here**.

Figure 3.2 presents a tree-based implementation of the pricing recursion (3.13).

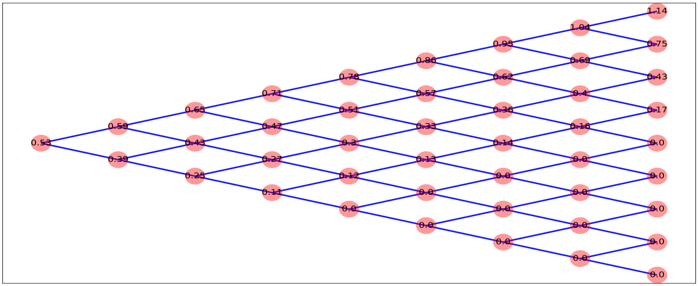


Fig. 3.2: Discrete-time call option pricing tree.

Note that the discrete-time recursion (3.13) can be connected to the continuous-time Black-Scholes PDE (6.2), cf. Exercises 6.15.

## 3.3 Hedging Contingent Claims

The basic idea of hedging is to allocate assets in a portfolio in order to protect oneself from a given risk. For example, a risk of increasing oil prices can be hedged by buying oil-related stocks, whose value should be positively correlated with the oil price. In this way, a loss connected to increasing oil prices could be compensated by an increase in the value of the corresponding portfolio.

In the setting of this chapter, hedging an attainable contingent claim with payoff $C$ means computing a self-financing portfolio strategy $\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N}$ such that

$$\bar{\xi}_N \cdot \bar{S}_N = C, \quad i.e. \quad \bar{\xi}_N \cdot \overline{X}_N = \widetilde{C}. \tag{3.15}$$

**Price, then hedge.**

The portfolio allocation $\bar{\xi}_N$ can be computed by first solving (3.15) for $\bar{\xi}_N$ from the payoff values $C$, based on the fact that the allocation $\bar{\xi}_N$ depends only on information up to time $N-1$, by the predictability of $\big(\bar{\xi}_k\big)_{1\leqslant k\leqslant N}$.

If the self-financing portfolio value $V_t$ is known, for example from (3.6), *i.e.*

$$V_t = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N, \tag{3.16}$$



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we may similarly compute $\bar{\xi}_t$ by solving $\bar{\xi}_t \cdot \overline{S}_t = V_t$ for all $t = 1, 2, \ldots, N$.

**Hedge, then price.**

If $V_t = \pi_t(C)$ has not been computed, we can use *backward induction* to compute a self-financing portfolio strategy. Starting from the values of $\bar{\xi}_N$ obtained by solving

$$\bar{\xi}_N \cdot \overline{S}_N = C,$$

we use the self-financing condition (3.17) to solve for $\bar{\xi}_{N-1}$, $\bar{\xi}_{N-2}$, $\ldots$, $\bar{\xi}_4$, down to $\bar{\xi}_3$, $\bar{\xi}_2$, and finally $\bar{\xi}_1$.

In order to implement this algorithm we can use the $N-1$ self-financing equations

$$\bar{\xi}_t \cdot \overline{X}_t = \bar{\xi}_{t+1} \cdot \overline{X}_t, \qquad t = 1, 2, \ldots, N-1, \tag{3.17}$$

allowing us in principle to compute the portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$.

Based on the values of $\bar{\xi}_N$ we can solve

$$\bar{\xi}_{N-1} \cdot \overline{S}_{N-1} = \bar{\xi}_N \cdot \overline{S}_{N-1}$$

for $\bar{\xi}_{N-1}$, then

$$\bar{\xi}_{N-2} \cdot \overline{S}_{N-2} = \bar{\xi}_{N-1} \cdot \overline{S}_{N-2}$$

for $\bar{\xi}_{N-2}$, and successively $\bar{\xi}_2$ down to $\bar{\xi}_1$. In Section 3.3 the backward induction (3.17) will be implemented in the CRR model, see the proof of Proposition 3.10, and Exercises 3.17 and 3.4 for an application in a two-step model.

The discounted value $\widetilde{V}_t$ at time $t$ of the portfolio claim can then be obtained from

$$\widetilde{V}_0 = \bar{\xi}_1 \cdot \overline{X}_0 \qquad \text{and} \qquad \widetilde{V}_t = \bar{\xi}_t \cdot \overline{X}_t, \qquad t = 1, 2, \ldots, N. \tag{3.18}$$

In addition we have shown in the proof of Theorem 3.5 that the price $\pi_t(C)$ of the claim payoff $C$ at time $t$ coincides with the value $V_t$ of any self-financing portfolio hedging the claim payoff $C$, *i.e.*

$$\pi_t(C) = V_t, \qquad t = 0, 1, \ldots, N,$$

as given by (3.18). Hence the price of the claim can be computed either algebraically by solving (3.15) and (3.17) using backward induction and then using (3.18), or by a probabilistic method by a direct evaluation of the discounted expected value (3.16).

The development of hedging algorithms has increased *credit exposure*, and *counterparty risk*, *e.g.* when one party is unable to deliver the option payoff stated in the contract.

## 3.4 Hedging Vanilla Options

In this section we implement the backward induction (3.17) of Section 3.3 for the hedging of contingent claims in the discrete-time Cox-Ross-Rubinstein model. Our aim is to compute a (predictable) self-financing portfolio strategy hedging a vanilla option with payoff of the form

$$C = h\big(S_N^{(1)}\big).$$

Since the discounted price $\widetilde{S}_t^{(0)}$ of the riskless asset satisfies

$$\widetilde{S}_t^{(0)} = (1+r)^{-t} S_t^{(0)} = S_0^{(0)},$$

we may sometimes write $S_0^{(0)}$ in place of $\widetilde{S}_t^{(0)}$. In Propositions 3.10 and 3.12 we present two different approaches to hedging and to the computation of the predictable process $\big(\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$, which is also called the *Delta*.

**Proposition 3.10.** Price, then hedge.* *The self-financing replicating portfolio strategy*

$$\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N} = \big(\xi_t^{(0)}\big(S_{t-1}^{(1)}\big), \xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\big)_{t=1,2,\ldots,N}$$

*hedging the contingent claim with payoff $C = h\big(S_N^{(1)}\big)$ is given by*

$$\begin{aligned} \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) &= \frac{v\big(t,(1+b)S_{t-1}^{(1)}\big) - v\big(t,(1+a)S_{t-1}^{(1)}\big)}{(b-a)S_{t-1}^{(1)}} \\ &= \frac{\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big) - \widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)}/(1+r)}, \end{aligned} \tag{3.19}$$

*where the function $v(t,x)$ is given by (3.11), and*

$$\begin{aligned} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) &= \frac{(1+b)v\big(t,(1+a)S_{t-1}^{(1)}\big) - (1+a)v\big(t,(1+b)S_{t-1}^{(1)}\big)}{(b-a)S_t^{(0)}} \\ &= \frac{(1+b)\widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big) - (1+a)\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big)}{(b-a)S_0^{(0)}}, \end{aligned} \tag{3.20}$$

*$t = 1,2,\ldots,N$, where the function $\widetilde{v}(t,x) = (1+r)^{-t}v(t,x)$ is given by* (3.11).

* Download the corresponding "price, then hedge" **IPython notebook** that can be run **here** or **here**.

*Proof.* We first compute the self-financing hedging strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ by solving

$$\bar{\xi}_t \bullet \overline{X}_t = \widetilde{V}_t, \qquad t = 1,2,\ldots,N,$$

from which we deduce the two equations

$$\begin{cases} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big)S_0^{(0)} + \xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\dfrac{1+a}{1+r}\widetilde{S}_{t-1}^{(1)} = \widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big) \\[2ex] \xi_t^{(0)}\big(S_{t-1}^{(1)}\big)S_0^{(0)} + \xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\dfrac{1+b}{1+r}\widetilde{S}_{t-1}^{(1)} = \widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big), \end{cases}$$

which can be solved as

$$\begin{cases} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) = \dfrac{(1+b)\widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big) - (1+a)\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big)}{(b-a)S_0^{(0)}} \\[2ex] \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = \dfrac{\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big) - \widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)}/(1+r)}, \end{cases}$$

$t = 1,2,\ldots,N$, which only depends on $S_{t-1}^{(1)}$, as expected, see also (1.20). This is consistent with the fact that $\xi_t^{(1)}$ represents the (possibly fractional) quantity of the risky asset to be present in the portfolio over the time period $[t-1,t]$ in order to hedge the claim payoff $C$ at time $N$, and is decided at time $t-1$. □

By applying (3.19) to the function $v(t,x)$ in (3.11), we find

$$\xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = \frac{1}{(1+r)^{N-t}}\sum_{k=0}^{N-t}\binom{N-t}{k}(p^*)^k(q^*)^{N-t-k}$$
$$\times \frac{h\big(S_{t-1}^{(1)}(1+b)^{k+1}(1+a)^{N-t-k}\big) - h\big(S_{t-1}^{(1)}(1+b)^k(1+a)^{N-t-k+1}\big)}{(b-a)S_t^{(1)}},$$

$t = 0,1,\ldots,N$.

The next Figure 3.3 presents a tree-based implementation of the risky hedging component (3.19).

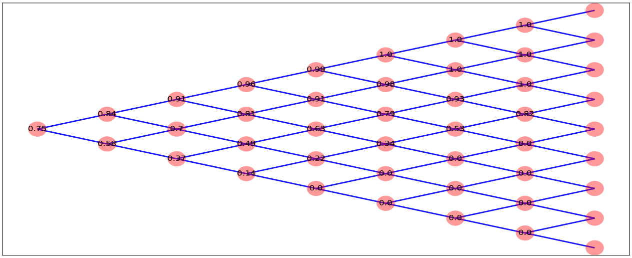


Fig. 3.3: Discrete-time call option hedging strategy (risky component).

The next Figure 3.4 presents a tree-based implementation of the riskless hedging component (3.20).

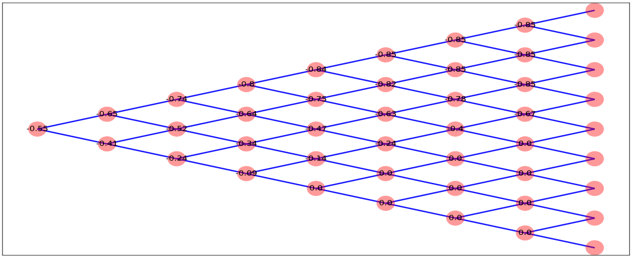


Fig. 3.4: Discrete-time call option hedging strategy (riskless component).

We can also check that the portfolio strategy

$$\big(\bar{\xi}_t\big)_{t=1,2,\ldots,N} = \big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N} = \big(\xi_t^{(0)}\big(S_{t-1}^{(1)}\big), \xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\big)_{t=1,2,\ldots,N}$$

given by (3.19)-(3.20) is self-financing, as follows:

$$\begin{aligned}
\bar{\xi}_{t+1} \cdot \overline{X}_t &= \xi_{t+1}^{(0)}\big(S_t^{(1)}\big) S_0^{(0)} + \xi_{t+1}^{(1)}\big(S_t^{(1)}\big) \widetilde{S}_t^{(1)} \\
&= S_0^{(0)} \frac{(1+b)\widetilde{v}\big(t+1, (1+a)S_t^{(1)}\big) - (1+a)\widetilde{v}\big(t+1, (1+b)S_t^{(1)}\big)}{(b-a)S_0^{(0)}} \\
&\quad + \widetilde{S}_t^{(1)} \frac{\widetilde{v}\big(t+1, (1+b)S_t^{(1)}\big) - \widetilde{v}\big(t+1, (1+a)S_t^{(1)}\big)}{(b-a)\widetilde{S}_t^{(1)}/(1+r)} \\
&= \frac{(1+b)\widetilde{v}\big(t+1, (1+a)S_t^{(1)}\big) - (1+a)\widetilde{v}\big(t+1, (1+b)S_t^{(1)}\big)}{b-a}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\widetilde{v}\big(t+1,(1+b)S_t^{(1)}\big)-\widetilde{v}\big(t+1,(1+a)S_t^{(1)}\big)}{(b-a)/(1+r)}\\
&=\frac{r-a}{b-a}\widetilde{v}\big(t+1,(1+b)S_t^{(1)}\big)+\frac{b-r}{b-a}\widetilde{v}\big(t+1,(1+a)S_t^{(1)}\big)\\
&=p^*\widetilde{v}\big(t+1,(1+b)S_t^{(1)}\big)+q^*\widetilde{v}\big(t+1,(1+a)S_t^{(1)}\big)\\
&=\widetilde{v}\big(t,S_t^{(1)}\big)\\
&=\xi_t^{(0)}\big(S_t^{(1)}\big)S_0^{(0)}+\xi_t^{(1)}\big(S_t^{(1)}\big)\widetilde{S}_t^{(1)}\\
&=\bar{\xi}_t\cdot\overline{X}_t,\qquad t=0,1,\ldots,N-1,
\end{aligned}
$$

where we used (3.14) or the martingale property of the discounted portfolio value process $\big(\tilde{v}(t,S_t^{(1)})\big)_{t=0,1,\ldots,N}$, see Lemma 3.2.

### Market terms and data

Delta. The *Delta* represents the quantity of underlying risky asset $S_t^{(1)}$ held in the portfolio over the time interval $[t-1,t]$. Here, it is denoted by $\xi_t^{(1)}\big(S_{t-1}^{(1)}\big)$ for $t=1,2,\ldots,N$.

