# Chapter 10
# Maximum of Brownian Motion

The probability distribution of the maximum of Brownian motion on a given interval can be computed in closed form using the reflection principle. As a consequence, the expected value of the running maximum of Brownian motion can also be computed explicitly. Those properties will be applied in the next Chapters 11 and 12 to the pricing of barrier and lookback options, whose payoffs may depend on extrema of the underlying asset price process $(S_t)_{t\in[0,T]}$, as well as on its terminal value $S_T$.



## 10.1 Running Maximum of Brownian Motion

Figure 10.1 represents the running maximum process

$$X_0^t := \operatorname*{Max}_{s\in[0,t]} W_s, \qquad t \geqslant 0,$$

of Brownian motion $(W_t)_{t\in\mathbb{R}_+}$.

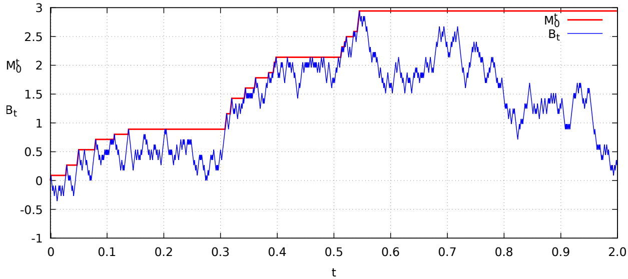


Fig. 10.1: Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ and its running maximum $(X_0^t)_{t\in\mathbb{R}_+}$.*

Note that Brownian motion admits (almost surely) no "point of increase". More precisely, there does not exist $t > 0$ and $\varepsilon > 0$ such that

$$\operatorname*{Max}_{s\in(t-\varepsilon,t)} W_s \leqslant W_t \leqslant \min_{s\in(t,t+\varepsilon)} W_s,$$

see, *e.g.*, Dvoretzky et al. (1961) and Burdzy (1990). This property is illustrated in Figure 10.2, see also (10.4)-(10.5) below.

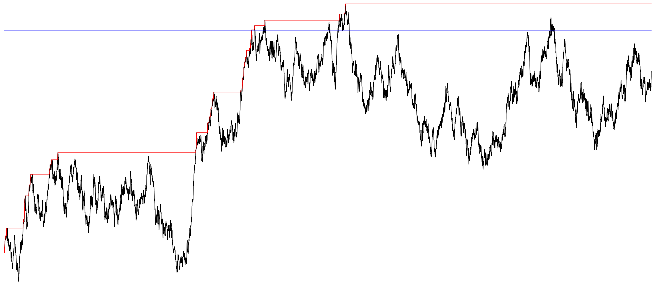

Fig. 10.2: Running maximum of Brownian motion.*

Related properties can be observed with the zeroes of Brownian motion which form an *uncountable* set (see *e.g.* Theorem 2.28 page 48 of Mörters and Peres (2010)) which has *zero measure* $\mathbb{P}$-almost surely, as we have

* The animation works in Acrobat Reader on the entire pdf file.

$$\mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{W_t=0\}}dt\right] = \int_0^\infty \mathbb{E}\left[\mathbb{1}_{\{W_t=0\}}\right]dt = \int_0^\infty \mathbb{P}(W_t=0)dt = 0,$$

see Figure 10.3.

Fig. 10.3: Zeroes of Brownian motion.*

See also the Cantor function presented in the next Figure 10.4, which is continuous on $[0,1]$ and flat (with a vanishing derivative) everywhere except on the *Cantor set*, which is an *uncountable* set of *zero measure* in $[0,1]$.

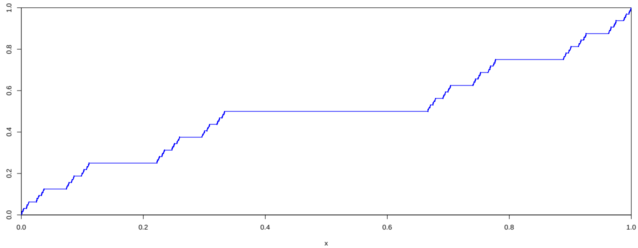


Fig. 10.4: Graph of the Cantor function.†

Examples of deterministic functions having no "last point of increase" can be built for some $\varepsilon \in (0,1)$ as

$$f(t) := (1-\varepsilon)\sum_{n\geqslant 1}\varepsilon^{n-1}\mathbb{1}_{[1-\varepsilon^n,1)}(t) + \mathbb{1}_{[1,\infty)}(t), \qquad t \geqslant 0,$$

* The animation works in Acrobat Reader on the entire pdf file.

† The animation works in Acrobat Reader on the entire pdf file.

which admits no "last" point of increase before $t = 1$, as illustrated in Figure 10.5 with $\varepsilon = 3/4$.

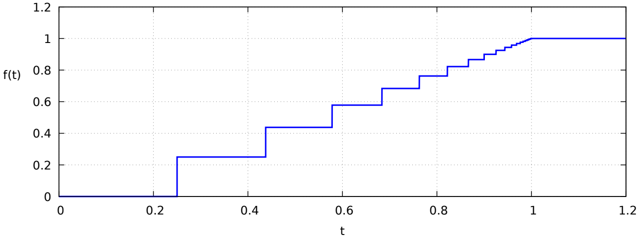


Fig. 10.5: A function with no last point of increase before $t = 1$.

## 10.2 The Reflection Principle

Let $(W_t)_{t\in\mathbb{R}_+}$ denote the standard Brownian motion started at $W_0 = 0$. While it is well-known that $W_T \simeq \mathcal{N}(0,T)$, computing the distribution of the maximum

$$X_0^T := \operatorname*{Max}_{t\in[0,T]} W_t$$

might seem a difficult problem. However, this is not the case, due to the *reflection principle*.

Note that since $W_0 = 0$, we have

$$X_0^T = \operatorname*{Max}_{t\in[0,T]} W_t \geqslant 0,$$

almost surely, *i.e.* with probability one. Given $a > W_0 = 0$, let

$$\tau_a = \inf\{t \geqslant 0 \ : \ W_t = a\}$$

denote the first time $(W_t)_{t\in\mathbb{R}_+}$ hits the level $a > 0$. Due to the spatial symmetry of Brownian motion we note the identity

$$\mathbb{P}(W_T \geqslant a \mid \tau_a \leqslant T) = \mathbb{P}(W_T > a \mid \tau_a \leqslant T) = \mathbb{P}(W_T \leqslant a \mid \tau_a \leqslant T) = \frac{1}{2}.$$

In addition, due to the relation

$$\{X_0^T \geqslant a\} = \{\tau_a \leqslant T\}, \tag{10.1}$$

we have

$$\begin{aligned}
\mathbb{P}(\tau_a \leqslant T) &= \mathbb{P}(\tau_a \leqslant T \text{ and } W_T > a) + \mathbb{P}(\tau_a \leqslant T \text{ and } W_T \leqslant a) \\
&= 2\mathbb{P}(\tau_a \leqslant T \text{ and } W_T \geqslant a) \\
&= 2\mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T \geqslant a\big) \\
&= 2\mathbb{P}(W_T \geqslant a) \\
&= \mathbb{P}(W_T \geqslant a) + \mathbb{P}(W_T \leqslant -a) \\
&= \mathbb{P}(|W_T| \geqslant a),
\end{aligned}$$

where we used the fact that

$$\{W_T \geqslant a\} \subset \big\{X_0^T \geqslant a \text{ and } W_T \geqslant a\big\} \subset \{W_T \geqslant a\}.$$

Figure 10.6 shows a graph of Brownian motion and its reflected path, with $0 < b < a < 2a - b$.