Effective gearing. The *effective gearing* at time $t=1,2,\ldots,N$ of the option with payoff $C=h\big(S_N^{(1)}\big)$ is defined as the ratio

$$
\begin{aligned}
\mathrm{EG}_t &:= \mathrm{G}_t\xi_t^{(1)}\\
&= \frac{S_t^{(1)}}{\pi_t(C)}\xi_t^{(1)}\\
&= \frac{S_t^{(1)}\big(v\big(t,(1+b)S_{t-1}^{(1)}\big)-v\big(t,(1+a)S_{t-1}^{(1)}\big)\big)}{S_{t-1}^{(1)}v\big(t,S_t^{(1)}\big)(b-a)}\\
&= \frac{\big(v\big(t,(1+b)S_{t-1}^{(1)}\big)-v\big(t,(1+a)S_{t-1}^{(1)}\big)\big)/v\big(t,S_t^{(1)}\big)}{S_{t-1}^{(1)}(b-a)/S_t^{(1)}},\quad t=1,2,\ldots,N.
\end{aligned}
$$

The effective gearing $\mathrm{EG}_t=\xi_tS_t^{(1)}/\pi_t(C)$ can be interpreted as the *hedge ratio*, *i.e.* the percentage of the portfolio which is invested on the risky asset. It also allows one to represent the percentage change in the option price in terms of the potential percentage change $S_{t-1}^{(1)}(b-a)/S_t^{(1)}$ in the underlying asset price when the asset return switches from $a$ to $b$, as

$$
\frac{\big(v\big(t,(1+b)S_{t-1}^{(1)}\big)-v\big(t,(1+a)S_{t-1}^{(1)}\big)\big)}{v\big(t,S_t^{(1)}\big)}=\mathrm{EG}_t\times\frac{S_{t-1}^{(1)}(b-a)}{S_t^{(1)}}.
$$



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By Proposition 3.8 we have the following remark.

**Remark 3.11.** *i) If the function $x \mapsto h(x)$ is non-decreasing,* e.g. *in the case of European call options, then the function $x \mapsto \widetilde{v}(t,x)$ is also non-decreasing for all fixed $t = 0,1,\ldots,N$, hence the portfolio strategy $\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ defined by (3.11) or (3.19) satisfies $\xi_t^{(1)} \geqslant 0$, $t = 1,2,\ldots,N$ and there is not short selling.*

*ii) Similarly, we can show that when $x \mapsto h(x)$ is a non-increasing function,* e.g. *in the case of European put options, the portfolio allocation $\xi_t^{(1)} \leqslant 0$ is negative, $t = 1,2,\ldots,N$, i.e. short selling always occurs.*

As a consequence of (3.20), the discounted amounts $\xi_t^{(0)} S_0^{(0)}$ and $\xi_t^{(1)} \widetilde{S}_t^{(1)}$ respectively invested on the riskless and risky assets are given by

$$S_0^{(0)} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) = \frac{(1+b)\widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big) - (1+a)\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big)}{b-a} \tag{3.21}$$

and

$$\widetilde{S}_t^{(1)} \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = (1+R_t)\frac{\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big) - \widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big)}{b-a},$$

$t = 1,2,\ldots,N$.

Regarding the quantity $\xi_t^{(0)}$ of the riskless asset in the portfolio at time $t$, from the relation

$$\widetilde{V}_t = \bar{\xi}_t \cdot \overline{X}_t = \xi_t^{(0)} \widetilde{S}_t^{(0)} + \xi_t^{(1)} \widetilde{S}_t^{(1)}, \qquad t = 1,2,\ldots,N,$$

we also obtain

$$\begin{aligned} \xi_t^{(0)} &= \frac{\widetilde{V}_t - \xi_t^{(1)} \widetilde{S}_t^{(1)}}{\widetilde{S}_t^{(0)}} \\ &= \frac{\widetilde{V}_t - \xi_t^{(1)} \widetilde{S}_t^{(1)}}{S_0^{(0)}} \\ &= \frac{\widetilde{v}\big(t,S_t^{(1)}\big) - \xi_t^{(1)} \widetilde{S}_t^{(1)}}{S_0^{(0)}}, \qquad t = 1,2,\ldots,N. \end{aligned}$$

In the next proposition we compute the hedging strategy by backward induction, starting from the relation

$$\xi_N^{(1)}\big(S_{N-1}^{(1)}\big) = \frac{h\big((1+b)S_{N-1}^{(1)}\big) - h\big((1+a)S_{N-1}^{(1)}\big)}{(b-a)S_{N-1}^{(1)}},$$

and

$$\xi_N^{(0)}\big(S_{N-1}^{(1)}\big) = \frac{(1+b)h\big((1+a)S_{N-1}^{(1)}\big) - (1+a)h\big((1+b)S_{N-1}^{(1)}\big)}{(b-a)S_0^{(0)}(1+r)^N},$$

that follow from (3.19)-(3.20) applied to the claim payoff function $h(\cdot)$.

**Proposition 3.12.** Hedge, then price.* *The self-financing replicating portfolio strategy*

$$\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N} = \big(\xi_t^{(0)}\big(S_{t-1}^{(1)}\big), \xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\big)_{t=1,2,\ldots,N}$$

*hedging the contingent claim with payoff* $C = h\big(S_N^{(1)}\big)$ *is given from* (3.19) *at time* $t = N$ *by*

$$\xi_N^{(1)}\big(S_{N-1}^{(1)}\big) = \frac{h\big((1+b)S_{N-1}^{(1)}\big) - h\big((1+a)S_{N-1}^{(1)}\big)}{(b-a)S_{N-1}^{(1)}}, \tag{3.22}$$

*and*

$$\xi_N^{(0)}\big(S_{N-1}^{(1)}\big) = \frac{(1+b)h\big((1+a)S_{N-1}^{(1)}\big) - (1+a)h\big((1+b)S_{N-1}^{(1)}\big)}{(b-a)S_N^{(0)}}, \tag{3.23}$$

*and then inductively by*

$$\begin{aligned}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = {} & \frac{(1+b)\xi_{t+1}^{(1)}((1+b)S_{t-1}^{(1)}) - (1+a)\xi_{t+1}^{(1)}((1+a)S_{t-1}^{(1)})}{b-a} \\ & + S_0^{(0)} \frac{\xi_{t+1}^{(0)}((1+b)S_{t-1}^{(1)}) - \xi_{t+1}^{(0)}((1+a)S_{t-1}^{(1)})}{(b-a)\widetilde{S}_{t-1}^{(1)}/(1+r)},\end{aligned} \tag{3.24}$$

*and*

$$\begin{aligned}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) = {} & \frac{(1+a)(1+b)\widetilde{S}_{t-1}^{(1)}\big(\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big) - \xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)\big)}{(b-a)(1+r)S_0^{(0)}} \\ & + \frac{(1+b)\xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big) - (1+a)\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)}{b-a},\end{aligned} \tag{3.25}$$

$t = 1, 2, \ldots, N-1.$

*The pricing function* $\widetilde{v}(t,x) = (1+r)^{-t} v(t,x)$ *is then given by*

* Download the corresponding "hedge, then price" **IPython notebook** that can be run **here** or **here**.

$$\widetilde{v}\big(t, S_t^{(1)}\big) = S_0^{(0)}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) + \widetilde{S}_t^{(1)}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big), \quad t = 1, 2, \ldots, N.$$

*Proof.* Relations (3.22)-(3.23) follow from (3.19)-(3.20) stated at time $t = N$. Next, by the self-financing condition (3.17), we have

$$\bar{\xi}_t \cdot \overline{X}_t = \bar{\xi}_{t+1} \cdot \overline{X}_t, \qquad t = 1, 2, \ldots, N-1,$$

*i.e.*

$$\begin{cases} S_0^{(0)}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) + \widetilde{S}_{t-1}^{(1)}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\dfrac{1+b}{1+r} \\ \quad = \xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)S_0^{(0)} + \xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)\widetilde{S}_{t-1}^{(1)}\dfrac{1+b}{1+r} \\ \\ S_0^{(0)}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) + \widetilde{S}_{t-1}^{(1)}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\dfrac{1+a}{1+r} \\ \quad = \xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big)S_0^{(0)} + \xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big)\widetilde{S}_{t-1}^{(1)}\dfrac{1+a}{1+r}, \end{cases}$$

which can be solved as

$$\begin{aligned} \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = {} & \frac{(1+b)\xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big) - (1+a)\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big)}{b-a} \\ & + (1+r)S_0^{(0)}\frac{\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big) - \xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)}}, \end{aligned}$$

and

$$\begin{aligned} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) = {} & \frac{(1+a)(1+b)\widetilde{S}_{t-1}^{(1)}\big(\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big) - \xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)\big)}{(b-a)(1+r)S_0^{(0)}} \\ & + \frac{(1+b)\xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big) - (1+a)\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)}{b-a}, \end{aligned}$$

$t = 1, 2, \ldots, N-1$. □

By (3.24)-(3.25), we can check that the corresponding discounted portfolio value process

$$\big(\widetilde{V}_t\big)_{t=1,2,\ldots,N} = \big(\bar{\xi}_t \cdot \overline{X}_t\big)_{t=1,2,\ldots,N}$$

is a martingale under $\mathbb{P}^*$:

$$\begin{aligned} \widetilde{V}_t &= \bar{\xi}_t \cdot \overline{X}_t \\ &= S_0^{(0)}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) + \widetilde{S}_t^{(1)}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big) \\ &= \frac{(1+a)(1+b)\widetilde{S}_{t-1}^{(1)}\big(\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big) - \xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)\big)}{(b-a)(1+r)} \end{aligned}$$

$$
\begin{aligned}
&+S_0^{(0)} \frac{(1+b)\xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big)-(1+a)\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)}{(b-a)}\\
&+\widetilde{S}_t^{(1)} \frac{(1+b)\xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)-(1+a)\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big)}{b-a}\\
&+(1+r)\widetilde{S}_t^{(1)}S_0^{(0)} \frac{\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)-\xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)}}\\
&= \frac{r-a}{b-a}S_0^{(0)}\xi_{t+1}^{(0)}\big(S_t^{(1)}\big)+\frac{b-r}{b-a}S_0^{(0)}\xi_{t+1}^{(0)}\big(S_t^{(1)}\big)\\
&\quad +\frac{(r-a)(1+b)}{(b-a)(1+r)}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}\big(S_t^{(1)}\big)+\frac{(b-r)(1+a)}{(b-a)(1+r)}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}\big(S_t^{(1)}\big)\\
&= p^*S_0^{(0)}\xi_{t+1}^{(0)}\big(S_t^{(1)}\big)+q^*S_0^{(0)}\xi_{t+1}^{(0)}\big(S_t^{(1)}\big)\\
&\quad +p^*\frac{1+b}{1+r}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}\big(S_t^{(1)}\big)+q^*\frac{1+a}{1+r}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}\big(S_t^{(1)}\big)\\
&= \mathbb{E}^*\big[S_0^{(0)}\xi_{t+1}^{(0)}\big(S_t^{(1)}\big)+\widetilde{S}_{t+1}^{(1)}\xi_{t+1}^{(1)}\big(S_t^{(1)}\big)\,\big|\,\mathcal{F}_t\big]\\
&= \mathbb{E}^*\big[\widetilde{V}_{t+1}\,\big|\,\mathcal{F}_t\big],
\end{aligned}
$$

$t=1,2,\ldots,N-1$, as in Remark 3.4.

The next Figure 3.5 presents a tree-based implementation of the riskless hedging component (3.20).*

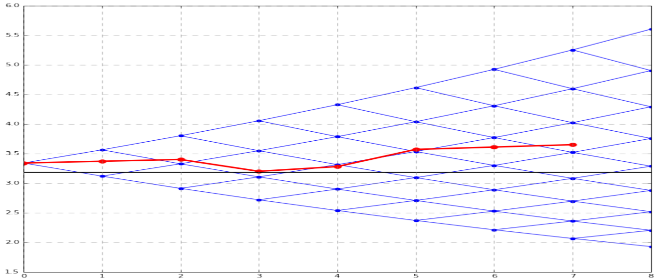



Fig. 3.5: Tree of asset prices in the CRR model.

The next Figure 3.6 presents a tree-based implementation of call option prices in the CRR model.

* Download the corresponding pricing and hedging **IPython (call)** or **IPython (put)** notebook that can be run **here** or **here**.

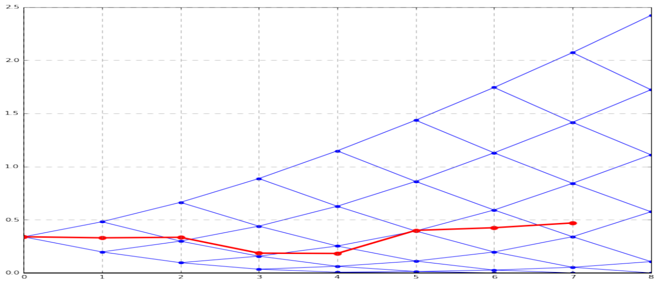



Fig. 3.6: Tree of option prices in the CRR model.

The next Figure 3.7 presents a tree-based implementation of risky hedging portfolio allocation in the CRR model.

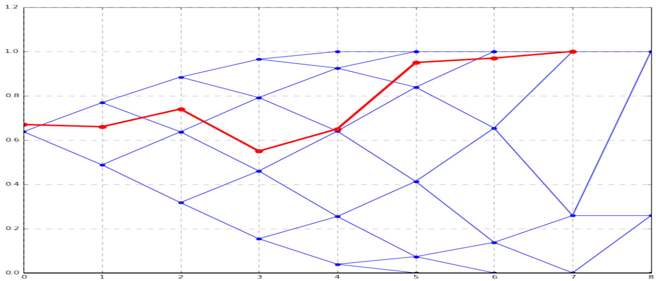



Fig. 3.7: Tree of hedging portfolio allocations in the CRR model.