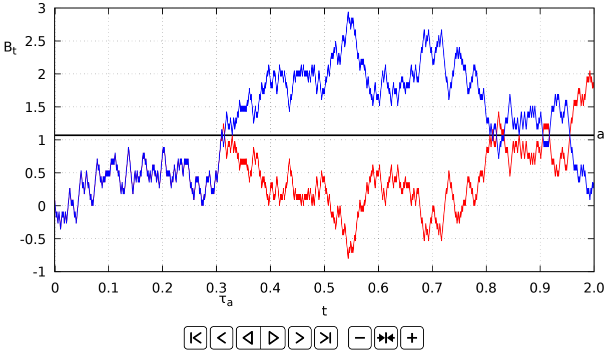


Fig. 10.6: Reflected Brownian motion with $a = 1.07$.*

As a consequence of the equality

$$\mathbb{P}(\tau_a \leqslant T) = \mathbb{P}(|W_T| \geqslant a), \qquad a > 0, \tag{10.2}$$

the maximum $X_0^T$ of Brownian motion has *same distribution* as the absolute value $|W_T|$ of $W_T$, which is a folded normal distribution, see Figure 10.7. Precisely, $X_0^T$ is a nonnegative random variable with cumulative distribution function given by

$$\begin{aligned}
\mathbb{P}\big(X_0^T < a\big) &= \mathbb{P}(\tau_a > T) \\
&= \mathbb{P}(|W_T| < a)
\end{aligned}$$

* The animation works in Acrobat Reader on the entire pdf file.

$$= \frac{1}{\sqrt{2\pi T}} \int_{-a}^{a} \mathrm{e}^{-x^2/(2T)} dx$$
$$= \frac{2}{\sqrt{2\pi T}} \int_{0}^{a} \mathrm{e}^{-x^2/(2T)} dx, \qquad a \geqslant 0,$$

*i.e.*

$$\mathbb{P}\big(X_0^T \leqslant a\big) = \frac{2}{\sqrt{2\pi T}} \int_{0}^{a} \mathrm{e}^{-x^2/(2T)} dx, \qquad a \geqslant 0,$$

and probability density function

$$\varphi_{X_0^T}(a) = \frac{d\mathbb{P}\big(X_0^T \leqslant a\big)}{da} = \sqrt{\frac{2}{\pi T}}\, \mathrm{e}^{-a^2/(2T)} \mathbb{1}_{[0,\infty)}(a), \quad a \in \mathbb{R}, \quad (10.3)$$

which vanishes over $a \in (-\infty, 0]$ because $X_0^T \geqslant 0$ almost surely.

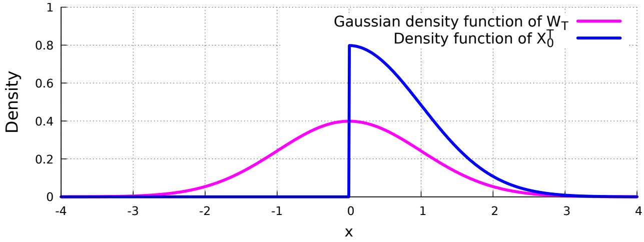


Fig. 10.7: Probability density of the maximum $X_0^1$ of Brownian motion over [0,1].

We note that, as a consequence of the existence of the probability density function (10.3), we have

$$\mathbb{P}(W_t \leqslant 0, \ \forall t \in [0, \varepsilon]) = \mathbb{P}(X_0^\varepsilon = 0) = \int_0^0 \varphi_{X_0^\varepsilon}(a) ds = 0, \qquad (10.4)$$

for all $\varepsilon > 0$. Similarly, by a symmetry argument, for all $\varepsilon > 0$ we find

$$\mathbb{P}(W_t \geqslant 0, \ \forall t \in [0, \varepsilon]) = 0, \qquad (10.5)$$

and similarly

$$\mathbb{P}(W_t \leqslant 0, \ \forall t \in [0, \varepsilon]) = 0.$$

Using the probability density function of $X_0^T$, we can price an option with payoff $\phi\big(X_0^T\big)$, as

$$\mathrm{e}^{-rT} \mathbb{E}^*\big[\phi\big(X_0^T\big)\big] = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \phi(x) d\mathbb{P}\big(X_0^T \leqslant x\big)$$



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$$= \mathrm{e}^{-rT} \sqrt{\frac{2}{\pi T}} \int_0^\infty \phi(x) \, \mathrm{e}^{-x^2/(2T)} dx.$$

**Proposition 10.1.** *Let* $\sigma > 0$ *and* $(S_t)_{t\in[0,T]} := \big(S_0 \, \mathrm{e}^{\sigma W_t}\big)_{t\in[0,T]}$. *The probability density function of the maximum*

$$M_0^T := \operatorname*{Max}_{t\in[0,T]} S_t$$

*of* $(S_t)_{t\in[0,T]}$ *over the time interval* $[0,T]$ *is given by the truncated lognormal probability density function*

$$\varphi_{M_0^T}(y) = \mathbb{1}_{[S_0,\infty)}(y) \frac{1}{\sigma y} \sqrt{\frac{2}{\pi T}} \exp\left( -\frac{1}{2\sigma^2 T} \left(\log\left(y/S_0\right)\right)^2 \right), \quad y > 0,$$

*see Figure 10.8.*

*Proof.* Since $\sigma > 0$, we have

$$\begin{aligned} M_0^T &= \operatorname*{Max}_{t\in[0,T]} S_t \\ &= S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t} \\ &= S_0 \, \mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,T]} W_t} \\ &= S_0 \, \mathrm{e}^{\sigma X_0^T}. \end{aligned}$$

Hence $M_0^T = h\big(X_0^T\big)$ with $h(x) = S_0 \, \mathrm{e}^{\sigma x}$, and

$$h'(x) = \sigma S_0 \, \mathrm{e}^{\sigma x}, \quad x \in \mathbb{R}, \ \text{ and } \ h^{-1}(y) = \frac{1}{\sigma} \log\left(\frac{y}{S_0}\right), \quad y > 0,$$

hence

$$\begin{aligned} \varphi_{M_0^T}(y) &= \frac{1}{|h'(h^{-1}(y))|} \varphi_{X_0^T}(h^{-1}(y)) \\ &= \mathbb{1}_{[0,\infty)}(h^{-1}(y)) \frac{\sqrt{2}}{|h'(h^{-1}(y))|\sqrt{\pi T}} \, \mathrm{e}^{-(h^{-1}(y))^2/(2T)} \\ &= \mathbb{1}_{[S_0,\infty)}(y) \frac{1}{\sigma y} \sqrt{\frac{2}{\pi T}} \exp\left( -\frac{1}{2\sigma^2 T} \left(\log\left(y/S_0\right)\right)^2 \right), \quad y > 0. \end{aligned}$$

□

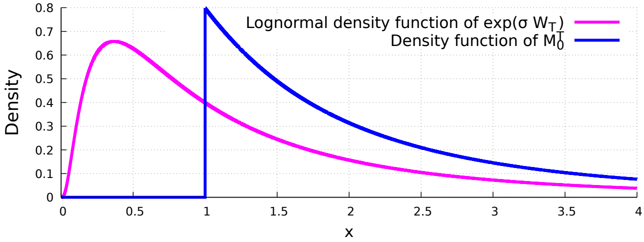


Fig. 10.8: Density of the maximum $M_0^T = \mathrm{Max}_{t\in[0,T]} S_t$ of geometric Brownian motion with $S_0 = 1$.

When the claim payoff takes the form $C = \phi(M_0^T)$, where $S_T = S_0 \, \mathrm{e}^{\sigma W_T}$, we have

$$C = \phi(M_0^T) = \phi\big(S_0 \, \mathrm{e}^{\sigma X_0^T}\big),$$

hence

$$\begin{aligned}
\mathrm{e}^{-rT}\mathbb{E}^*[C] &= \mathrm{e}^{-rT}\mathbb{E}^*\big[\phi\big(S_0 \, \mathrm{e}^{\sigma X_0^T}\big)\big] \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty}\phi(S_0 \, \mathrm{e}^{\sigma x}) d\mathbb{P}(X_0^T \leqslant x) \\
&= \sqrt{\frac{2}{\pi T}}\, \mathrm{e}^{-rT}\int_0^{\infty}\phi(S_0 \, \mathrm{e}^{\sigma x}) \, \mathrm{e}^{-x^2/(2T)} dx \\
&= \sqrt{\frac{2}{\pi\sigma^2 T}}\, \mathrm{e}^{-rT}\int_1^{\infty}\phi(y)\exp\left(-\frac{1}{2\sigma^2 T}\left(\log\left(y/S_0\right)\right)^2\right)\frac{dy}{y},
\end{aligned}$$

after the change of variable $y = S_0 \, \mathrm{e}^{\sigma x}$ with $dx = dy/(\sigma y)$.

The above computation is however not sufficient for practical applications as it imposes the condition $r = \sigma^2/2$. In order to do away with this condition we need to consider the maximum of *drifted* Brownian motion, and for this we have to compute the *joint* probability density function of $X_0^T$ and $W_T$.

## 10.3 Maximum of Drifted Brownian Motion

The reflection principle also allows us to compute the *joint* probability density function of Brownian motion $W_T$ and its maximum $X_0^T = \underset{t\in[0,T]}{\mathrm{Max}} W_t$. Recall that the probability density function $\varphi_{X_0^T, W_T}$ can be recovered from the joint cumulative distribution function

$$\begin{aligned}
(x, y) \longmapsto F_{X_0^T, W_T}(x, y) &:= \mathbb{P}\big(X_0^T \leqslant x \text{ and } W_T \leqslant y\big) \\
&= \int_{-\infty}^{x}\int_{-\infty}^{y}\varphi_{X_0^T, W_T}(s, t) ds dt,
\end{aligned}$$



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and

$$(x, y) \longmapsto \mathbb{P}\big(X_0^T \geqslant x \text{ and } W_T \geqslant y\big) = \int_x^\infty \int_y^\infty \varphi_{X_0^T, W_T}(s,t) ds dt,$$

as

$$\varphi_{X_0^T, W_T}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X_0^T, W_T}(x,y) \quad (10.6)$$

$$= \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^x \int_{-\infty}^y \varphi_{X_0^T, W_T}(s,t) ds dt \quad (10.7)$$

$$= \frac{\partial^2}{\partial x \partial y} \int_x^\infty \int_y^\infty \varphi_{X_0^T, W_T}(s,t) ds dt, \qquad x, y \in \mathbb{R}.$$

The probability densities $\varphi_{X_0^T} : \mathbb{R} \longrightarrow \mathbb{R}_+$ and $\varphi_{W_T} : \mathbb{R} \longrightarrow \mathbb{R}_+$ of $X_0^T$ and $W_T$ are called the marginal densities of $\big(X_0^T, W_T\big)$, and are given by

$$\varphi_{X_0^T}(x) = \int_{-\infty}^\infty \varphi_{X_0^T, W_T}(x,y) dy, \qquad x \in \mathbb{R},$$

and

$$\varphi_{W_T}(y) = \int_{-\infty}^\infty \varphi_{X_0^T, W_T}(x,y) dx, \qquad y \in \mathbb{R}.$$

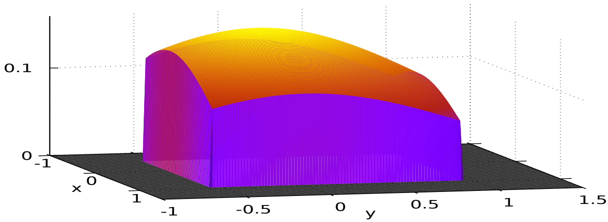


Fig. 10.9: Probability $\mathbb{P}((X,Y) \in [-0.5,1] \times [-0.5,1])$ computed as a volume integral.

In order to compute the *joint* probability density function of Brownian motion $W_T$ and its maximum $X_0^T = \operatorname{Max}_{t \in [0,T]} W_t$ by the reflection principle, we note that for any $b \leqslant a$ we have

$$\mathbb{P}(W_T < b \mid \tau_a < T) = \mathbb{P}(W_T > a + (a-b) \mid \tau_a < T)$$

as shown in Figure 10.10, *i.e.*

$$\mathbb{P}(W_T < b \text{ and } \tau_a < T) = \mathbb{P}(W_T > 2a - b \text{ and } \tau_a < T),$$

or, by (10.1),

$$\mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T < b\big) = \mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T > 2a - b\big).$$

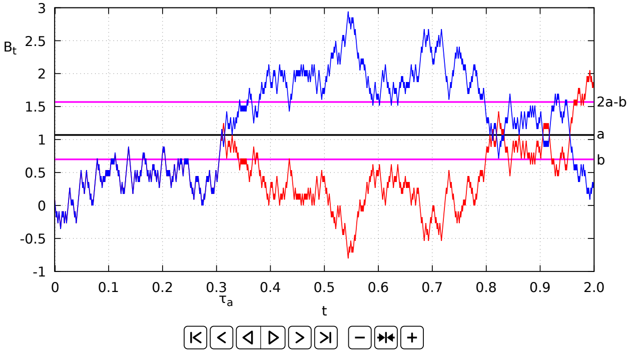


Fig. 10.10: Reflected Brownian motion with $a = 1.07$.*

Hence, since $2a - b \geqslant a$ we have

$$\mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T < b\big) = \mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T > 2a - b\big) = \mathbb{P}(W_T \geqslant 2a - b), \tag{10.8}$$

where we used the fact that

$$\begin{aligned}\{W_T \geqslant 2a - b\} &\subset \big\{X_0^T \geqslant 2a - b \text{ and } W_T > 2a - b\big\} \\ &\subset \big\{X_0^T \geqslant a \text{ and } W_T > 2a - b\big\} \subset \{W_T > 2a - b\},\end{aligned}$$

which shows that

$$\{W_T \geqslant 2a - b\} = \big\{X_0^T \geqslant a \text{ and } \ W_T > 2a - b\big\}.$$

Consequently, by (10.8) we find

$$\begin{aligned}\mathbb{P}\big(X_0^T > a \text{ and } W_T \leqslant b\big) &= \mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T < b\big) \\ &= \mathbb{P}(W_T \geqslant 2a - b) \\ &= \frac{1}{\sqrt{2\pi T}} \int_{2a-b}^{\infty} \mathrm{e}^{-x^2/(2T)} dx, \end{aligned} \tag{10.9}$$

* The animation works in Acrobat Reader on the entire pdf file.

$0 \leqslant b \leqslant a$, which yields the joint probability density function

$$\begin{aligned}
\varphi_{X_0^T, W_T}(a, b) &= \frac{\partial^2}{\partial a \partial b} \mathbb{P}\big(X_0^T \leqslant a \text{ and } W_T \leqslant b\big) \\
&= \frac{\partial^2}{\partial a \partial b} \big(\mathbb{P}\big(W_T \leqslant b\big) - \mathbb{P}\big(X_0^T > a \text{ and } W_T \leqslant b\big)\big) \\
&= -\frac{d\mathbb{P}\big(X_0^T > a \text{ and } W_T \leqslant b\big)}{dadb}, \qquad a, b \in \mathbb{R}.
\end{aligned}$$

By (10.9), we obtain the following proposition.

**Proposition 10.2.** *The joint probability density function $\varphi_{X_0^T, W_T}$ of Brownian motion $W_T$ and its maximum $X_0^T = \underset{t \in [0,T]}{\mathrm{Max}} W_t$ is given by*

$$\varphi_{X_0^T, W_T}(a, b) = \sqrt{\frac{2}{\pi}} \frac{(2a - b)}{T^{3/2}} \mathrm{e}^{-(2a-b)^2/(2T)} \mathbb{1}_{\{a \geqslant \mathrm{Max}(b,0)\}} \tag{10.10}$$

$$= \begin{cases} \sqrt{\dfrac{2}{\pi}} \dfrac{(2a - b)}{T^{3/2}} \mathrm{e}^{-(2a-b)^2/(2T)}, & a > \mathrm{Max}(b, 0), \\[2ex] 0, & a < \mathrm{Max}(b, 0). \end{cases}$$

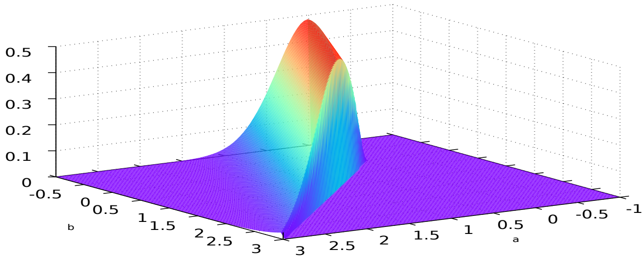


Fig. 10.11: Joint probability density of $W_1$ and the maximum $X_0^1$ over [0,1].

Figure 10.12 presents the *heat map* of Figure 10.11, as viewed from above.

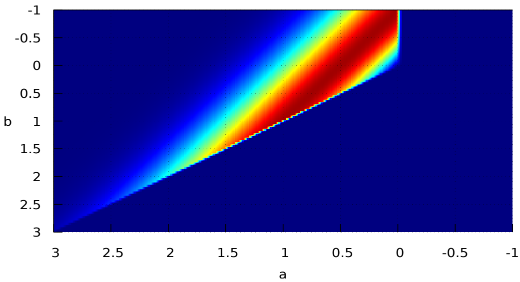


Fig. 10.12: Heat map of the joint density of $W_1$ and its maximum $\widehat{X}_0^1$ over [0,1].

See Relation (4.44) in Borodin (2017) for the joint distribution of the minimum $\min_{t\in[0,T]} W_t$, the maximum $\operatorname{Max}_{t\in[0,T]} W_t$ and the endpoint $W_t$ of Brownian motion.

### Maximum of drifted Brownian motion

Using the Girsanov Theorem, it is even possible to compute the probability density function of the maximum

$$\widehat{X}_0^T := \operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t = \operatorname*{Max}_{t\in[0,T]} (W_t + \mu t)$$

of drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0, T]$, for any $\mu \in \mathbb{R}$.