## 3.5 Hedging Exotic Options

In this section we take $p = p^*$ given by (3.12) and we consider the hedging of path-dependent options. Here we choose to use the finite difference gradient and the discrete Clark-Ocone formula of stochastic analysis, see also §15-1 of Williams (1991), Lamberton and Lapeyre (1996), Ruiz de Chávez (2001), Föllmer and Schied (2004), Privault (2008), or Chapter 1 of Privault (2009). See Di Nunno et al. (2009) and Section 8.2 of Privault (2009) for a similar approach in continuous time. Given

$$\omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \Omega = \{-1, 1\}^N,$$

and $r = 1, 2, \ldots, N$, let

$$\omega_+^t := (\omega_1, \omega_2, \ldots, \omega_{t-1}, +1, \omega_{t+1}, \ldots, \omega_N)$$

and

$$\omega_-^t := (\omega_1, \omega_2, \ldots, \omega_{t-1}, -1, \omega_{t+1}, \ldots, \omega_N).$$

We also assume that the return $R_t(\omega)$ takes only two possible values

$$R_t(\omega_+^t) = b \quad \text{and} \quad R_t(\omega_-^t) = a, \qquad t = 1, 2, \ldots, N, \quad \omega \in \Omega.$$

**Definition 3.13.** *The operator $D_t$ is defined on any random variable $F$ by*

$$D_t F(\omega) = F(\omega_+^t) - F(\omega_-^t), \qquad t = 1, 2, \ldots, N. \tag{3.26}$$

We define the centered and normalized return $Y_t$ by

$$Y_t := \frac{R_t - r}{b - a} = \begin{cases} \dfrac{b - r}{b - a} = q^*, & \omega_t = +1, \\ \\ \dfrac{a - r}{b - a} = -p^*, & \omega_t = -1, \end{cases} \qquad t = 1, 2, \ldots, N.$$

Note that under the risk-neutral probability measure $\mathbb{P}^*$ we have

$$\begin{aligned} \mathbb{E}^*[Y_t] &= \mathbb{E}^*\left[\frac{R_t - r}{b - a}\right] \\ &= \frac{a - r}{b - a}\mathbb{P}^*(R_t = a) + \frac{b - r}{b - a}\mathbb{P}^*(R_t = b) \\ &= \frac{a - r}{b - a} \times \frac{b - r}{b - a} + \frac{b - r}{b - a} \times \frac{r - a}{b - a} \\ &= 0, \end{aligned}$$

and

$$\mathrm{Var}\,[Y_t] = p^*(q^*)^2 + q^*(p^*)^2 = p^* q^*, \qquad t = 1, 2, \ldots, N.$$

In addition, the discounted asset price increment reads

$$\begin{aligned} \widetilde{S}_t^{(1)} - \widetilde{S}_{t-1}^{(1)} &= \widetilde{S}_{t-1}^{(1)} \frac{1 + R_t}{1 + r} - \widetilde{S}_{t-1}^{(1)} \\ &= \frac{R_t - r}{1 + r} \widetilde{S}_{t-1}^{(1)} \\ &= \frac{b - a}{1 + r} Y_t \widetilde{S}_{t-1}^{(1)}, \qquad t = 1, 2, \ldots, N. \end{aligned}$$

We also have

$$D_t Y_t = \frac{b - r}{b - a} - \left(\frac{a - r}{b - a}\right) = 1, \qquad t = 1, 2, \ldots, N,$$

and

$$\begin{aligned} D_t S_N^{(1)} &= S_0^{(1)}(1+b)\prod_{\substack{k=1\\k\neq t}}^N(1+R_k) - S_0^{(1)}(1+a)\prod_{\substack{k=1\\k\neq t}}^N(1+R_k) \\ &= (b-a)S_0^{(1)}\prod_{\substack{k=1\\k\neq t}}^N(1+R_k) \\ &= S_0^{(1)}\frac{b-a}{1+R_t}\prod_{k=1}^N(1+R_k) \\ &= \frac{b-a}{1+R_t}S_N^{(1)}, \qquad t=1,2,\ldots,N. \end{aligned}$$

The following stochastic integral decomposition formula for the functionals of the binomial process is known as the Clark-Ocone formula in discrete time, cf. *e.g.* Privault (2009), Proposition 1.7.1.

**Proposition 3.14.** *For any square-integrable random variables $F$ on $\Omega$, we have*

$$F = \mathbb{E}^*[F] + \sum_{k\geqslant 1} Y_k \mathbb{E}^*[D_k F \mid \mathcal{F}_{k-1}]. \tag{3.27}$$

The Clark-Ocone formula (3.27) has the following consequence.

**Corollary 3.15.** *Assume that $(M_k)_{k\in\mathbb{N}}$ is a square-integrable $(\mathcal{F}_k)_{k\in\mathbb{N}}$-martingale. Then, we have*

$$M_N = \mathbb{E}^*[M_N] + \sum_{k=1}^N Y_k D_k M_k, \qquad N \geqslant 0.$$

*Proof.* We have

$$\begin{aligned} M_N &= \mathbb{E}^*[M_N] + \sum_{k\geqslant 1} Y_k \mathbb{E}^*[D_k M_N \mid \mathcal{F}_{k-1}] \\ &= \mathbb{E}^*[M_N] + \sum_{k\geqslant 1} Y_k D_k \mathbb{E}^*[M_N \mid \mathcal{F}_k] \\ &= \mathbb{E}^*[M_N] + \sum_{k\geqslant 1} Y_k D_k M_k \\ &= \mathbb{E}^*[M_N] + \sum_{k=1}^N Y_k D_k M_k. \end{aligned}$$

□

In addition to the Clark-Ocone formula we also state a discrete-time analog of Itô's change of variable formula, which can be useful for option hedging. The next result extends Proposition 1.13.1 of Privault (2009) by removing the unnecessary martingale requirement on $(M_t)_{n\in\mathbb{N}}$.

**Proposition 3.16.** *Let* $(Z_n)_{n\in\mathbb{N}}$ *be an* $(\mathcal{F}_n)_{n\in\mathbb{N}}$*-adapted process and let* $f : \mathbb{R}\times\mathbb{N} \longrightarrow \mathbb{R}$ *be a given function. We have*

$$f(Z_t,t) = f(Z_0,0) + \sum_{k=1}^{t} D_k f(Z_k,k) Y_k + \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big), \qquad t \geqslant 0. \quad (3.28)$$

*Proof.* First, we note that the process

$$M_t := f(Z_t,t) - \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big),$$

$t = 0,1,\ldots,N$, is a martingale under $\mathbb{P}^*$. Indeed, we have

$$\begin{aligned}
&\mathbb{E}^*\left[ f(Z_t,t) - \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big) \;\Big|\; \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}^*[f(Z_t,t) \mid \mathcal{F}_{t-1}] \\
&\quad - \sum_{k=1}^{t} \big(\mathbb{E}^*[\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{t-1}] - \mathbb{E}^*[\mathbb{E}^*[f(Z_{k-1},k-1) \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_{t-1}]\big) \\
&= \mathbb{E}^*[f(Z_t,t) \mid \mathcal{F}_{t-1}] - \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big) \\
&= f(Z_{t-1},t-1) - \sum_{k=1}^{t-1} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big), \qquad t \geqslant 1.
\end{aligned}$$

Next, applying Corollary 3.15 to the martingale $(M_t)_{t=0,1,\ldots,N}$, we have

$$\begin{aligned}
f(Z_t,t) &= M_t + \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big) \\
&= \mathbb{E}^*[M_t] + \sum_{k=1}^{t} Y_k D_k M_k + \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big) \\
&= f(Z_0,0) + \sum_{k=1}^{t} Y_k D_k f(Z_k,k) + \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big),
\end{aligned}$$

$t \geqslant 0$, as

$$D_k\big(\mathbb{E}^*[f(Z_k,k) \mid \mathcal{F}_{k-1}] - f(Z_{k-1},k-1)\big) = 0, \qquad k \geqslant 1.$$

□

Note that if $(Z_t)_{t\in\mathbb{N}}$ is a discrete-time $(\mathcal{F}_t)_{t\in\mathbb{N}}$-martingale in $L^2(\Omega)$ written as

$$Z_t = Z_0 + \sum_{k=1}^{t} u_k Y_k, \qquad t \geqslant 0,$$

where $(u_t)_{t\in\mathbb{N}}$ is an $(\mathcal{F}_t)_{t\in\mathbb{N}}$-predictable process locally* in $L^2(\Omega\times\mathbb{N})$, then we have

$$D_t f(Z_t, t) = f\big(Z_{t-1} + q u_t, t\big) - f\big(Z_{t-1} - p u_t, t\big), \tag{3.29}$$

$t = 1, 2, \ldots, N$. On the other hand, the term

$$\mathbb{E}^*\big[f(Z_t, t) - f(Z_{t-1}, t-1) \mid \mathcal{F}_{t-1}\big]$$

in (3.28) is analog to the finite variation part in the continuous-time Itô formula, and can be written as

$$p f\big(Z_{t-1} + q u_t, t\big) + q f\big(Z_{t-1} - p u_t, t\big) - f\big(Z_{t-1}, t-1\big).$$

When $(f(Z_t, t))_{t\in\mathbb{N}}$ is a martingale, Proposition 3.16 naturally recovers the decomposition

$$\begin{aligned} f(Z_t, t) &= f(Z_0, 0) \\ &\quad + \sum_{k=1}^{t} \big(f\big(Z_{k-1} + q u_k, k\big) - f\big(Z_{k-1} - p u_k, k\big)\big) Y_k \\ &= f(Z_0, 0) + \sum_{k=1}^{t} Y_k D_k f(Z_k, k), \end{aligned} \tag{3.30}$$

that follows from Corollary 3.15 as well as from Proposition 3.14. In this case, the Clark-Ocone formula (3.27) and the change of variable formula (3.30) both coincide and we have in particular

$$D_k f(Z_k, k) = \mathbb{E}^*\big[D_k f(Z_N, N) \mid \mathcal{F}_{k-1}\big],$$

$k = 1, 2, \ldots, N$. For example, this recovers the martingale representation

$$\begin{aligned} \widetilde{S}_t^{(1)} &= S_0^{(1)} + \sum_{k=1}^{t} Y_k D_k \widetilde{S}_k^{(1)} \\ &= S_0^{(1)} + \frac{b-a}{1+r} \sum_{k=1}^{t} \widetilde{S}_{k-1}^{(1)} Y_k \\ &= S_0^{(1)} + \sum_{k=1}^{t} \widetilde{S}_{k-1}^{(1)} \frac{R_k - r}{1+r} \end{aligned}$$

* *i.e.* $u(\cdot)\mathbb{1}_{[0,N]}(\cdot) \in L^2(\Omega\times\mathbb{N})$ for all $N > 0$.

$$= S_0^{(1)} + \sum_{k=1}^{t} \big(\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}\big),$$

of the discounted asset price.

Our goal is to hedge an arbitrary claim payoff $C$ on $\Omega$, *i.e.* given an $\mathcal{F}_N$-measurable random variable $C$ we search for a portfolio strategy $\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ such that the equality

$$C = V_N = \xi_N^{(0)} S_N^{(0)} + \xi_N^{(1)} S_N^{(1)} \tag{3.31}$$

holds, where $S_N^{(0)} = S_0^{(0)}(1+r)^N$ denotes the value of the riskless asset at time $N \geqslant 0$.

The next proposition is the main result of this section, and provides a solution to the hedging problem under the constraint (3.31).

**Proposition 3.17.** Hedge, then price. *Given a contingent claim with payoff $C$, let $\xi_0^{(1)} = 0$,*

$$\xi_t^{(1)} = \frac{(1+r)^{-(N-t)}}{(b-a)S_{t-1}^{(1)}} \mathbb{E}^*[D_t C \mid \mathcal{F}_{t-1}], \qquad t = 1,2,\ldots,N, \tag{3.32}$$

*and*

$$\xi_t^{(0)} = \frac{1}{S_t^{(0)}} \big((1+r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t] - \xi_t^{(1)} S_t^{(1)}\big), \tag{3.33}$$

*$t = 0,1,\ldots,N$. Then, the portfolio strategy $\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ is predictable and self financing, and we have*

$$V_t = \xi_t^{(0)} S_t^{(0)} + \xi_t^{(1)} S_t^{(1)} = (1+r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t], \quad t = 0,1,\ldots,N.$$

*In particular, we have $V_N = C$, hence $\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ is a hedging strategy leading to $C$.*

*Proof.* Let $\big(\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ be defined by (3.32), and consider the process $\big(\xi_t^{(0)}\big)_{t=0,1,\ldots,N}$ recursively defined by

$$\xi_0^{(0)} = (1+r)^{-N} \frac{\mathbb{E}^*[C]}{S_0^{(1)}} \quad \text{and} \quad \xi_{t+1}^{(0)} = \xi_t^{(0)} - \frac{\big(\xi_{t+1}^{(1)} - \xi_t^{(1)}\big) S_t^{(1)}}{S_t^{(0)}},$$

$t = 0,1,\ldots,N-1$. Then $\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ satisfies the self-financing condition

$$S_t^{(0)}\big(\xi_{t+1}^{(0)}-\xi_t^{(0)}\big)+S_t^{(1)}\big(\xi_{t+1}^{(1)}-\xi_t^{(1)}\big)=0, \qquad t=1,2,\ldots,N-1.$$

Let now

$$V_0 := \frac{1}{(1+r)^N}\mathbb{E}^*[C], \qquad V_t := \xi_t^{(0)}S_t^{(0)}+\xi_t^{(1)}S_t^{(1)}, \quad t=1,2,\ldots,N,$$

and

$$\widetilde{V}_t = \frac{V_t}{(1+r)^t} \qquad t=0,1,\ldots,N.$$

Since $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ is self-financing, by Lemma 3.2 we have

$$\widetilde{V}_t = \widetilde{V}_0 + (b-a)\sum_{k=1}^{t}\frac{1}{(1+r)^k}Y_k\xi_k^{(1)}S_{k-1}^{(1)}, \tag{3.34}$$

$t=1,2,\ldots,N$. On the other hand, from the Clark-Ocone formula (3.27) and the definition of $\big(\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ we have

$$\begin{aligned}
&\frac{1}{(1+r)^N}\mathbb{E}^*[C \mid \mathcal{F}_t]\\
&\quad = \frac{1}{(1+r)^N}\mathbb{E}^*\left[\mathbb{E}^*[C]+\sum_{k=0}^{N}Y_k\mathbb{E}^*[D_kC \mid \mathcal{F}_{k-1}]\,\Big|\,\mathcal{F}_t\right]\\
&\quad = \frac{1}{(1+r)^N}\mathbb{E}^*[C]+\frac{1}{(1+r)^N}\sum_{k=0}^{t}\mathbb{E}^*[D_kC \mid \mathcal{F}_{k-1}]Y_k\\
&\quad = \frac{1}{(1+r)^N}\mathbb{E}^*[C]+(b-a)\sum_{k=0}^{t}\frac{1}{(1+r)^k}\xi_k^{(1)}S_{k-1}^{(1)}Y_k\\
&\quad = \widetilde{V}_t
\end{aligned}$$

from (3.34). Hence

$$\widetilde{V}_t = \frac{1}{(1+r)^N}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t=0,1,\ldots,N,$$

and

$$V_t = (1+r)^{-(N-t)}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t=0,1,\ldots,N. \tag{3.35}$$