**Proposition 10.3.** *The joint probability density function* $\varphi_{\widehat{X}_0^T, \widetilde{W}_T}$ *of the drifted Brownian motion* $\widetilde{W}_T := W_T + \mu T$ *and its maximum* $\widehat{X}_0^T = \operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t$ *is given by*

$$\varphi_{\widehat{X}_0^T, \widetilde{W}_T}(a, b) = \mathbb{1}_{\{a \geqslant \operatorname{Max}(b,0)\}} \frac{1}{T}\sqrt{\frac{2}{\pi T}}(2a - b)\, \mathrm{e}^{\mu b - (2a-b)^2/(2T) - \mu^2 T/2} \tag{10.11}$$

$$= \begin{cases} \dfrac{1}{T}\sqrt{\dfrac{2}{\pi T}}(2a - b)\, \mathrm{e}^{-\mu^2 T/2 + \mu b - (2a-b)^2/(2T)}, & a > \operatorname{Max}(b, 0), \\ 0, & a < \operatorname{Max}(b, 0). \end{cases}$$

*Proof.* The arguments previously applied to the standard Brownian motion $(W_t)_{t\in[0,T]}$ cannot be directly applied to $\big(\widetilde{W}_t\big)_{t\in[0,T]}$ because drifted Brownian motion is no longer symmetric in space when $\mu \neq 0$. On the other hand, the drifted process $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined from the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := e^{-\mu W_T - \mu^2 T/2}, \tag{10.12}$$

and the joint probability density function of $\big(\widehat{X}_0^T, \widetilde{W}_T\big)$ under $\widetilde{\mathbb{P}}$ is given by (10.10). Now, using the probability density function (10.12) and the relation $\widetilde{W}_t := W_t + \mu t$, we get

$$\begin{aligned}
&\mathbb{P}\big(\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\big) = \mathbb{E}\left[\mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \int_\Omega \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}} d\mathbb{P} \\
&= \int_\Omega \frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}} d\widetilde{\mathbb{P}} \\
&= \widetilde{\mathbb{E}}\left[\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{\mu W_T + \mu^2 T/2} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{\mu \widetilde{W}_T - \mu^2 T/2} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^b \mathbb{1}_{(-\infty,x]}(y)\, e^{\mu y - \mu^2 T/2} \frac{(2x-y)}{T}\, e^{-(2x-y)^2/(2T)} dy dx,
\end{aligned}$$

$0 \leqslant b \leqslant a$, which yields the joint probability density function (10.11) from the differentiation

$$\varphi_{\widehat{X}_0^T, \widetilde{W}_T}(a,b) = \frac{d\mathbb{P}\big(\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\big)}{da db}.$$

□

The following proposition is consistent with (10.3) in case $\mu = 0$.

**Proposition 10.4.** *The cumulative distribution function of the maximum*

$$\widehat{X}_0^T := \operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t = \operatorname*{Max}_{t\in[0,T]} (W_t + \mu t)$$

*of drifted Brownian motion* $\widetilde{W}_t = W_t + \mu t$ *over* $t \in [0,T]$ *is given by*

$$\mathbb{P}\big(\widehat{X}_0^T \leqslant a\big) = \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) - \mathrm{e}^{2\mu a}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right), \quad a \geqslant 0, \qquad (10.13)$$

*and the probability density function* $\varphi_{\widehat{X}_0^T}$ *of* $\widehat{X}_0^T$ *satisfies*

$$\varphi_{\widehat{X}_0^T}(a) = \sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(a-\mu T)^2/(2T)} - 2\mu\,\mathrm{e}^{2\mu a}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right), \quad a \geqslant 0. \quad (10.14)$$

*Proof.* Letting $\underline{a \vee b := \mathrm{Max}(a,b)}$, $a, b \in \mathbb{R}$, since the condition ($y \leqslant x$ and $0 \leqslant x \leqslant a$) is equivalent to the condition ($y \vee 0 \leqslant x \leqslant a$), we have

$$\begin{aligned}
\mathbb{P}\big(\widehat{X}_0^T \leqslant a\big) &= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^\infty \mathbb{1}_{(-\infty,x]}(y)\,\mathrm{e}^{\mu y-\mu^2T/2}\,\frac{(2x-y)}{T}\,\mathrm{e}^{-(2x-y)^2/(2T)}dydx \\
&= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^x \mathrm{e}^{\mu y-\mu^2T/2}\,\frac{(2x-y)}{T}\,\mathrm{e}^{-(2x-y)^2/(2T)}dydx \\
&= \sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-\mu^2T/2} \int_{-\infty}^a \mathrm{e}^{\mu y} \int_{y\vee 0}^a \frac{(2x-y)}{T}\,\mathrm{e}^{-(2x-y)^2/(2T)}dxdy.
\end{aligned}$$

Next, since

$$2(y\vee 0)^2 - y = \begin{cases} 2\times 0 - y = -y, & y \leqslant 0, \\ 2y - y = y, & y \geqslant 0, \end{cases}$$

and using the "completion of the square" identity

$$\mu y - \frac{(2a-y)^2}{2T} - \frac{\mu^2 T}{2} = 2a\mu - \frac{1}{2T}(y-(\mu T+2a))^2$$

and a standard changes of variables, we have

$$\begin{aligned}
&\mathbb{P}\big(\widehat{X}_0^T \leqslant a\big) = \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^\infty \mathbb{1}_{(-\infty,x]}(y)\,\mathrm{e}^{\mu y-\mu^2T/2}\,\frac{(2x-y)}{T}\,\mathrm{e}^{-(2x-y)^2/(2T)}dydx \\
&= \frac{1}{\sqrt{2\pi T}}\,\mathrm{e}^{-\mu^2T/2}\int_{-\infty}^a \big(\mathrm{e}^{\mu y-(2(y\vee 0)-y)^2/(2T)} - \mathrm{e}^{\mu y-(2a-y)^2/(2T)}\big)dy \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^a \big(\mathrm{e}^{\mu y-y^2/(2T)-\mu^2T/2} - \mathrm{e}^{\mu y-(2a-y)^2/(2T)-\mu^2T/2}\big)dy \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^a \big(\mathrm{e}^{-(y-\mu T)^2/(2T)} - \mathrm{e}^{-(y-(\mu T+2a))^2/(2T)+2a\mu}\big)dy \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^a \mathrm{e}^{-(y-\mu T)^2/(2T)}dy - \mathrm{e}^{2a\mu}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^a \mathrm{e}^{-(y-(\mu T+2a))^2/(2T)}dy \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{a-\mu T} \mathrm{e}^{-y^2/(2T)}dy - \mathrm{e}^{2a\mu}\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{-a-\mu T} \mathrm{e}^{-y^2/(2T)}dy \\
&= \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) - \mathrm{e}^{2\mu a}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right), \qquad a \geqslant 0,
\end{aligned}$$



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cf. Corollary 7.2.2 and pages 297-299 of Shreve (2004) for another derivation. □

See Profeta et al. (2010) for interpretations of (10.13) and (10.15) in terms of the Black-Scholes formula.

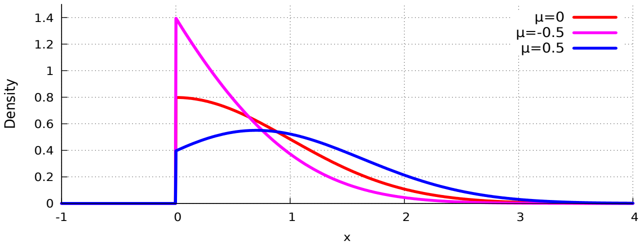


Fig. 10.13: Probability density of the maximum $\widehat{X}_0^T$ of drifted Brownian motion.

We note from Figure 10.13 that small values of the maximum are more likely to occur when $\mu$ takes large negative values. As $T$ tends to infinity, Proposition 10.4 also shows that when $\mu < 0$, the maximum of drifted Brownian motion $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+} = (W_t + \mu t)_{t\in\mathbb{R}_+}$ over all time has an exponential distribution with parameter $2|\mu|$, *i.e.*

$$\varphi_{\widehat{X}_0^\infty}(a) = 2|\mu|\,\mathrm{e}^{-2|\mu|a}, \qquad a \geqslant 0.$$

Relation (10.13), resp. Relation (10.16) below, will be used for the pricing of lookback call, resp. put options in Section 10.4. See also Exercise 10.8 for the joint probability density function of geometric Brownian motion $S_T := S_0\,\mathrm{e}^{\sigma W_T + (r-\sigma^2/2)T}$ and its maximum $M_0^T := \underset{t\in[0,T]}{\mathrm{Max}}\, S_t$.

**Corollary 10.5.** *The cumulative distribution function of the maximum*

$$M_0^T := \underset{t\in[0,T]}{\mathrm{Max}}\, S_t = S_0 \underset{t\in[0,T]}{\mathrm{Max}}\, \mathrm{e}^{\sigma W_t + (r-\sigma^2/2)t}$$

*of geometric Brownian motion over $t\in[0,T]$ is given by*

$$\mathbb{P}\big(M_0^T \leqslant x\big) = \Phi\left(\frac{-(r-\sigma^2/2)T + \log(x/S_0)}{\sigma\sqrt{T}}\right) \qquad (10.15)$$
$$- \left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{-(r-\sigma^2/2)T - \log(x/S_0)}{\sigma\sqrt{T}}\right), \quad x \geqslant S_0,$$

*and the probability density function $\varphi_{M_0^T}$ of $M_0^T$ satisfies*

$$
\begin{aligned}
\varphi_{M_0^T}(x) &= \frac{1}{\sigma x\sqrt{2\pi T}}\exp\left(-\frac{(-(r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right)\\
&+\frac{1}{\sigma x\sqrt{2\pi T}}\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\exp\left(-\frac{((r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right)\\
&+\frac{1}{x}\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{-(r-\sigma^2/2)T-\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad x \geqslant S_0.
\end{aligned}
$$

*Proof.* Taking

$$
\widetilde{W}_t := W_t + \mu t = W_t + \frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)t
$$

with $\mu := r/\sigma - \sigma/2$, by (10.13) we find

$$
\begin{aligned}
&\mathbb{P}\big(M_0^T \leqslant x\big) = \mathbb{P}\left(\mathrm{e}^{\sigma\widehat{X}_0^T} \leqslant \frac{x}{S_0}\right)\\
&= \mathbb{P}\left(\widehat{X}_0^T \leqslant \frac{1}{\sigma}\log\frac{x}{S_0}\right)\\
&= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) - \mathrm{e}^{2\mu\sigma^{-1}\log(x/S_0)}\Phi\left(\frac{-\mu T-\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)\\
&= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) - \left(\frac{x}{S_0}\right)^{2\mu/\sigma}\Phi\left(\frac{-\mu T-\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)\\
&= \Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right)\\
&\quad - \left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{-(r-\sigma^2/2)T-\log(x/S_0)}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

□

### Minimum of drifted Brownian motion

**Proposition 10.6.** *The joint probability density function* $\varphi_{\check{X}_0^T,\widetilde{W}_T}$ *of the minimum of the drifted Brownian motion* $\widetilde{W}_t := W_t + \mu t$ *and its value* $\widetilde{W}_T$ *at time* $T$ *is given by*

$$\varphi_{\breve{X}_0^T,\widetilde{W}_T}(a,b) = \mathbb{1}_{\{a\leqslant \min(b,0)\}}\frac{1}{T}\sqrt{\frac{2}{\pi T}}(b-2a)\,\mathrm{e}^{\mu b-(2a-b)^2/(2T)-\mu^2T/2}$$

$$= \begin{cases} \dfrac{1}{T}\sqrt{\dfrac{2}{\pi T}}(b-2a)\,\mathrm{e}^{-\mu^2T/2+\mu b-(2a-b)^2/(2T)}, & a < \min(b,0), \\ 0, & a > \min(b,0). \end{cases}$$

*Proof.* We use the relations

$$\min_{t\in[0,T]} \widetilde{W}_t = -\operatorname*{Max}_{t\in[0,T]}\big(-\widetilde{W}_t\big),$$

and

$$\begin{aligned} \breve{X}_0^T &:= \min_{t\in[0,T]} \widetilde{W}_t \\ &= \min_{t\in[0,T]}(W_t+\mu t) \\ &= -\operatorname*{Max}_{t\in[0,T]}\big(-\widetilde{W}_t\big) \\ &= -\operatorname*{Max}_{t\in[0,T]}(-W_t-\mu t) \\ &\simeq -\operatorname*{Max}_{t\in[0,T]}(W_t-\mu t), \end{aligned}$$

where the last equality "$\simeq$" follows from the identity in distribution of $(W_t)_{t\in\mathbb{R}_+}$ and $(-W_t)_{t\in\mathbb{R}_+}$, and we conclude by applying the change of variables $(a,b,\mu)\mapsto(-a,-b,-\mu)$ to (10.11). □

Similarly to the above, the following proposition holds for the minimum drifted Brownian motion, and Relation (10.17) below can be obtained by changing the signs of both $a$ and $\mu$ in Proposition 10.4.

**Proposition 10.7.** *The cumulative distribution function and probability density function of the minimum*

$$\breve{X}_0^T := \min_{t\in[0,T]} \widetilde{W}_t = \min_{t\in[0,T]}(W_t+\mu t)$$

*of the drifted Brownian motion* $\widetilde{W}_t = W_t+\mu t$ *over* $t\in[0,T]$ *are given by*

$$\mathbb{P}\big(\breve{X}_0^T \leqslant a\big) = \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right), \qquad a\leqslant 0, \qquad (10.16)$$

*and*

$$\varphi_{\breve{X}_0^T}(a) = \sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(a-\mu T)^2/(2T)} + 2\mu\,\mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right), \quad a \leqslant 0. \quad (10.17)$$

*Proof.* From (10.13), the cumulative distribution function of the minimum of drifted Brownian motion can be expressed as

$$\begin{aligned}
\mathbb{P}\big(\breve{X}_0^T \leqslant a\big) &= \mathbb{P}\left(\min_{t\in[0,T]} \widetilde{W}_t \leqslant a\right) \\
&= \mathbb{P}\left(\min_{t\in[0,T]} (W_t + \mu t) \leqslant a\right) \\
&= \mathbb{P}\left(-\operatorname*{Max}_{t\in[0,T]} (-W_t - \mu t) \leqslant a\right) \\
&= \mathbb{P}\left(-\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t) \leqslant a\right) \\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t) \geqslant -a\right) \\
&= 1 - \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t) \leqslant -a\right) \\
&= 1 - \Phi\left(\frac{-a+\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right) \\
&= \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right), \qquad a \leqslant 0,
\end{aligned}$$

where we used the identity in distribution of $(W_t)_{t\in\mathbb{R}_+}$ and $(-W_t)_{t\in\mathbb{R}_+}$, hence the probability density function of the minimum of drifted Brownian motion is given by (10.17). □

From (10.16), we also have

$$\mathbb{P}\big(\breve{X}_0^T > a\big) = \Phi\left(\frac{\mu T - a}{\sqrt{T}}\right) - \mathrm{e}^{2a\mu}\Phi\left(\frac{\mu T + a}{\sqrt{T}}\right), \quad a \leqslant 0.$$

If $\mu > 0$, letting $T$ tend to infinity we find that the minimum of the positively drifted Brownian motion $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+} = (W_t + \mu t)_{t\in\mathbb{R}_+}$ over all time has an exponential distribution with parameter $2\mu$ on $\mathbb{R}_-$, *i.e.*

$$\varphi_{\breve{X}_0^\infty}(a) = 2\mu\,\mathrm{e}^{2\mu a}, \qquad a \leqslant 0.$$

In addition, as in Corollary 10.5, we have the following result.

**Corollary 10.8.** *The cumulative distribution function of the minimum*



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$$m_0^T := \min_{t\in[0,T]} S_t = S_0 \min_{t\in[0,T]} \mathrm{e}^{\sigma W_t+(r-\sigma^2/2)t}$$

*of geometric Brownian motion over $t \in [0,T]$ is given by*

$$\mathbb{P}\big(m_0^T \leqslant x\big) = \Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \tag{10.18}$$
$$+\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad 0 < x \leqslant S_0,$$

*and the probability density function $\varphi_{m_0^T}$ of $m_0^T$ satisfies*

$$\varphi_{m_0^T}(x) = \frac{1}{\sigma x\sqrt{2\pi T}} \exp\left(-\frac{(-(r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right)$$
$$+\frac{1}{\sigma x\sqrt{2\pi T}}\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \exp\left(-\frac{((r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right)$$
$$+\frac{1}{x}\left(\frac{2r}{\sigma^2}-1\right)\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad 0 < x \leqslant S_0.$$

*Proof.* From (10.16) we have

$$\begin{aligned}
&\mathbb{P}\big(m_0^T \leqslant x\big) \\
&= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) + \mathrm{e}^{2\mu\sigma^{-1}\log(x/S_0)}\Phi\left(\frac{\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) \\
&= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) + \left(\frac{x}{S_0}\right)^{2\mu/\sigma} \Phi\left(\frac{\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) \\
&= \Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad +\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad 0 < x \leqslant S_0,
\end{aligned}$$

with $\mu := r/\sigma - \sigma/2$. The probability density function $\varphi_{m_0^T}$ is computed from

$$\varphi_{m_0^T}(x) = \frac{\partial}{\partial x}\mathbb{P}\big(m_0^T \leqslant x\big), \qquad 0 < x \leqslant S_0.$$

□

## 10.4 Average of Geometric Brownian Extrema

Let

$$m_s^t = \min_{u\in[s,t]} S_u \quad \text{and} \quad M_s^t = \operatorname*{Max}_{u\in[s,t]} S_u,$$

$0 \leqslant s \leqslant t \leqslant T$, and let $\mathcal{M}_s^t$ be either $m_s^t$ or $M_s^t$. In the lookback option case the payoff $\phi(S_T, \mathcal{M}_0^T)$ depends not only on the price of the underlying asset at maturity but it also depends on all price values of the underlying asset over the period which starts from the initial time and ends at maturity.