In particular, (3.35) shows that we have $V_N = C$. To conclude the proof we note that from the relation $V_t = \xi_t^{(0)}S_t^{(0)}+\xi_t^{(1)}S_t^{(1)}$, $t=1,2,\ldots,N$, the process $\big(\xi_t^{(0)}\big)_{t=1,2,\ldots,N}$ coincides with $\big(\xi_t^{(0)}\big)_{t=1,2,\ldots,N}$ defined by (3.33). $\square$

**Example - Vanilla options**

From Proposition 3.8, the price $\pi_t(C)$ of the contingent claim with payoff $C = h\big(S_N^{(1)}\big)$ is given by

$$\pi_t(C) = v\big(t, S_t^{(1)}\big),$$

where the function $v(t,x)$ is given by

$$\begin{aligned} v\big(t, S_t^{(1)}\big) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t] \\ &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\left[ h\left( x \prod_{j=t+1}^{N} (1+R_j) \right) \right]_{x = S_t^{(1)}}. \end{aligned}$$

Note that in this case we have $C = v\big(N, S_N^{(1)}\big)$, $\mathbb{E}[C] = v(0, M_0)$, and the discounted claim payoff $\widetilde{C} = C/(1+r)^N = \widetilde{v}\big(N, S_N^{(1)}\big)$ satisfies

$$\begin{aligned} \widetilde{C} &= \mathbb{E}^*\big[\widetilde{C}\big] + \sum_{t=1}^{N} Y_t \mathbb{E}^*\big[D_t \widetilde{v}\big(N, S_N^{(1)}\big) \,\big|\, \mathcal{F}_{t-1}\big] \\ &= \mathbb{E}^*\big[\widetilde{C}\big] + \sum_{t=1}^{N} Y_t D_t \widetilde{v}\big(t, S_t^{(1)}\big) \\ &= \mathbb{E}^*\big[\widetilde{C}\big] + \sum_{t=1}^{N} \frac{1}{(1+r)^t} Y_t D_t v\big(t, S_t^{(1)}\big) \\ &= \mathbb{E}^*\big[\widetilde{C}\big] + \sum_{t=1}^{N} Y_t D_t \mathbb{E}^*\big[\widetilde{v}\big(N, S_N^{(1)}\big) \,\big|\, \mathcal{F}_t\big] \\ &= \mathbb{E}^*\big[\widetilde{C}\big] + \frac{1}{(1+r)^N} \sum_{t=1}^{N} Y_t D_t \mathbb{E}[C \mid \mathcal{F}_t], \end{aligned}$$

hence we have

$$\mathbb{E}^*\big[D_t v\big(N, S_N^{(1)}\big) \,\big|\, \mathcal{F}_{t-1}\big] = (1+r)^{N-t} D_t v\big(t, S_t^{(1)}\big), \quad t = 1, 2, \ldots, N,$$

and by Proposition 3.17 the hedging strategy for $C = h\big(S_N^{(1)}\big)$ is given by

$$\begin{aligned} \xi_t^{(1)} &= \frac{(1+r)^{-(N-t)}}{(b-a) S_{t-1}^{(1)}} \mathbb{E}^*\big[D_t h\big(S_N^{(1)}\big) \,\big|\, \mathcal{F}_{t-1}\big] \\ &= \frac{(1+r)^{-(N-t)}}{(b-a) S_{t-1}^{(1)}} \mathbb{E}^*\big[D_t v\big(N, S_N^{(1)}\big) \,\big|\, \mathcal{F}_{t-1}\big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(b-a)S_{t-1}^{(1)}} D_t v\big(t, S_t^{(1)}\big) \\
&= \frac{1}{(b-a)S_{t-1}^{(1)}} \big(v\big(t, S_{t-1}^{(1)}(1+b)\big) - v\big(t, S_{t-1}^{(1)}(1+a)\big)\big) \\
&= \frac{1}{(b-a)\widetilde{S}_{t-1}^{(1)}/(1+r)} \big(\widetilde{v}\big(t, S_{t-1}^{(1)}(1+b)\big) - \widetilde{v}\big(t, S_{t-1}^{(1)}(1+a)\big)\big),
\end{aligned}
$$

$t = 1, 2, \ldots, N$, which recovers Proposition 3.10 as a particular case. Note that $\xi_t^{(1)}$ is nonnegative (*i.e.* there is no short selling) when $f$ is a non-decreasing function, because $a < b$. This is in particular true in the case of the European call option, for which we have $f(x) = (x - K)^+$.

## 3.6 Convergence of the CRR Model

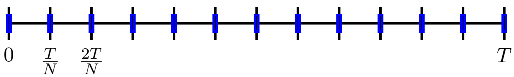
As the pricing formulas (3.11) in the CRR model can be difficult to implement for large values on $N$, in this section we consider the convergence of the discrete-time model to the continuous-time Black Scholes model.

### Continuous compounding - riskless asset

Consider the discretization

$$
\left[0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N}, T\right]
$$

of the time interval $[0, T]$ into $N$ time steps.




Note that

$$
\lim_{N\to\infty} (1 + r)^N = \infty,
$$

when $r > 0$, thus we need to renormalize $r$ so that the interest rate on each time interval becomes $r_N$, with $\lim_{N\to\infty} r_N = 0$. It turns out that the correct renormalization is

$$
r_N := r\frac{T}{N}, \tag{3.36}
$$

so that for $T \geqslant 0$,

$$
\begin{aligned}
\lim_{N\to\infty} (1 + r_N)^N &= \lim_{N\to\infty} \left(1 + r\frac{T}{N}\right)^N \\
&= \lim_{N\to\infty} \exp\left(N \log\left(1 + r\frac{T}{N}\right)\right)
\end{aligned}
$$

$$= e^{rT}. \tag{3.37}$$

Hence the price $S_t^{(0)}$ of the riskless asset is given by

$$S_t^{(0)} = S_0^{(0)} e^{rt}, \qquad t \geqslant 0, \tag{3.38}$$

which solves the differential equation

$$\frac{dS_t^{(0)}}{dt} = rS_t^{(0)}, \qquad S_0^{(0)} = 1, \quad t \geqslant 0. \tag{3.39}$$

We can also write

$$dS_t^{(0)} = rS_t^{(0)} dt, \qquad \text{or} \qquad \frac{dS_t^{(0)}}{S_t^{(0)}} = rdt, \tag{3.40}$$

and using $dS_t^{(0)} \simeq S_{t+dt}^{(0)} - S_t^{(0)}$ we can discretize this equation by saying that the *infinitesimal return* $(S_{t+dt}^{(0)} - S_t^{(0)})/S_t^{(0)}$ of the riskless asset equals $rdt$ on the small time interval $[t, t+dt]$, *i.e.*

$$\frac{S_{t+dt}^{(0)} - S_t^{(0)}}{S_t^{(0)}} = rdt.$$

In this sense, the rate $r$ is the instantaneous interest rate per unit of time.

The same equation rewrites in *integral form* as

$$S_T^{(0)} - S_0^{(0)} = \int_0^T dS_t^{(0)} = r \int_0^T S_t^{(0)} dt.$$

### Continuous compounding - risky asset

We recall the central limit theorem.

**Theorem 3.18.** *Let $(X_n)_{n \geqslant 1}$ be a sequence of independent and identically distributed random variables with finite mean $\mu = \mathbb{E}[X_1]$ and variance $\sigma^2 := \mathrm{Var}[X_1] < \infty$. We have the convergence* in distribution

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}} = \mathcal{N}(0, \sigma^2),$$

*or equivalently*

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} = \mathcal{N}(0, 1).$$

The convergence in distribution of Theorem 3.18 is illustrated by the Galton board simulation of Figure 3.8, which shows the convergence of the binomial random walk to a Gaussian distribution in large time.

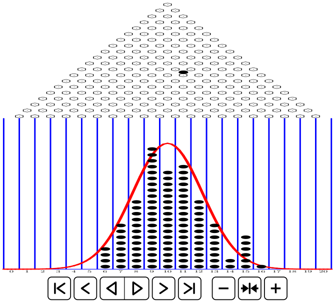


Fig. 3.8: Galton board simulation.*

Figure 3.9 pictures a real-life Galton board.

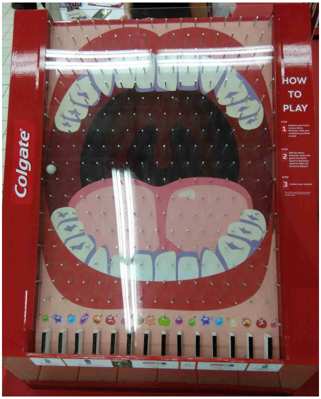



Fig. 3.9: A real-life Galton board at Jurong Point # 03-01.

In the CRR model we need to replace the standard Galton board by its multiplicative version, which shows that as $N$ tends to infinity the distribution of $S_N^{(1)}$ converges to the *lognormal distribution* with probability density function of the form

* The animation works in Acrobat Reader on the entire pdf file.

$$x \longmapsto f(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp\left( -\frac{\left(-\left(r-\sigma^2/2\right)T + \log\left(x/S_0^{(1)}\right)\right)^2}{2\sigma^2 T} \right),$$

$x > 0$, with location parameter $(r-\sigma^2/2)T + \log S_0^{(1)}$ and scale parameter $\sigma\sqrt{T}$, or log-variance $\sigma^2 T$, as illustrated in the modified Galton board of Figure 3.10 below, see also Figure 5.6 and Exercise 5.1.

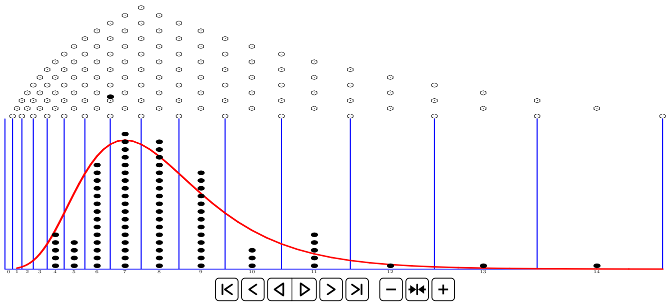


Fig. 3.10: Multiplicative Galton board simulation.*

Exercise: Check that $f$ is the probability density function of $\mathrm{e}^{\sigma X + (r-\sigma^2/2)T}$ where $X \simeq \mathcal{N}(0, T)$ is a centered Gaussian random variable with variance $T > 0$.

In addition to the renormalization (3.36) for the interest rate $r_N := rT/N$, we need to apply a similar renormalization to the coefficients $a$ and $b$ of the CRR model. In what follows, we let $\sigma > 0$ denote a positive parameter called the volatility, which quantifies the range of random fluctuations.

**Definition 3.19.** *We let the normalized market returns $a_N$, $b_N$ be defined as*

$$a_N = (1+r_N)\left(1-\sigma\sqrt{\frac{T}{N}}\right) - 1 \quad \textit{and} \quad b_N = (1+r_N)\left(1+\sigma\sqrt{\frac{T}{N}}\right) - 1. \tag{3.41}$$

We note that the returns $a_N$, $b_N$ satisfy the relations

$$\frac{1+a_N}{1+r_N} = 1-\sigma\sqrt{\frac{T}{N}} \quad \text{and} \quad \frac{1+b_N}{1+r_N} = 1+\sigma\sqrt{\frac{T}{N}}.$$

* The animation works in Acrobat Reader on the entire pdf file.

Consider the random return $R_k^{(N)} \in \{a_N, b_N\}$ and the price process defined as

$$S_{t,N}^{(1)} = S_0^{(1)} \prod_{k=1}^{t} (1 + R_k^{(N)}), \qquad t = 1, 2, \ldots, N. \tag{3.42}$$

Note that the risk-neutral probabilities are given by

$$\begin{aligned} \mathbb{P}^*\big(R_t^{(N)} = a_N\big) &= \frac{b_N - r_N}{b_N - a_N} \\ &= \frac{(1+r_N)\big(1+\sigma\sqrt{T/N}\big) - 1 - r_N}{(1+r_N)\big(1+\sigma\sqrt{T/N}\big) - (1+r_N)\big(1-\sigma\sqrt{T/N}\big)} \\ &= \frac{1}{2}, \quad t = 1, 2, \ldots, N, \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} \mathbb{P}^*\big(R_t^{(N)} = b_N\big) &= \frac{r_N - a_N}{b_N - a_N} \\ &= \frac{r_N - (1+r_N)\big(1-\sigma\sqrt{T/N}\big) + 1}{(1+r_N)\big(1+\sigma\sqrt{T/N}\big) - (1+r_N)\big(1-\sigma\sqrt{T/N}\big)} \\ &= \frac{1}{2}, \qquad t = 1, 2, \ldots, N. \end{aligned} \tag{3.44}$$

### Continuous-time limit in distribution

We have the following convergence result.