The payoff of such of an option is of the form $\phi(S_T, \mathcal{M}_0^T)$ with $\phi(x,y) = x - y$ in the case of lookback call options, and $\phi(x,y) = y - x$ in the case of lookback put options. We let

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\big(S_T, \mathcal{M}_0^T\big)\big|\mathcal{F}_t\right]$$

denote the price at time $t \in [0,T]$ of such an option.

### Maximum selling price over $[0, T]$

In the next proposition we start by computing the average of the maximum selling price $M_0^T := \underset{t\in[0,T]}{\operatorname{Max}} S_t$ of $(S_t)_{t\in[0,T]}$ over the time interval $[0,T]$. We denote

$$\delta_\pm^\tau(s) := \frac{1}{\sigma\sqrt{\tau}}\left(\log s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right), \qquad s > 0. \tag{10.19}$$

**Proposition 10.9.** *The average maximum value of $(S_t)_{t\in[0,T]}$ over $[0,T]$ is given by*

$$\begin{aligned}
&\mathbb{E}^*\left[M_0^T \,\middle|\, \mathcal{F}_t\right] \qquad (10.20)\\
&\quad = M_0^t \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t\,\mathrm{e}^{(T-t)r}\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)\\
&\qquad - S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right),
\end{aligned}$$

*where $\delta_\pm^{T-t}$ is defined in* (10.19).

When $t = 0$ we have $S_0 = M_0^0$, and given that

$$\delta_\pm^T(1) = \frac{r \pm \sigma^2/2}{\sigma}\sqrt{T}, \tag{10.21}$$

the formula (10.20) simplifies to

$$\begin{aligned}
&\mathbb{E}^*\left[M_0^T\right]\\
&= S_0\left(1 - \frac{\sigma^2}{2r}\right)\Phi\left(\frac{\sigma^2/2 - r}{\sigma}\sqrt{T}\right) + S_0\,\mathrm{e}^{rT}\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left(\frac{\sigma^2/2 + r}{\sigma}\sqrt{T}\right),
\end{aligned}$$



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with

$$\mathbb{E}^*\big[M_0^T\big] = 2S_0\left(1+\frac{\sigma^2 T}{4}\Phi\left(\sigma\frac{\sqrt{T}}{2}\right)\right) + \sigma S_0\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8}$$

when $r = 0$, cf. Exercise 12.2.

In general, when $T$ tends to infinity we find that

$$\lim_{T\to\infty}\frac{\mathbb{E}^*\big[M_0^T \mid \mathcal{F}_t\big]}{\mathbb{E}^*[S_T \mid \mathcal{F}_t]} = \begin{cases} 1+\dfrac{\sigma^2}{2r} & \text{if } r>0, \\ \infty & \text{if } r=0, \end{cases}$$

see Exercise 10.3-(d) in the case $r = \sigma^2/2$.

**Proof of Proposition 10.9**. We have

$$\begin{aligned}
\mathbb{E}^*\big[M_0^T \mid \mathcal{F}_t\big] &= \mathbb{E}^*\big[\operatorname{Max}\left(M_0^t, M_t^T\right) \mid \mathcal{F}_t\big] \\
&= \mathbb{E}^*\big[M_0^t \mathbb{1}_{\{M_0^t > M_t^T\}} \mid \mathcal{F}_t\big] + \mathbb{E}^*\big[M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \mid \mathcal{F}_t\big] \\
&= M_0^t \mathbb{E}^*\big[\mathbb{1}_{\{M_0^t > M_t^T\}} \mid \mathcal{F}_t\big] + \mathbb{E}^*\big[M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \mid \mathcal{F}_t\big] \\
&= M_0^t \mathbb{P}\big(M_0^t > M_t^T \mid \mathcal{F}_t\big) + \mathbb{E}^*\big[M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \mid \mathcal{F}_t\big].
\end{aligned}$$

Next, we have

$$\begin{aligned}
\mathbb{P}\big(M_0^t > M_t^T \mid \mathcal{F}_t\big) &= \mathbb{P}\left(\frac{M_0^t}{S_t} > \frac{M_t^T}{S_t} \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{P}\left(x > \frac{M_t^T}{S_t} \,\middle|\, \mathcal{F}_t\right)_{x = M_0^t/S_t} \\
&= \mathbb{P}\left(\frac{M_0^{T-t}}{S_0} < x\right)_{x = M_0^t/S_t}.
\end{aligned}$$

On the other hand, letting $\mu := r/\sigma - \sigma/2$, from (10.13) or (10.15) in Corollary 10.5 we have

$$\begin{aligned}
\mathbb{P}\left(\frac{M_0^T}{S_0} < x\right) &= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t + rt - \sigma^2 t/2} < x\right) \\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{(W_t+\mu t)\sigma} < x\right) \\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma \widetilde{W}_t} < x\right) \\
&= \mathbb{P}\left(\mathrm{e}^{\sigma \widehat{X}_0^T} < x\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left(\widehat{X}_T < \frac{1}{\sigma}\log x\right) \\
&= \Phi\left(\frac{-\mu T + \sigma^{-1}\log x}{\sqrt{T}}\right) - \mathrm{e}^{2\mu\sigma^{-1}\log x}\Phi\left(\frac{-\mu T - \sigma^{-1}\log x}{\sqrt{T}}\right) \\
&= \Phi\left(-\delta_-^T\left(\frac{1}{x}\right)\right) - x^{-1+2r/\sigma^2}\Phi\left(-\delta_-^T(x)\right).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\mathbb{P}\big(M_0^t > M_t^T \bigm| \mathcal{F}_t\big) &= \mathbb{P}\left(\frac{M_0^{T-t}}{S_0} < x\right)_{x = M_0^t/S_t} \\
&= \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{M_0^t}{S_t}\right)^{-1+2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&\mathbb{E}^*\left[M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \Bigm| \mathcal{F}_t\right] = S_t\mathbb{E}^*\left[\frac{M_t^T}{S_t}\mathbb{1}_{\{M_t^T/S_t > M_0^t/S_t\}} \Bigm| \mathcal{F}_t\right] \\
&= S_t\mathbb{E}^*\left[\mathbb{1}_{\{\mathrm{Max}_{u\in[t,T]} S_u/S_t > x\}} \operatorname*{Max}_{u\in[t,T]}\frac{S_u}{S_t} \Bigm| \mathcal{F}_t\right]_{x = M_0^t/S_t} \\
&= S_t\mathbb{E}^*\left[\mathbb{1}_{\{\mathrm{Max}_{u\in[0,T-t]} S_u/S_0 > x\}} \operatorname*{Max}_{u\in[0,T-t]}\frac{S_u}{S_0}\right]_{x = M_0^t/S_t},
\end{aligned}
$$

and by Proposition 10.4 we have

$$
\begin{aligned}
&\mathbb{E}^*\left[\mathbb{1}_{\{\mathrm{Max}_{u\in[0,T]} S_u/S_0 > x\}} \operatorname*{Max}_{u\in[0,T]}\frac{S_u}{S_0}\right] \qquad (10.22) \\
&= \mathbb{E}^*\left[\mathbb{1}_{\{\mathrm{Max}_{u\in[0,T]}\, \mathrm{e}^{\sigma\widehat{W}_u} > x\}} \operatorname*{Max}_{u\in[0,T]} \mathrm{e}^{\sigma\widehat{W}_u}\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{\sigma\,\mathrm{Max}_{u\in[0,T]}\widehat{W}_u}\mathbb{1}_{\{\mathrm{Max}_{u\in[0,T]}\widehat{W}_u > \sigma^{-1}\log x\}}\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{\sigma\widehat{X}_T}\mathbb{1}_{\{\widehat{X}_T > \sigma^{-1}\log x\}}\right] \\
&= \int_{\sigma^{-1}\log x}^\infty \mathrm{e}^{\sigma z} f_{\widehat{X}_T}(z)dz \\
&= \int_{\sigma^{-1}\log x}^\infty \mathrm{e}^{\sigma z}\left(\sqrt{\frac{2}{\pi T}}\mathrm{e}^{-(z-\mu T)^2/(2T)} - 2\mu\,\mathrm{e}^{2\mu z}\Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right)\right)dz \\
&= \sqrt{\frac{2}{\pi T}}\int_{\sigma^{-1}\log x}^\infty \mathrm{e}^{\sigma z - (z-\mu T)^2/(2T)}dz - 2\mu\int_{\sigma^{-1}\log x}^\infty \mathrm{e}^{(\sigma+2\mu)z}\Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right)dz.
\end{aligned}
$$

By a standard "completion of the square" argument, we find

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log x}^{\infty} \mathrm{e}^{\sigma z-(z-\mu T)^2/(2T)} dz \\
&= \frac{1}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log x}^{\infty} \mathrm{e}^{-(z^2+\mu^2T^2-2(\mu+\sigma)Tz)/(2T)} dz \\
&= \frac{1}{\sqrt{2\pi T}} \mathrm{e}^{\sigma^2 T/2+\mu\sigma T} \int_{\sigma^{-1}\log x}^{\infty} \mathrm{e}^{-(z-(\mu+\sigma)T)^2/(2T)} dz \\
&= \frac{1}{\sqrt{2\pi T}} \mathrm{e}^{rT} \int_{-(\mu+\sigma)T+\sigma^{-1}\log x}^{\infty} \mathrm{e}^{-z^2/(2T)} dz \\
&= \mathrm{e}^{rT}\Phi\left(\delta_+^T\left(\frac{1}{x}\right)\right),
\end{aligned}$$

since $\mu\sigma + \sigma^2/2 = r$. The second integral

$$\int_{\sigma^{-1}\log x}^{\infty} \mathrm{e}^{(\sigma+2\mu)z}\Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) dz$$

can be computed by integration by parts using the identity

$$\int_a^\infty v'(z)u(z)dz = u(+\infty)v(+\infty) - u(a)v(a) - \int_a^\infty v(z)u'(z)dz,$$

with $a := \sigma^{-1}\log x$. We let

$$u(z) = \Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) \quad \text{and} \quad v'(z) = \mathrm{e}^{(\sigma+2\mu)z}$$

which satisfy

$$u'(z) = -\frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-(z+\mu T)^2/(2T)} \quad \text{and} \quad v(z) = \frac{1}{\sigma+2\mu}\mathrm{e}^{(\sigma+2\mu)z},$$

and using the completion of square identity

$$\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{\gamma y-y^2/(2T)}dy = \mathrm{e}^{\gamma^2T/2}\left(\Phi\left(\frac{-c+\gamma T}{\sqrt{T}}\right) - \Phi\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right) \tag{10.23}$$

for $b = +\infty$, we find

$$\begin{aligned}
&\int_a^\infty \mathrm{e}^{(\sigma+2\mu)z}\Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) dz = \int_a^\infty v'(z)u(z)dz \\
&\quad = u(+\infty)v(+\infty) - u(a)v(a) - \int_a^\infty v(z)u'(z)dz \\
&\quad = -\frac{1}{\sigma+2\mu}\mathrm{e}^{a(\sigma+2\mu)}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&\qquad + \frac{1}{(\sigma+2\mu)\sqrt{2\pi T}}\int_a^\infty \mathrm{e}^{(\sigma+2\mu)z}\,\mathrm{e}^{-(z+\mu T)^2/(2T)}dz
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\sigma+2\mu}\,\mathrm{e}^{a(\sigma+2\mu)}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right)\\
&\quad+\frac{1}{(\sigma+\mu)\sqrt{2\pi T}}\,\mathrm{e}^{(T(\sigma+\mu)^2-\mu^2T)/2}\int_a^\infty \mathrm{e}^{-(z-T(\sigma+\mu))^2/(2T)}dz\\
&= -\frac{1}{\sigma+2\mu}\,\mathrm{e}^{a(\sigma+2\mu)}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right)\\
&\quad+\frac{1}{(\sigma+2\mu)\sqrt{2\pi}}\,\mathrm{e}^{(T(\sigma+\mu)^2-\mu^2T)/2}\int_{(a-T(\sigma+\mu))/\sqrt{T}}^\infty \mathrm{e}^{-z^2/2}dz\\
&= -\frac{1}{\sigma+2\mu}\,\mathrm{e}^{a(\sigma+2\mu)}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right)\\
&\quad+\frac{1}{\sigma+2\mu}\,\mathrm{e}^{(T(\sigma+\mu)^2-\mu^2T)/2}\Phi\left(\frac{-a+T(\sigma+\mu)}{\sqrt{T}}\right)\\
&= -\frac{2r}{\sigma}\,(x)^{2r/\sigma^2}\,\Phi\left(\frac{-(r/\sigma-\sigma/2)T-\sigma^{-1}\log x}{\sqrt{T}}\right)\\
&\quad+\frac{2r}{\sigma}\,\mathrm{e}^{\sigma T(\sigma+2\mu)/2}\Phi\left(\frac{T(r/\sigma+\sigma/2)-\sigma^{-1}\log x}{\sqrt{T}}\right)\\
&= \frac{\sigma}{2r}\,\mathrm{e}^{rT}\Phi\left(\delta_+^T\left(\frac{1}{x}\right)\right)-\frac{\sigma}{2r}x^{2r/\sigma^2}\Phi\left(-\delta_-^T\,(x)\right),
\end{aligned}
$$

cf. pages 317-319 of Shreve (2004) for a different derivation using double integrals. Hence we have

$$
\begin{aligned}
&\mathbb{E}^*\left[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}}\,\middle|\,\mathcal{F}_t\right]=S_t\mathbb{E}^*\left[\mathbb{1}_{\{\mathrm{Max}_{u\in[0,T-t]}S_u/S_0>x\}}\operatorname*{Max}_{u\in[0,T-t]}\frac{S_u}{S_0}\right]_{x=M_0^t/S_t}\\
&\quad=2S_t\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-S_t\frac{\mu\sigma}{r}\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)\\
&\qquad+S_t\frac{\mu\sigma}{r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right),
\end{aligned}
$$

and consequently this yields, since $\mu\sigma/r=1-\sigma^2/(2r)$,

$$
\begin{aligned}
&\mathbb{E}^*\big[M_0^T\,\big|\,\mathcal{F}_t\big]=\mathbb{E}^*\big[M_0^T\,\big|\,M_0^t\big]\\
&\quad=M_0^t\mathbb{P}\big(M_0^t>M_t^T\,\big|\,M_0^t\big)+\mathbb{E}^*\big[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}}\,\big|\,M_0^t\big]\\
&\quad=M_0^t\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-S_t\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\\
&\qquad+2S_t\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)\\
&\qquad-S_t\left(1-\frac{\sigma^2}{2r}\right)\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)
\end{aligned}
$$

$$+S_t\left(1-\frac{\sigma^2}{2r}\right)\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)$$

$$= M_0^t\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)+S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)$$

$$-S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).$$

This concludes the proof of Proposition 10.9. □

### Minimum buying price over $[0, T]$

In the next proposition we compute the average of the minimum buying price $m_0^T := \min_{t\in[0,T]} S_t$ of $(S_t)_{t\in[0,T]}$ over the time interval $[0, T]$. In particular, this extends Exercise 10.6-(a) for the computation of the average minimum $\mathbb{E}^*[m_0^T] = \mathbb{E}^*\left[\min_{t\in[0,T]} S_t\right]$.

**Proposition 10.10.** *The average minimum value of $(S_t)_{t\in[0,T]}$ over $[0, T]$ is given by*

$$\mathbb{E}^*\left[m_0^T \mid \mathcal{F}_t\right] \tag{10.24}$$

$$= m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)-S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)$$

$$+S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right),$$

*where $\delta_\pm^{T-t}$ is defined in (10.19).*

We note a certain symmetry between the expressions (10.20) and (10.24).

When $t = 0$ we have $S_0 = m_0^0$, and given (10.21) the formula (10.24) simplifies to

$$\mathbb{E}^*\left[m_0^T\right] = S_0\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right)-S_0\frac{\sigma^2}{2r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right)$$

$$+S_0\,\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{\sigma^2/2+r}{\sigma}\sqrt{T}\right),$$

with

$$\mathbb{E}^*\left[m_0^T\right] = 2S_0\left(1+\frac{\sigma^2 T}{4}\right)\Phi\left(-\frac{\sigma^2 T/2}{\sigma\sqrt{T}}\right)-\sigma S_0\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8}.$$

when $r = 0$, cf. Exercise 12.1.

In general, when $T$ tends to infinity we find that

$$\lim_{T\to\infty} \frac{\mathbb{E}^*[m_0^T \mid \mathcal{F}_t]}{\mathbb{E}^*[S_T \mid \mathcal{F}_t]} = 0, \qquad r \geqslant 0,$$

see Exercise 10.3-(f) in the case $r = \sigma^2/2$.