**Proposition 3.20.** *Let $h$ be a continuous and bounded function on $\mathbb{R}$. The price at time $t = 0$ of a contingent claim with payoff $C = h\big(S_{N,N}^{(1)}\big)$ converges as follows:*

$$\lim_{N\to\infty} \frac{1}{(1+rT/N)^N} \mathbb{E}^*\big[h\big(S_{N,N}^{(1)}\big)\big] = \mathrm{e}^{-rT}\mathbb{E}\big[h\big(S_0^{(1)}\,\mathrm{e}^{\sigma X + rT - \sigma^2 T/2}\big)\big] \tag{3.45}$$

*where $X \simeq \mathcal{N}(0, T)$ is a centered Gaussian random variable with variance $T > 0$.*

*Proof.* This result is a consequence of the weak convergence in distribution of the sequence $\big(S_{N,N}^{(1)}\big)_{N\geqslant 1}$ to a lognormal distribution, see *e.g.* Theorem 5.53 page 261 of Föllmer and Schied (2004). Informally, using the Taylor expansion of the log function and (3.41), by (3.42) we have

$$\begin{aligned}\log S^{(1)}_{N,N} &= \log S^{(1)}_0 + \sum_{k=1}^N \log\big(1+R^{(N)}_k\big)\\ &= \log S^{(1)}_0 + \sum_{k=1}^N \log(1+r_N) + \sum_{k=1}^N \log\frac{1+R^{(N)}_k}{1+r_N}\\ &= \log S^{(1)}_0 + \sum_{k=1}^N \log\left(1+\frac{rT}{N}\right) + \sum_{k=1}^N \log\left(1\pm\sigma\sqrt{\frac{T}{N}}\right)\\ &= \log S^{(1)}_0 + \sum_{k=1}^N \frac{rT}{N} + \sum_{k=1}^N \left(\pm\sigma\sqrt{\frac{T}{N}} - \frac{\sigma^2 T}{2N} + o\left(\frac{T}{N}\right)\right)\\ &= \log S^{(1)}_0 + rT - \frac{\sigma^2 T}{2} + \frac{1}{\sqrt{N}}\sum_{k=1}^N \pm\sqrt{\sigma^2 T} + o(1).\end{aligned}$$

Next, we note that by the Central Limit Theorem (CLT), the normalized sum

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N \pm\sqrt{\sigma^2 T}$$

of independent Bernoulli random variables, with variance obtained from (3.43)-(3.44) as

$$\begin{aligned}\mathrm{Var}\left[\frac{1}{\sqrt{N}}\sum_{k=1}^N \pm\sqrt{\sigma^2 T}\right] &= 4\frac{\sigma^2 T}{N}\sum_{k=1}^N \big(1-\mathbb{P}^*\big(R^{(N)}_t = b_N\big)\big)\mathbb{P}^*\big(R^{(N)}_t = a_N\big)\\ &\simeq \sigma^2 T, \qquad [N\to\infty],\end{aligned}$$

converges in distribution to a centered $\mathcal{N}(0,\sigma^2 T)$ Gaussian random variable with variance $\sigma^2 T$. Finally, the convergence of the discount factor $(1+rT/N)^N$ to $\mathrm{e}^{-rT}$ follows from (3.37). □

Note that the expectation (3.45) can be written as the Gaussian integral

$$\mathrm{e}^{-rT}\mathbb{E}\big[f\big(S^{(1)}_0\,\mathrm{e}^{\sigma X + rT - \sigma^2 T/2}\big)\big] = \mathrm{e}^{-rT}\int_{-\infty}^{\infty} f\big(S^{(1)}_0\,\mathrm{e}^{\sigma\sqrt{T}x + rT - \sigma^2 T/2}\big)\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}dx,$$

see also Lemma 7.7 in Chapter 7, hence we have

$$\lim_{N\to\infty}\frac{1}{(1+rT/N)^N}\mathbb{E}^*\big[h\big(S^{(1)}_{N,N}\big)\big] = \mathrm{e}^{-rT}\int_{-\infty}^{\infty} f\big(S^{(1)}_0\,\mathrm{e}^{\sigma x\sqrt{T} + rT - \sigma^2 T/2}\big)\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}dx.$$

It is a remarkable fact that in case $h(x) = (x-K)^+$, *i.e.* when

$$C = \big(S^{(1)}_T - K\big)^+$$

is the payoff of the European call option with strike price $K$, the above integral can be computed according to the *Black-Scholes formula*, as

$$e^{-rT}\mathbb{E}\big[\big(S_0^{(1)} e^{\sigma X + rT - \sigma^2 T/2} - K\big)^+\big] = S_0^{(1)}\Phi(d_+) - K e^{-rT}\Phi(d_-),$$

where

$$d_- = \frac{(r-\sigma^2/2)T + \log\big(S_0^{(1)}/K\big)}{\sigma\sqrt{T}}, \qquad d_+ = d_- + \sigma\sqrt{T},$$

and

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

is the Gaussian cumulative distribution function, see Proposition 6.3.

The Black-Scholes formula will be derived explicitly in the subsequent chapters using both PDE and probabilistic methods, cf. Propositions 6.11 and 7.6. It can be regarded as a building block for the pricing of financial derivatives, and its importance is not restricted to the pricing of options on stocks. Indeed, the complexity of the interest rate models makes it in general difficult to obtain closed-form expressions, and in many situations one has to rely on the Black-Scholes framework in order to find pricing formulas, for example in the case of interest rate derivatives as in the Black caplet formula of the BGM model, see Proposition 19.5 in Section 19.3.

Our aim later on will be to price and hedge options directly in continuous-time using stochastic calculus, instead of applying the limit procedure described in the previous section. In addition to the construction of the riskless asset price $(A_t)_{t\in\mathbb{R}_+}$ via (3.38) and (3.39) we now need to construct a mathematical model for the price of the risky asset in continuous time.

In addition to modeling the return of the riskless asset $S_t^{(0)}$ as in (3.40), the return of the risky asset $S_t^{(1)}$ over the time interval $[t, d+dt]$ will be modeled as

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu dt + \sigma dB_t,$$

where in comparison with (3.40), we add a "small" Gaussian random fluctuation $\sigma dB_t$ which accounts for market volatility. Here, the Brownian increment $dB_t$ is multiplied by the volatility parameter $\sigma > 0$. In the next Chapter 4 we will turn to the formal definition of the stochastic process $(B_t)_{t\in\mathbb{R}_+}$ which will be used for the modeling of risky assets in continuous time.



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## Exercises

Exercise 3.1 (Exercise 2.7 continued). Consider a two-step trinomial market model $\big(S_t^{(1)}\big)_{t=0,1,2}$ with $r=0$ and three possible return rates $R_t=-1,0,1$, and the risk-neutral probability measure $\mathbb{P}^*$ given by

$$\mathbb{P}^*(R_t=-1):=p^*, \quad \mathbb{P}^*(R_t=0):=1-2p^*, \quad \mathbb{P}^*(R_t=1):=p^*.$$

Taking $S_0^{(1)}=1$ and using Proposition 3.7, price the European put option with strike price $K=1$ and maturity $N=2$ at times $t=0$ and $t=1$.

Exercise 3.2 Consider a two-step binomial market model $(S_t)_{t=0,1,2}$ with $S_0=1$ and stock return rates $a=0$, $b=1$, and a riskless account priced $A_t=(1+r)^t$ at times $t=0,1,2$, where $r=0.5$. Price and hedge the *tunnel option* whose payoff $C$ at time $t=2$ is given by

$$C=\begin{cases} 3 & \text{if } S_2=4, \\ \\ 1 & \text{if } S_2=2, \\ \\ 3 & \text{if } S_2=1. \end{cases}$$

Exercise 3.3 In a two-step trinomial market model $(S_t)_{t=0,1,2}$ with interest rate $r=0$ and three return rates $R_t=-0.5,0,1$, we consider a down-an-out barrier call option with exercise date $N=2$, strike price $K$ and barrier level $B$, whose payoff $C$ is given by

$$C=\big(S_N-K\big)^+\mathbb{1}_{\left\{\displaystyle\min_{t=1,2,\ldots,N} S_t>B\right\}}=\begin{cases} \big(S_N-K\big)^+ & \text{if } \displaystyle\min_{t=1,2,\ldots,N} S_t>B, \\ \\ 0 & \text{if } \displaystyle\min_{t=1,2,\ldots,N} S_t\leqslant B. \end{cases}$$

a) Show that $\mathbb{P}^*$ given by $r^*=\mathbb{P}^*(R_t=-0.5):=1/2$, $q^*=\mathbb{P}^*(R_t=0):=1/4$, $p^*=\mathbb{P}^*(R_t=1):=1/4$ is a risk-neutral probability measure.
b) Taking $S_0=1$, compute the possible values of the down-an-out barrier call option payoff $C$ with strike price $K=1.5$ and barrier level $B=1$, at maturity $N=2$.
c) Price the down-an-out barrier call option with exercise date $N=2$, strike price $K=1.5$ and barrier level $B=1$, at time $t=0$ and $t=1$.

*Hint:* Use the formula

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*[C \mid S_t], \qquad t = 0, 1, \ldots, N,$$

where $N$ denotes maturity time and $C$ is the option payoff.

d) Is this market complete? Is every contingent claim attainable?

Exercise 3.4 Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a = 0$ and $b = 200\%$, and a riskless asset $A_k = A_0(1 + r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$, and $S_0 = A_0 = 1$, under the risk-neutral probabilities $p^* = (r - a)/(b - a) = 1/2$ and $q^* = (b - r)/(b - a) = 1/2$.

a) Draw a binomial tree for the possible values of $(S_k)_{k=0,1,2}$, and compute the values $V_k$ at times $k = 0, 1, 2$ of the portfolio hedging the European *call* option on $S_N$ with strike price $K = 8$ and maturity $N = 2$.

   *Hint*: Consider three cases when $k = 2$, and two cases when $k = 1$.

b) Price, then hedge. Compute the self-financing hedging portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ with values

$$V_0 = \xi_1 S_0 + \eta_1 A_0,\ V_1 = \xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \ \text{ and }\ V_2 = \xi_2 S_2 + \eta_2 A_2,$$

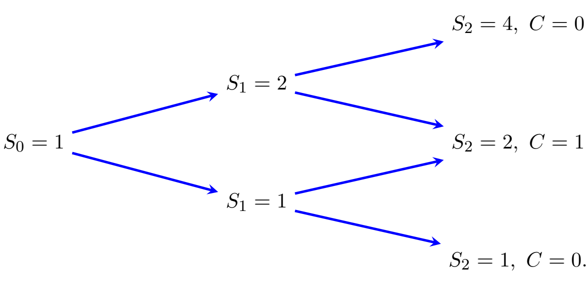
   hedging the European *call* option with strike price $K = 8$ and maturity $N = 2$.

   *Hint*: Consider two separate cases for $k = 2$ and one case for $k = 1$.

c) Hedge, then price. Compute the hedging portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ from the self-financing condition (3.17), and use it to recover the result of part (a).

Exercise 3.5 We consider a two-step binomial market model $(S_t)_{t=0,1,2}$ with $S_0 = 1$ and return rates $R_t = (S_t - S_{t-1})/S_{t-1}$, $t = 1, 2$, taking the values $a = 0$, $b = 1$, and assume that

$$p := \mathbb{P}(R_t = 1) > 0, \qquad q := \mathbb{P}(R_t = 0) > 0, \qquad t = 1, 2.$$



The riskless account is $A_t = \$1$ and the risk-free interest rate is $r = 0$. We consider the *tunnel option* whose payoff $C$ at time $t = 2$ is given by

$$C = \begin{cases} 0 & \text{if } S_2 = 4, \\ \\ \$1 & \text{if } S_2 = 2, \\ \\ 0 & \text{if } S_2 = 1. \end{cases}$$

a) Build a hedging portfolio for the claim $C$ at time $t = 1$ depending on the value of $S_1$.
b) Price the claim $C$ at time $t = 1$ depending on the value of $S_1$.
c) Build a hedging portfolio for the claim $C$ at time $t = 0$.
d) Price the claim $C$ at time $t = 0$.
e) Does this model admit an equivalent risk-neutral measure in the sense of Definitions 2.12-2.14?
f) Is the model without arbitrage according to Theorem 2.15?

Exercise 3.6 Consider a claim $C$ with maturity time $N \geqslant 1$ and affine payoff function

$$C = h(S_N) = \alpha + \beta S_N,$$

where $\alpha, \beta \in \mathbb{R}$ are constants, in a discrete-time market model made of a riskless asset priced $A_k = (1+r)^k$ with the risk-free interest rate $r$, and a risky asset with price $S_k$, $k \geqslant 0$, such that the discounted asset price process $((1+r)^{-k} S_k)_{k \geqslant 0}$ is a martingale under a risk-neutral probability measure $\mathbb{P}^*$. Using Theorem 3.5, compute the arbitrage-free price $\pi_k(C)$ at time $k = 0, 1, \ldots, N$ of the claim $C$, and give its hedging strategy.

Exercise 3.7 Call-put parity.

a) Show that the relation $(x - K)^+ = x - K + (K - x)^+$ holds for any $K, x \in \mathbb{R}$.
b) From part (a), find a relation between the prices of call and put options with strike price $K > 0$ and maturity $N \geqslant 1$ in a market with risk-free rate $r > 0$.

   *Hints:*

   i) Recall that an option with payoff $\phi(S_N)$ and maturity $N \geqslant 1$ is priced at times $k = 0, 1, \ldots, N$ as $(1+r)^{-(N-k)} \mathbb{E}^* \left[ \phi(S_N) \mid \mathcal{F}_k \right]$ under the risk-neutral measure $\mathbb{P}^*$.
   ii) The payoff at maturity of a European call (resp. put) option with strike price $K$ is $(S_N - K)^+$, resp. $(K - S_N)^+$.

Exercise 3.8 We consider a range forward contract having the payoff



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$$S_T - F + (K_1 - S_T)^+ - (S_T - K_2)^+,$$

on an underlying asset priced $S_T$ at maturity $T$, where $0 < K_1 < F < K_2$.

a) Show that this range forward contract can be realized as a portfolio containing $S_T$, a call option, a put option, and a certain (positive or negative) amount in cash. Specify the quantity of every asset held in the portfolio.
b) Draw the graph of the payoff function of the range forward contract by taking $K_1 := \$80$, $F := \$100$, and $K_2 := \$110$.

Exercise 3.9 Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a = -50\%$ and $b = 150\%$, and a riskless asset $A_k = A_0(1+r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$, and $S_0 = A_0 = 1$, under the risk-neutral probabilities $p^* = (r-a)/(b-a) = 3/4$ and $q^* = (b-r)/(b-a) = 1/4$.

a) Draw a binomial tree for the values of $(S_k)_{k=0,1,2}$.
b) Compute the values $V_k$ at times $k = 0, 1, 2$ of the hedging portfolio of the European *put* option with strike price $K = 5/4$ and maturity $N = 2$ on $S_N$.
c) Compute the self-financing hedging portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ with values

$$V_0 = \xi_1 S_0 + \eta_1 A_0, \; V_1 = \xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \text{ and } V_2 = \xi_2 S_2 + \eta_2 A_2,$$

hedging the European *put* option on $S_N$ with strike price $K := 5/4$ and maturity $N := 2$.

Exercise 3.10 Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a := -50\%$ and $b := 200\%$, and a riskless asset $A_k := A_0(1+r)^k$, $k = 0, 1, 2$ with interest rate $r := 100\%$, $S_0 := \$4$, and $A_0 := \$1$.

Price and hedge the European **put** option on $S_N$ with strike price $K := \$11$ and maturity $N = 2$.

Write your answers using **simplified fractions** only. For example, write 7/4 instead of 14/8 or 1.75.

Exercise 3.11 Analysis of a binary option trading website.

a) In a one-step model with risky asset prices $S_0$, $S_1$ at times $t = 0$ and $t = 1$, compute the price at time $t = 0$ of the binary call option with payoff

$$C = \mathbb{1}_{[K,\infty)}(S_1) = \begin{cases} \$1 & \text{if } S_1 \geqslant K, \\ \\ 0 & \text{if } S_1 < K, \end{cases}$$

in terms of the probability $p^* = \mathbb{P}^*(S_1 \geqslant K)$ and of the risk-free interest rate $r$.

b) Compute the two potential net returns obtained by purchasing one binary call option.
c) Compute the corresponding expected (net) return.
d) A website proposes to pay a return of 86% in case the binary call option matures "in the money", *i.e.* when $S_1 \geqslant K$. Compute the corresponding expected (net) return. What do you conclude?

Exercise 3.12 A *put spread collar* option requires its holder to *sell* an asset at the price $f(S)$ when its market price is at the level $S$, where $f(S)$ is the function plotted in Figure 3.11, with $K_1 := 80$, $K_2 := 90$, and $K_3 := 110$.

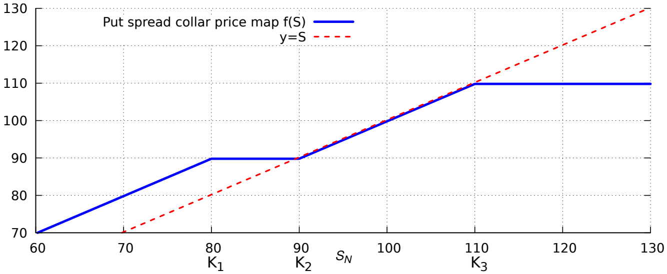



Fig. 3.11: Put spread collar price map.

a) Draw the *payoff function* of the put spread collar as a function of the underlying asset price at maturity. See *e.g.* https://optioncreator.com/.
b) Show that this put spread collar option can be realized by purchasing and/or issuing standard European call and put options with strike prices to be specified.

*Hints:* Recall that an option with payoff $\phi(S_N)$ is priced $(1+r)^{-N}\mathbb{E}^*\big[\phi(S_N)\big]$ at time 0. The payoff of the European call (resp. put) option with strike price $K$ is $(S_N - K)^+$, resp. $(K - S_N)^+$.

Exercise 3.13 A *call spread collar* option requires its holder to *buy* an asset at the price $f(S)$ when its market price is at the level $S$, where $f(S)$ is the function plotted in Figure 3.11, with $K_1 := 80$, $K_2 := 100$, and $K_3 := 110$.

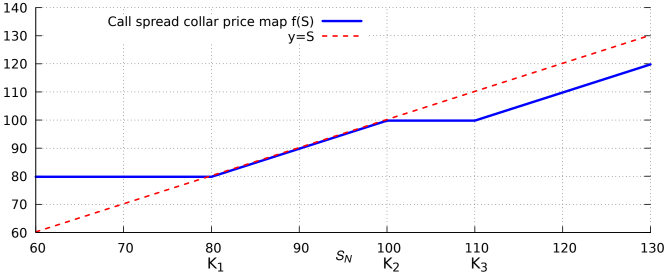



Fig. 3.12: Call spread collar price map.

a) Draw the *payoff function* of the call spread collar as a function of the underlying asset price at maturity. See *e.g.* https://optioncreator.com/.
b) Show that this call spread collar option can be realized by purchasing and/or issuing standard European call and put options with strike prices to be specified.

*Hints:* Recall that an option with payoff $\phi(S_N)$ is priced $(1+r)^{-N}\mathbb{E}^*\left[\phi(S_N)\right]$ at time 0. The payoff of the European call (resp. put) option with strike price $K$ is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 3.14 Consider an asset price $(S_n)_{n=0,1,\ldots,N}$ which is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, with respect to the filtration $(\mathcal{F}_n)_{n=0,1,\ldots,N}$. Given the (convex) function $\phi(x) := (x-K)^+$, show that the price of an Asian option with payoff

$$\phi\left(\frac{S_1+\cdots+S_N}{N}\right)$$

and maturity $N \geqslant 1$ is always lower than the price of the corresponding European *call* option, *i.e.* show that

$$\mathbb{E}^*\left[\phi\left(\frac{S_1+S_2+\cdots+S_N}{N}\right)\right] \leqslant \mathbb{E}^*[\phi(S_N)].$$

*Hint*: Use in the following order:

(i) the convexity inequality $\phi(x_1/N+\cdots+x_N/N) \leqslant \phi(x_1)/N+\cdots+\phi(x_N)/N$,
(ii) the martingale property $S_k = \mathbb{E}^*[S_N \mid \mathcal{F}_k]$, $k=1,2,\ldots,N$.
(iii) The Jensen (1906) inequality

$$\phi(\mathbb{E}^*[S_N \mid \mathcal{F}_k]) \leqslant \mathbb{E}^*[\phi(S_N) \mid \mathcal{F}_k], \qquad k=1,2,\ldots,N,$$

(iv) the tower property $\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathcal{F}_k]] = \mathbb{E}^*[\phi(S_N)]$ of conditional expectations, $k = 1, 2, \ldots, N$.

Exercise 3.15 (Exercise 2.11 continued).

a) We consider a forward contract on $S_N$ with strike price $K$ and payoff

$$C := S_N - K.$$

Find a portfolio allocation $(\eta_N, \xi_N)$ with value

$$V_N = \eta_N \pi_N + \xi_N S_N$$

at time $N$, such that

$$V_N = C, \tag{3.46}$$

by writing Condition (3.46) as a $2 \times 2$ system of equations.

b) Find a portfolio allocation $(\eta_{N-1}, \xi_{N-1})$ with value

$$V_{N-1} = \eta_{N-1}\pi_{N-1} + \xi_{N-1} S_{N-1}$$

at time $N - 1$, and verifying the self-financing condition

$$V_{N-1} = \eta_N \pi_{N-1} + \xi_N S_{N-1}.$$

Next, at all times $t = 1, 2, \ldots, N - 1$, find a portfolio allocation $(\eta_t, \xi_t)$ with value $V_t = \eta_t \pi_t + \xi_t S_t$ verifying (3.46) and the self-financing condition

$$V_t = \eta_{t+1}\pi_t + \xi_{t+1} S_t,$$

where $\eta_t$, *resp.* $\xi_t$, represents the quantity of the riskless, *resp.* risky, asset in the portfolio over the time period $[t - 1, t]$, $t = 1, 2, \ldots, N - 1$.

c) Compute the arbitrage-free price $\pi_t(C) = V_t$ of the forward contract $C$, at time $t = 0, 1, \ldots, N$.

d) Check that the arbitrage-free price $\pi_t(C)$ satisfies the relation

$$\pi_t(C) = \frac{1}{(1 + r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0, 1, \ldots, N.$$

Exercise 3.16 Power option. Let $(S_n)_{n \in \mathbb{N}}$ denote a binomial price process with returns $-50\%$ and $+50\%$, and let the riskless asset be valued $A_k = \$1$, $k \in \mathbb{N}$. We consider a power option with payoff $C := (S_N)^2$, and a *predictable* self-financing portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\ldots,N}$ with value

$$V_k = \xi_k S_k + \eta_k A_0, \qquad k = 1, 2, \ldots, N.$$

a) Find the portfolio allocation $(\xi_N, \eta_N)$ that matches the payoff $C = (S_N)^2$ at time $N$, *i.e.* that satisfies

$$V_N = (S_N)^2.$$

*Hint:* We have $\eta_N = -3(S_{N-1})^2/4$.

b) In the following questions we use the risk-neutral probability $p^* = 1/2$ of a $+50\%$ return.

   i) Compute the portfolio value

$$V_{N-1} = \mathbb{E}^*[C \mid \mathcal{F}_{N-1}].$$

   ii) Find the portfolio allocation $(\eta_{N-1}, \xi_{N-1})$ at time $N-1$ from the relation

$$V_{N-1} = \xi_{N-1} S_{N-1} + \eta_{N-1} A_0.$$

   *Hint:* We have $\eta_{N-1} = -15(S_{N-2})^2/16$.

   iii) Check that the portfolio satisfies the self-financing condition

$$V_{N-1} = \xi_{N-1} S_{N-1} + \eta_{N-1} A_0 = \xi_N S_{N-1} + \eta_N A_0.$$

Exercise 3.17 Consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$. The price $S_t^0$ of the riskless asset evolves as $S_t^0 = \pi^0 (1+r)^t$, $t = 0, 1, \ldots, N$. The *return* of the risky asset, defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, 2, \ldots, N,$$

is random and allowed to take only two values $a$ and $b$, with $-1 < a < r < b$.

The discounted asset price is given by $\widetilde{S}_t := S_t/(1+r)^t$, $t = 0, 1, \ldots, N$.

a) Show that this model admits a unique risk-neutral probability measure $\mathbb{P}^*$ and explicitly compute $\mathbb{P}^*(R_t = a)$ and $\mathbb{P}(R_t = b)$ for all $t = 1, 2, \ldots, N$, with $a = 2\%$, $b = 7\%$, $r = 5\%$.

b) Does there exist arbitrage opportunities in this model? Explain why.

c) Is this market model complete? Explain why.

d) Consider a contingent claim with payoff*

$$C = (S_N)^2.$$

Compute the discounted arbitrage-free price $\widetilde{V}_t$, $t = 0, 1, \ldots, N$, of a self-financing portfolio hedging the claim payoff $C$, *i.e.* such that

$$V_N = C = (S_N)^2, \quad \text{or} \quad \widetilde{V}_N = \widetilde{C} = \frac{(S_N)^2}{(1+r)^N}.$$

* This is the payoff of a power call option with strike price $K = 0$.

e) Compute the portfolio strategy

$$(\bar{\xi}_t)_{t=1,2,\ldots,N} = (\xi_t^0, \xi_t^1)_{t=1,2,\ldots,N}$$

associated to $\widetilde{V}_t$, *i.e.* such that

$$\widetilde{V}_t = \bar{\xi}_t \cdot \overline{X}_t = \xi_t^0 X_t^0 + \xi_t^1 X_t^1, \qquad t = 1, 2, \ldots, N.$$

f) Check that the above portfolio strategy is self-financing, *i.e.*

$$\bar{\xi}_t \cdot \overline{S}_t = \bar{\xi}_{t+1} \cdot \overline{S}_t, \qquad t = 1, 2, \ldots, N-1.$$

Exercise 3.18 We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$.

The price $\pi_t$ of the riskless asset evolves as $\pi_t = \pi_0(1+r)^t$, $t = 0, 1, \ldots, N$. The evolution of $S_{t-1}$ to $S_t$ is given by

$$S_t = \begin{cases} (1+b)S_{t-1} & \text{if } R_t = b, \\ \\ (1+a)S_{t-1} & \text{if } R_t = a, \end{cases}$$

with $-1 < a < r < b$. The *return* of the risky asset is defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, 2, \ldots, N.$$

Let $\xi_t$, *resp.* $\eta_t$, denote the (possibly fractional) quantities of the risky, *resp.* riskless, asset held over the time period $[t-1, t]$ in the portfolio with value

$$V_t = \xi_t S_t + \eta_t \pi_t, \qquad t = 0, 1, \ldots, N. \tag{3.47}$$

a) Show that

$$V_t = (1 + R_t)\xi_t S_{t-1} + (1+r)\eta_t \pi_{t-1}, \qquad t = 1, 2, \ldots, N. \tag{3.48}$$

b) Show that under the probability $\mathbb{P}^*$ defined by

$$\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) = \frac{b-r}{b-a}, \qquad \mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) = \frac{r-a}{b-a},$$

where $\mathcal{F}_{t-1}$ represents the information generated by $\{R_1, R_2, \ldots, R_{t-1}\}$, we have

$$\mathbb{E}^*[R_t \mid \mathcal{F}_{t-1}] = r.$$

c) Under the self-financing condition

$$V_{t-1} = \xi_t S_{t-1} + \eta_t \pi_{t-1}, \qquad t = 1, 2, \ldots, N, \tag{3.49}$$

recover the martingale property

$$V_{t-1} = \frac{1}{1+r}\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}],$$

using the result of Question (a).

d) Let $a = 5\%$, $b = 25\%$ and $r = 15\%$. Assume that the value $V_t$ at time $t$ of the portfolio is \$3 if $R_t = a$ and \$8 if $R_t = b$, and compute the value $V_{t-1}$ of the portfolio at time $t-1$.

Problem 3.19 CRR model with transaction costs. Stock broker income is generated by commissions or transaction costs representing the difference between ask prices (at which they are willing to sell an asset to their client), and bid prices (at which they are willing to buy an asset from their client).

We consider a discrete-time Cox-Ross-Rubinstein model with one risky asset priced $S_t$ at time $t = 0, 1, \ldots, N$. The price $A_t$ of the riskless asset evolves as

$$A_t = \rho^t, \qquad t = 0, 1, \ldots, N,$$

with $A_0 := 1$ and $\rho > 0$, and the random evolution of $S_{t-1}$ to $S_t$ is given by two possible returns $\alpha$, $\beta$ as

$$S_t = \begin{cases} \beta S_{t-1} \\ \\ \alpha S_{t-1} \end{cases}$$

$t = 1, \ldots, N$, with $0 < \alpha < \beta$.

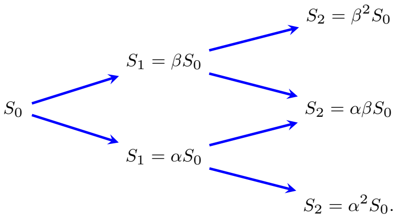
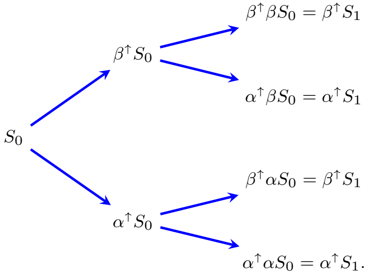



Fig. 3.13: Tree of market prices with $N = 2$.

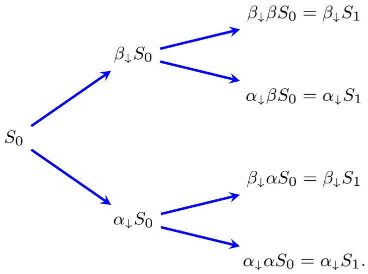
The ask and bid prices of the risky asset quoted $S_t$ on the market are respectively given by $(1+\lambda)S_t$, and $(1-\lambda)S_t$ for some $\lambda \in [0, 1)$, such that

$$\begin{cases} \alpha^\uparrow := \alpha(1+\lambda) < \beta(1-\lambda) =: \beta_\downarrow, \\ \alpha_\downarrow := \alpha(1-\lambda) < \beta(1-\lambda) := \beta_\downarrow, \\ \alpha^\uparrow := \alpha(1+\lambda) < \beta(1+\lambda) =: \beta^\uparrow, \end{cases}$$

*i.e.*, transaction costs are charged at the rate $\lambda \in [0,1)$, proportionally to the traded amount.




(a) Tree of ask prices.




(b) Tree of bid prices.

Fig. 3.14: Trees of bid and ask prices with $N = 2$.

The riskless asset is not subject to transaction costs or bid/ask prices, and is priced $A_t$ at time $t = 0, 1, \ldots, N$. We consider a predictable, self-financing replicating portfolio strategy

$$(\eta_t(S_{t-1}), \xi_t(S_{t-1}))_{t=1,2,\ldots,N}.$$

made of $\eta_t(S_{t-1})$ of the riskless asset $A_t$ and of $\xi_t(S_{t-1})$ units of the risky asset $S_t$ at time $t = 1, 2, \ldots, N$.

Our goal is to derive a backward recursion giving $\xi_t(S_{t-1})$, $\eta_t(S_{t-1})$ from $\xi_{t+1}(S_t)$, $\eta_{t+1}(S_t)$ for $t = N-1, N-2 \ldots, 1$. The following questions are interdependent and should be treated in sequence.

a) We consider a portfolio reallocation $(\eta_t(S_{t-1}), \xi_t(S_{t-1})) \to (\eta_{t+1}(S_t), \xi_{t+1}(S_t))$ at time $t \in \{1, \ldots, N-1\}$. Write down the self-financing condition in the event of:

   i) an *increase* in the stock position from $\xi_t(S_{t-1})$ to $\xi_{t+1}(S_t)$,
   ii) a *decrease* in the stock position from $\xi_t(S_{t-1})$ to $\xi_{t+1}(S_t)$.

   The conditions are written using $\eta_t(S_{t-1})$, $\eta_{t+1}(S_t)$, $\xi_t(S_{t-1})$, $\xi_{t+1}(S_t)$, $A_t$, $S_t$ and $\lambda$.

b) From Questions ai)-aii), deduce two self-financing equations in case $S_t = \alpha S_{t-1}$, and two self-financing equations in case $S_t = \beta S_{t-1}$.

c) Using the functions

$$g_\alpha(x,y) := \begin{cases} \alpha^\uparrow & \text{if } x \leqslant y, \\ \alpha_\downarrow & \text{if } x > y, \end{cases} \quad \text{and} \quad g_\beta(x,y) := \begin{cases} \beta^\uparrow & \text{if } x \leqslant y, \\ \beta_\downarrow & \text{if } x > y, \end{cases}$$



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rewrite the equations of Question (b) into a single equation in case $S_t = \alpha S_{t-1}$, and a single equation in case $S_t = \beta S_{t-1}$.

d) From the result of Question (c), derive an equation satisfied by $\xi_t(S_{t-1})$, and show that it admits a unique solution $\xi_t(S_{t-1})$.

*Hint:* Show that the *piecewise affine* function

$$x \mapsto f(x, S_{t-1}) := g_\beta(x, \xi_{t+1}(\beta S_{t-1}))(x - \xi_{t+1}(\beta S_{t-1})) \\ -g_\alpha(x, \xi_{t+1}(\alpha S_{t-1}))(x - \xi_{t+1}(\alpha S_{t-1})) - \rho \frac{\eta_{t+1}(\beta S_{t-1}) - \eta_{t+1}(\alpha S_{t-1})}{\widetilde{S}_{t-1}}$$

is strictly increasing in $x \in \mathbb{R}$.

e) Find the expressions of $\xi_t(S_{t-1})$ and $\eta_t(S_{t-1})$ by solving the $2 \times 2$ system of equations of Question (c).

*Hint:* The expressions have to use the quantities

$$g_\alpha(\xi_t(S_{t-1}), \xi_{t+1}(\alpha S_{t-1})), \qquad g_\beta(\xi_t(S_{t-1}), \xi_{t+1}(\beta S_{t-1})),$$

and they should be consistent with Proposition 3.12 when $\lambda = 0$, *i.e.* when $\alpha^\uparrow = \alpha_\downarrow = 1 + a$ and $\beta^\uparrow = \beta_\downarrow = 1 + b$, with $\rho = 1 + r$.

f) Find the value of $g_\alpha(\xi_t(S_{t-1}), \xi_{t+1}(\alpha S_{t-1}))$ in the following two cases:

i) $f(\xi_{t+1}(\alpha S_{t-1}), S_{t-1}) \geqslant 0$,
ii) $f(\xi_{t+1}(\alpha S_{t-1}), S_{t-1}) < 0$,

and the value of $g_\beta(\xi_t(S_{t-1}), \xi_{t+1}(\beta S_{t-1}))$ in the following two cases:

i) $f(\xi_{t+1}(\beta S_{t-1}), S_{t-1}) \geqslant 0$,
ii) $f(\xi_{t+1}(\beta S_{t-1}), S_{t-1}) < 0$.

g) Hedge and price the call option with strike price $K = \$2$ and $N = 2$ when $S_0 = 8$, $\rho = 1$, $\alpha = 0.5$, $\beta = 2$, and the transaction cost rate is $\lambda = 12.5\%$. Provide sufficient details of hand calculations.

*Remark:* The evaluation of the terminal payoff uses the value of $S_2$ only, and is not affected by bid/ask prices.

h) Modify the attached IPython notebook in order to include the treatment of transaction costs.

Grouping: 0.10

| | | |
|---|---|---|
| 6746.80 | 6.800000 | 27.307670 |
| 6746.50 | 2.077389 | 20.507670 |
| 6746.30 | 0.500000 | 18.430281 |
| 6746.20 | 3.058234 | 17.930281 |
| 6746.10 | 0.199108 | 14.872047 |
| 6745.40 | 2.004892 | 14.672939 |
| 6745.30 | 2.600000 | 12.668047 |
| 6745.20 | 3.400000 | 10.068047 |
| 6745.10 | 2.144932 | 6.668047 |
| 6745.00 | 0.941114 | 4.523115 |
| 6744.90 | 0.009213 | 3.582001 |
| 6744.80 | 0.082913 | 3.572788 |
| 6744.70 | 2.461683 | 3.489875 |
| 6744.60 | 0.880170 | 1.028192 |
| 6744.50 | 0.148022 | 0.148022 |
| 6744.43 USDT | | |
| 6744.00 | 0.173899 | 0.173899 |
| 6743.80 | 0.007414 | 0.181313 |
| 6743.70 | 0.142790 | 0.324103 |
| 6743.60 | 0.050000 | 0.374103 |
| 6743.20 | 0.203218 | 0.577321 |
| 6742.40 | 0.500000 | 1.077321 |
| 6742.10 | 2.500000 | 3.577321 |
| 6741.70 | 1.725744 | 5.303065 |
| 6741.40 | 0.726119 | 6.029184 |
| 6741.00 | 0.060000 | 6.089184 |
| 6740.80 | 7.634518 | 13.723702 |
| 6740.60 | 0.254281 | 13.977983 |
| 6740.20 | 0.299791 | 14.277774 |
| 6740.00 | 0.269737 | 14.547511 |
| 6739.90 | 0.600000 | 15.147511 |

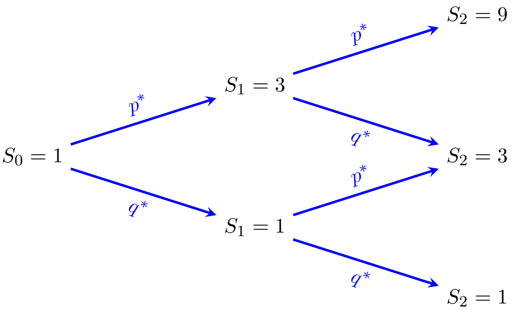
Fig. 3.15: ₿TC/USD order book example.[*]

In the above figure, ask prices are marked in red and bid prices are marked in green. The center column gives the quantity of the asset available at that row's bid or ask price, and the right column represents the cumulative volume of orders from the last-traded price until the current bid/ask price level. The large number in the center shows the last-traded price.

Problem 3.20 CRR model with dividends (1). Consider a two-step binomial model for a stock paying a dividend at the *rate* $\alpha \in (0,1)$ at times $k = 1$ and $k = 2$, and the following recombining tree represents the *ex-dividend*[†] prices $S_k$ at times $k = 1, 2$, starting from $S_0 = \$1$.

[*] The animation works in Acrobat Reader on the entire pdf file.

[†] "Ex-dividend" means after dividend payment.




```
install.packages("quantmod");library(quantmod)
getDividends("Z74.SI",from="2018-01-01",to="2018-12-31",src="yahoo")
getSymbols("Z74.SI",from="2018-11-16",to="2018-12-19",src="yahoo")
dev.new(width=16,height=7)
myPars <- chart_pars();myPars$cex<-1.8
myTheme <- chart_theme();myTheme$col$line.col <- "purple"
myTheme$rylab <- FALSE
chart_Series(Op(`Z74.SI`),name="Opening prices (purple) - Closing prices
      (blue)",lty=4,lwd=6,pars=myPars,theme=myTheme)
add_TA(Cl(`Z74.SI`),lwd=3,lty=5,legend='Difference',col="blue",on = 1)
```

```
           Z74.SI.div
2018-07-26 0.107
2018-12-17 0.068
2018-12-18 0.068
```

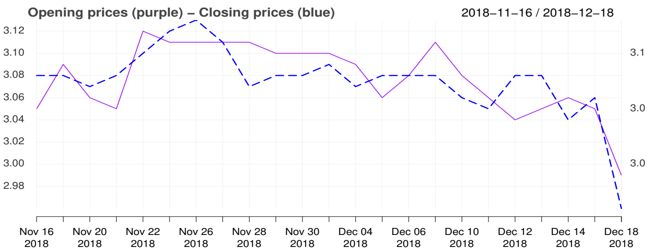



Fig. 3.16: SGD0.068 dividend detached on 18 Dec 2018 on Z74.SI.

The difference between the closing price on Dec 17 (\$3.06) and the opening price on Dec 18 (\$2.99) is \$3.06 − \$2.99 = \$0.07. The adjusted price on Dec 17 (\$2.992) is the closing price (\$3.06) minus the dividend (\$0.068).

| Z74.SI | Open | High | Low | Close | Volume | Adjusted (ex-dividend) |
|---|---|---|---|---|---|---|
| 2018-12-17 | 3.05 | 3.08 | 3.05 | **3.06** | 17441000 | **2.992** |
| 2018-12-18 | **2.99** | 2.99 | 2.96 | 2.96 | 28456400 | 2.960 |

The dividend rate $\alpha$ is given by $\alpha = 0.068/3.06 = 2.22\%$.

We consider a riskless asset $A_k = A_0(1+r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$ and $A_0 = 1$, and two portfolio allocations $(\xi_1, \eta_1)$ at time $k = 0$ and $(\xi_2, \eta_2)$ at time $k = 1$, with the values

$$V_1 = \xi_2 S_1 + \eta_2 A_1 \tag{3.50}$$

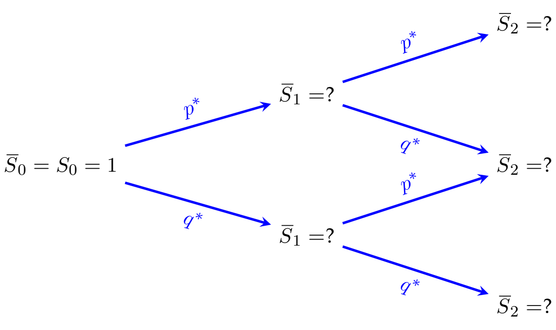
and

$$V_0 = \xi_1 S_0 + \eta_1 A_0. \tag{3.51}$$

We make the following three assumptions:

**[A]** All dividends are reinvested.
**[B]** The portfolio strategies are self-financing.
**[C]** The portfolio value $V_2$ at time $k = 2$ hedges the European *call* option with payoff $C = (S_T - K)^+$, strike price $K = 8$, and maturity $T = 2$.

a) Using (3.50) and **[A]**, express $V_2$ in terms of $\xi_2$, $\eta_2$, $S_2$, $A_2$ and $\alpha$.
b) Using (3.51) and **[A]**-**[B]**, express $V_1$ in terms of $\xi_1$, $\eta_1$, $S_1$, $A_1$ and $\alpha$.
c) Using Assumption **[C]** and the result of Question (a), compute the portfolio allocation $(\xi_2, \eta_2)$ in cases $S_1 = 1$ and $S_1 = 3$.
d) Using (3.50) and the portfolio allocation $(\xi_2, \eta_2)$ obtained in Question (c), compute the portfolio value $V_1$ in cases $S_1 = 1$ and $S_1 = 3$.
e) From the results of Questions (b) and (d), compute the initial portfolio allocation $(\xi_1, \eta_1)$.
f) Compute the initial portfolio value $V_0$ from the result of Question (e).
g) Knowing that the dividend rate is $\alpha = 25\%$, draw the tree of asset prices $(\overline{S}_k)_{k=1,2}$ *before* (*i.e.* without) dividend payments.



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h) Compute the risk-neutral probabilities $p^*$ and $q^*$ under which the conditional expected return of $(\overline{S}_k)_{k=0,1,2}$ is the risk-free interest rate $r = 100\%$.

i) ✓ Check that the portfolio value $V_1$ found in Question (d) satisfies

$$V_1 = \frac{1}{1+r}\mathbb{E}^*[(S_2 - K)^+ \mid S_1].$$

j) ✓ Check that the portfolio value $V_0$ found in Question (f) satisfies

$$V_0 = \frac{1}{(1+r)^2}\mathbb{E}^*\big[(S_2 - K)^+\big] \quad \text{and} \quad V_0 = \frac{1}{1+r}\mathbb{E}^*\big[V_1\big].$$

Problem 3.21 CRR model with dividends (2). We consider a riskless asset priced as

$$S_k^{(0)} = S_0^{(0)}(1+r)^k, \qquad k = 0, 1, \ldots, N,$$

with $r > -1$, and a risky asset $S^{(1)}$ whose *return* is given by

$$R_k := \frac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}, \qquad k = 1, 2, \ldots, N,$$

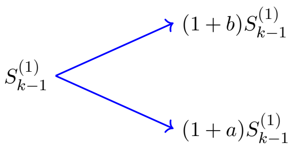
with $N+1$ time instants $k = 0, 1, \ldots, N$ and $d = 1$. In the CRR model the return $R_k$ is random and allowed to take two values $a$ and $b$ at each time step, *i.e.*

$$R_k \in \{a, b\}, \qquad k = 1, 2, \ldots, N,$$

with $-1 < a < 0 < b$, and the random evolution of $S_{k-1}^{(1)}$ to $S_k^{(1)}$ is given by

$$S_k^{(1)} = \left\{ \begin{array}{ll} (1+b)S_{k-1}^{(1)} & \text{if } R_k = b \\ \\ (1+a)S_{k-1}^{(1)} & \text{if } R_k = a \end{array} \right\} = (1+R_k)S_{k-1}^{(1)}, \qquad k = 1, 2, \ldots, N, \tag{3.52}$$

according to the tree




and we have

$$S_k^{(1)} = S_0^{(1)} \prod_{i=1}^{k} (1 + R_i), \qquad k = 0, 1, \ldots, N.$$

The information $\mathcal{F}_k$ known to the market up to time $k$ is given by the knowledge of $S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}$, *i.e.* we write

$$\mathcal{F}_k = \sigma\big(S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}\big) = \sigma(R_1, R_2, \ldots, R_k),$$

$k = 0, 1, \ldots, N$, where $S_0^{(1)}$ is a constant and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ contains no information.

Under the risk-neutral probability measure $\mathbb{P}^*$ defined by

$$p^* := \mathbb{P}^*(R_k = b) = \frac{r-a}{b-a} > 0, \quad q^* := \mathbb{P}^*(R_k = a) = \frac{b-r}{b-a} > 0,$$

$k = 1, 2, \ldots, N$, the asset returns $(R_k)_{k=1,2,\ldots,N}$ form a sequence of independent identically distributed random variables.

In what follows we assume that the stock $S_k$ pays a dividend *rate* $\alpha > 0$ at times $k = 1, 2, \ldots, N$. At the beginning of every time step $k = 1, 2, \ldots, N$, the price $S_k$ is immediately adjusted to its *ex-dividend* level by losing $\alpha\%$ of its value. The following ten questions are interdependent and should be treated in sequence.

a) Rewrite the evolution (3.52) of $S_{k-1}^{(1)}$ to $S_k^{(1)}$ in the presence of a *daily* dividend rate $\alpha > 0$.
b) Express the dividend amount as a percentage of the ex-dividend price $S_k^{(1)}$, and show that under the risk-neutral probability measure the return of the risky asset satisfies

$$\mathbb{E}^*\left[\frac{S_{k+1}^{(1)}}{1-\alpha}\,\middle|\,\mathcal{F}_k\right] = (1+r)S_k^{(1)}, \qquad k = 0, 1, \ldots, N-1.$$

c) We consider a (predictable) portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\ldots,N}$ with value process

$$V_k = \xi_{k+1}S_k^{(1)} + \eta_{k+1}S_k^{(0)}$$

at time $k = 0, 1, \ldots, N-1$. Write down the self-financing condition for the portfolio value process $(V_k)_{k=0,1,\ldots,N}$ by taking into account the reinvested dividends, and give the expression of $V_N$.
d) Show that under the self-financing condition, the discounted portfolio value process

$$\widetilde{V}_k := \frac{V_k}{S_k^{(0)}}, \qquad k = 0, 1, \ldots, N,$$

is a martingale under the risk-neutral probability measure $\mathbb{P}^*$.
e) Show that the price at time $k = 0, 1, \ldots, N$ of a claim with random payoff $C$ can be written as

$$V_k = \frac{1}{(1+r)^{N-k}} \mathbb{E}^*[C \mid \mathcal{F}_k], \qquad k = 0, 1, \ldots, N,$$

assuming that the claim $C$ is attained at time $N$ by the portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\ldots,N}$.

f) Compute the price at time $t = 0, 1, \ldots, N$ of a vanilla option with payoff $h\big(S_N^{(1)}\big)$ using the pricing function

$$C_0\big(k, x, N, a, b, r\big)$$
$$:= \frac{1}{(1+r)^{N-k}} \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^l (q^*)^{N-k-l} h\big(x(1+b)^l(1+a)^{N-k-l}\big)$$

of a vanilla claim with payoff $h\big(S_N^{(1)}\big)$.

g) Show that the price at time $t = 0, 1, \ldots, N$ of a vanilla option with payoff function $h\big(S_N^{(1)}\big)$ can be rewritten as

$$V_k = C_\alpha\big(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big) := (1-\alpha)^{N-k} C_0\big(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big),$$

$k = 0, 1, \ldots, N$, where the coefficients $a_\alpha, b_\alpha, r_\alpha$ will be determined explicitly.

h) Find a recurrence relation between the functions $C_\alpha\big(k, x, N, a_\alpha, b_\alpha, r_\alpha\big)$ and $C_\alpha\big(k+1, x, N, a_\alpha, b_\alpha, r_\alpha\big)$ using the martingale property of the discounted portfolio value process $(\widetilde{V}_k)_{k=0,1,\ldots,N}$ under the risk-neutral probability measure $\mathbb{P}^*$.

i) Using the function $C_0\big(k, x, N, a_\alpha, b_\alpha, r_\alpha\big)$, compute the quantity $\xi_k$ of risky asset $S_k^{(1)}$ allocated on the time interval $[k-1, k)$ in a self-financing portfolio hedging the claim $C = h\big(S_N^{(1)}\big)$.

j) How are the dividends reinvested in the self-financing hedging portfolio?

Problem 3.22 We consider a *ternary tree* (or *trinomial*) model with $N+1$ time instants $k = 0, 1, \ldots, N$ and $d = 1$ risky asset. The price $S_k^{(0)}$ of the riskless asset evolves as

$$S_k^{(0)} = S_0^{(0)} (1+r)^k, \qquad k = 0, 1, \ldots, N,$$

with $r > -1$. Let the *return* of the risky asset $S^{(1)}$ be defined as

$$R_k := \frac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}, \qquad k = 1, 2, \ldots, N.$$

In this ternary tree model, the return $R_k$ is random and allowed to take only three values $a$, $0$ and $b$ at each time step, *i.e.*

$$R_k \in \{a, 0, b\}, \qquad k = 1, 2, \ldots, N,$$

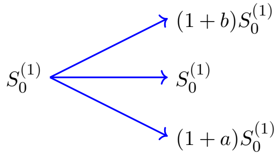
with $-1 < a < 0 < b$. That means, the evolution of $S_{k-1}^{(1)}$ to $S_k^{(1)}$ is random and given by

$$S_k^{(1)} = \begin{cases} (1+b)S_{k-1}^{(1)} & \text{if } R_k = b \\ \\ S_{k-1}^{(1)} & \text{if } R_k = 0 \\ \\ (1+a)S_{k-1}^{(1)} & \text{if } R_k = a \end{cases} = (1+R_k)S_{k-1}^{(1)}, \quad k = 1, 2, \ldots, N,$$

and

$$S_k^{(1)} = S_0^{(1)} \prod_{i=1}^{k} (1+R_i), \qquad k = 0, 1, \ldots, N.$$

The price process $\big(S_k^{(1)}\big)_{k=0,1,\ldots,N}$ evolves on a ternary tree of the form:




The information $\mathcal{F}_k$ known to the market up to time $k$ is given by the knowledge of $S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}$, *i.e.* we write

$$\mathcal{F}_k = \sigma\big(S_1^{(1)}, S_2^{(1)}, \ldots, S_k^{(1)}\big) = \sigma(R_1, R_2, \ldots, R_k),$$

$k = 0, 1, \ldots, N$, where, as a convention, $S_0^{(1)}$ is a constant and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ contains no information. In what follows we will consider that $(R_k)_{k=1,2,\ldots,N}$ is a sequence of independent identically distributed random variables under any risk-neutral probability measure $\mathbb{P}^*$, and we denote

$$\begin{cases} p^* := \mathbb{P}^*(R_k = b) > 0, \\ \\ \theta^* := \mathbb{P}^*(R_k = 0) > 0, \\ \\ q^* := \mathbb{P}^*(R_k = a) > 0, \quad k = 1, 2, \ldots, N. \end{cases}$$

a) Determine all possible risk-neutral probability measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in terms of the parameter $\theta^* \in (0, 1)$.
b) Give a necessary and sufficient condition for absence of arbitrage in this ternary tree model.

Hint: Use your intuition of the market to find what the condition should be, and then prove that it is necessary and sufficient. Note that we have $a < 0$ and $b > 0$, and the condition should only depend on the model parameters $a$, $b$ and $r$.

c) When the model parameters allow for arbitrage opportunities, explain how you would exploit them if you joined the market with zero money to invest.
d) Is this ternary tree market model complete?
e) In this question we assume that the conditional variance

$$\mathrm{Var}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \,\middle|\, \mathcal{F}_k\right] = \sigma^2 > 0$$

of the asset return $(S_{k+1}^{(1)} - S_k^{(1)})/S_k^{(1)}$ given $\mathcal{F}_k$ is constant and equal to $\sigma^2$, $k = 0, 1, \ldots, N-1$. Show that this condition determines a unique value of $\theta^*$ and a unique risk-neutral probability measure $\mathbb{P}_\sigma^*$ to be written explicitly, under a certain condition on $a, b, r$ and $\sigma$.

f) In this question and in the following we impose the condition $(1+a)(1+b) = 1$, *i.e.* we let $a := -b/(b+1)$. What does this imply on this ternary tree model and on the risk-neutral probability measure $\mathbb{P}^*$?
g) We consider a vanilla financial claim with payoff $C = h(S_N)$ and maturity $N$, priced as time $k$ as

$$\begin{aligned} f\big(k, S_k^{(1)}\big) &= \frac{1}{(1+r)^{N-k}} \mathbb{E}_\theta^*\big[h(S_N) \,\big|\, \mathcal{F}_k\big] \\ &= \frac{1}{(1+r)^{N-k}} \mathbb{E}_\theta^*\big[h(S_N) \,\big|\, S_k^{(1)}\big], \end{aligned}$$

$k = 0, 1, \ldots, N$, under the risk-neutral probability measure $\mathbb{P}_\theta^*$. Find a recurrence equation between the functions $f(k, \cdot)$ and $f(k+1, \cdot)$, $k = 0, \ldots, N-1$.

Hint: Use the tower property of conditional expectations.

h) Assuming that $C$ is the payoff of the European *put* option with strike price $K$, give the expression of $f(N, x)$.
i) Modify the attached binomial Python code in order to make it deal with the trinomial model.
j) Taking $S_0^{(1)} = 1$, $r = 0.1$, $b = 1$, $(1+a)(1+b) = 1$, compute the price at time $k = 0$ of the European *put* option with strike price $K = 1$ and maturity $N = 2$ using the code of Question (i) with $\theta = 0.5$.

Download* and install the Anaconda distribution from https://www.anaconda.com/distribution/ or try it online at https://jupyter.org/try.

* Download the corresponding **IPython notebook** that can be run **here** or **here**.

```
%matplotlib inline
import networkx as nx
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
N=2;S0=1
r = 0.1;a=-0.5;b=1; # change
# add definition of theta
p = (r-a)/(b-a) # change
q = (b-r)/(b-a) # change
def plot_tree(g):
    plt.figure(figsize=(20,10))
    pos={};lab={}
    for n in g.nodes():
        pos[n]=(n[0],n[1])
        if g.nodes[n]['value'] is not None: lab[n]=float("{0:.2f}".format(g.nodes[n]['value']))
    elarge=g.edges(data=True)
    nx.draw_networkx_labels(g,pos,lab,font_size=15)
    nx.draw_networkx_nodes(g,pos,node_color='lightblue',alpha=0.4,node_size=1000)
    nx.draw_networkx_edges(g,pos,edge_color='blue',alpha=0.7,width=3,edgelist=elarge)
    plt.ylim(-N-0.5,N+0.5)
    plt.xlim(-0.5,N+0.5)
    plt.show()
```

```
def graph_stock():
    S=nx.Graph()
    for k in range(0,N):
        for l in range(-k,k+2,2): # change range and step size
            S.add_edge((k,l),(k+1,l+1)) # add edge
            S.add_edge((k,l),(k+1,l-1))
    for n in S.nodes():
        k=n[0]
        l=n[1]
        S.nodes[n]['value']=S0*((1.0+b)**((k+l)/2))*((1.0+a)**((k-l)/2))
    return S
plot_tree(graph_stock())
```

```
def European_call_price(K):
    price = nx.Graph()
    hedge = nx.Graph()
    S = graph_stock()
    for k in range(0,N):
            for l in range(-k,k+2,2): # change range and step size
                price.add_edge((k,l),(k+1,l+1)) # add edge
                price.add_edge((k,l),(k+1,l-1))
    for l in range(-N,N+2,2): # change range and step size
        price.nodes[(N,l)]['value'] = np.maximum(S.nodes[(N,l)]['value']-K,0)

    for k in reversed(range(0,N)):
        for l in range(-k,k+2,2): # change range and step size
            price.nodes[(k,l)]['value'] = (price.nodes[(k+1,l+1)]['value']*p
                  +price.nodes[(k+1,l-1)]['value']*q)/(1+r) # add theta
    return price
```

```
K = input("Strike K=")
call_price = European_call_price(float(K))
print('Underlying asset prices:')
plot_tree(graph_stock())
print('European call option prices:')
plot_tree(call_price)
print('Price at time 0 of the European call option:',
      float("{0:.4f}".format(call_price.nodes[(0,0)]['value'])))
```