**Proof of Proposition 10.10**. We have

$$\begin{aligned}
\mathbb{E}^*\big[m_0^T \,\big|\, \mathcal{F}_t\big] &= \mathbb{E}^*\big[\min\big(m_0^t, m_t^T\big) \,\big|\, \mathcal{F}_t\big] \\
&= \mathbb{E}^*\big[m_0^t \mathbb{1}_{\{m_0^t < m_t^T\}} \,\big|\, \mathcal{F}_t\big] + \mathbb{E}^*\big[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\big|\, \mathcal{F}_t\big] \\
&= m_0^t \mathbb{E}^*\big[\mathbb{1}_{\{m_0^t < m_t^T\}} \,\big|\, \mathcal{F}_t\big] + \mathbb{E}^*\big[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\big|\, \mathcal{F}_t\big] \\
&= m_0^t \mathbb{P}\big(m_0^t < m_t^T \,\big|\, \mathcal{F}_t\big) + \mathbb{E}^*\big[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\big|\, \mathcal{F}_t\big].
\end{aligned}$$

By (10.17) we find the cumulative distribution function

$$\mathbb{P}\left(\frac{m_0^{T-t}}{S_0} > x\right)_{x = m_0^t/S_t} = \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{-1+2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right),$$

of the minimum $m_0^{T-t}$ of $(S_t)_{t\in\mathbb{R}_+}$ over the time interval $[0, T-t]$, hence

$$\begin{aligned}
\mathbb{P}\big(m_0^t < m_t^T \,\big|\, \mathcal{F}_t\big) &= \mathbb{P}\left(\frac{m_0^t}{S_t} < \frac{m_t^T}{S_t} \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{P}\left(x < \frac{m_t^T}{S_t} \,\middle|\, \mathcal{F}_t\right)_{x = m_0^t/S_t} \\
&= \mathbb{P}\left(\frac{m_0^{T-t}}{S_0} > x\right)_{x = m_0^t/S_t} \\
&= \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{-1+2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right).
\end{aligned}$$

Next, by integration with respect to the probability density function (10.16) as in (10.22) in the proof of Proposition 10.9, we find

$$\begin{aligned}
\mathbb{E}^*\big[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\big|\, \mathcal{F}_t\big] &= S_t \mathbb{E}^*\left[\mathbb{1}_{\{m_0^t/S_t > m_t^T/S_t\}} \min_{u\in[t,T]} \frac{S_u}{S_t}\right]_{x = m_0^t, y = S_t} \\
&= S_t \mathbb{E}^*\left[\mathbb{1}_{\{\min_{u\in[t,T]} S_u/S_t < x\}} \min_{u\in[t,T]} \frac{S_u}{S_t}\right]_{x = m_0^t/S_t} \\
&= S_t \mathbb{E}^*\left[\mathbb{1}_{\{\min_{u\in[0,T-t]} S_u/S_0 < x\}} \min_{u\in[0,T-t]} \frac{S_u}{S_0}\right]_{x = m_0^t/S_t}
\end{aligned}$$

$$= 2S_t \,\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - S_t\frac{\mu\sigma}{r}\,\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)$$
$$+S_t\frac{\mu\sigma}{r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right).$$

Given the relation $\mu\sigma/r = 1-\sigma^2/(2r)$, this yields

$$\begin{aligned}
\mathbb{E}^*\left[m_0^T \mid \mathcal{F}_t\right] &= m_0^t\mathbb{P}\left(\frac{m_0^{T-t}}{S_0} > x\right)_{x=m_0^t/S_t} \\
&\quad +S_t\mathbb{E}^*\left[\mathbb{1}_{\{\min_{u\in[0,T-t]} S_u/S_0<x\}}\min_{u\in[0,T-t]}\frac{S_u}{S_0}\right]_{x=m_0^t/S_t} \\
&= m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - m_0^t\left(\frac{m_0^t}{S_t}\right)^{-1+2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) \\
&\quad +2S_t\,\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - S_t\,\mathrm{e}^{(T-t)r}\frac{\mu\sigma}{r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\
&\quad +S_t\frac{\mu\sigma}{r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) \\
&= m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) + S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\
&\quad -S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right).
\end{aligned}$$

□

## Exercises

Exercise 10.1 Let $(W_t)_{t\in\mathbb{R}_+}$ be standard Brownian motion, and let $a > W_0 = 0$.

a) Using the equality (10.2), find the probability density function $\varphi_{\tau_a}$ of the first time
$$\tau_a := \inf\{t \geqslant 0 \ : \ W_t = a\}$$
that $(W_t)_{t\in\mathbb{R}_+}$ hits the level $a > 0$.

b) Let $\mu \in \mathbb{R}$. By Proposition 10.4, find the probability density function $\varphi_{\tau_a}$ of the first time
$$\tilde{\tau}_a := \inf\{t \geqslant 0 \ : \ \widetilde{W}_t = a\}$$
that the drifted Brownian motion $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+} := (W_t+\mu t)_{t\in\mathbb{R}_+}$ hits the level $a > 0$.

c) Let $\sigma > 0$ and $r \in \mathbb{R}$. By Corollary 10.5, find the probability density function $\varphi_{\tau_a}$ of the first time

$$\hat{\tau}_a := \inf\{t \geqslant 0 \ : \ S_t = a\}$$

that the geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+} := \left(e^{\sigma W_t + rt - \sigma^2 t/2}\right)_{t\in\mathbb{R}_+}$ hits the level $a > 0$.

Exercise 10.2

a) Compute the mean value

$$\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t\right] = \mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} (\sigma W_t + \mu t)\right]$$

of the maximum of drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0, T]$, for $\sigma > 0$ and $\mu \in \mathbb{R}$. The probability density function of the maximum is given in (10.14).

b) Compute the mean value $\mathbb{E}\left[\min_{t\in[0,T]} \widetilde{W}_t\right] = \mathbb{E}\left[\min_{t\in[0,T]} (\sigma W_t + \mu t)\right]$ of the *minimum* of drifted Brownian motion $\widetilde{W}_t = \sigma W_t + \mu t$ over $t \in [0, T]$, for $\sigma > 0$ and $\mu \in \mathbb{R}$. The probability density function of the minimum is given in (10.17).

Exercise 10.3 Consider a risky asset whose price $S_t$ is given by

$$dS_t = \sigma S_t dW_t + \frac{\sigma^2}{2} S_t dt, \tag{10.25}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Solve the stochastic differential equation (10.25).
b) Compute the expected stock price value $\mathbb{E}^*[S_T]$ at time $T$.
c) What is the probability distribution of the maximum $\operatorname*{Max}_{t\in[0,T]} W_t$ over the interval $[0, T]$?
d) Compute the expected value $\mathbb{E}^*\left[M_0^T\right]$ of the maximum

$$M_0^T := \operatorname*{Max}_{t\in[0,T]} S_t = S_0 \operatorname*{Max}_{t\in[0,T]} e^{\sigma W_t} = S_0 \exp\left(\sigma \operatorname*{Max}_{t\in[0,T]} W_t\right).$$

of the stock price over the interval $[0, T]$.

e) What is the probability distribution of the *minimum* $\min_{t\in[0,T]} W_t$ over the interval $[0, T]$?
f) Compute the expected value $\mathbb{E}^*\left[m_0^T\right]$ of the *minimum*



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$$m_0^T := \min_{t\in[0,T]} S_t = S_0 \min_{t\in[0,T]} \mathrm{e}^{\sigma W_t} = S_0 \exp\left(\sigma \min_{t\in[0,T]} W_t\right).$$

of the stock price over the interval $[0,T]$.

Exercise 10.4 Arcsine law. Let $\tau$ denote the first time a standard Brownian motion $(B_t)_{t\in[0,T]}$ reaches its maximum over $[0,T]$.

a) Write down $\mathbb{P}(\tau \leqslant t)$ using two independent Gaussian random variables $Z_1 \sim \mathcal{N}(0,t)$ and $Z_2 \sim \mathcal{N}(0,T-t)$.

   *Hint*: By (10.3), $\mathrm{Max}_{s\in[0,t]} B_s$ has same distribution as $|Z_1|$.

b) Write down $\mathbb{P}(\tau \leqslant t)$ as an integral.

   *Hint*: Use Answer 2 on https://math.stackexchange.com/questions/3534598/let-x-y-be-independent-normally-distributed-random-variables-find-the-density.

Exercise 10.5 (Exercise 10.3 continued).

a) Compute the "optimal call option" prices $\mathbb{E}\big[\big(M_0^T - K\big)^+\big]$ estimated by optimally exercising at the maximum value $M_0^T$ of $(S_t)_{t\in[0,T]}$ before maturity $T$.

b) Compute the "optimal put option" prices $\mathbb{E}\big[\big(K - m_0^T\big)^+\big]$ estimated by optimally exercising at the minimum value $m_0^T$ of $(S_t)_{t\in[0,T]}$ before maturity $T$.

Exercise 10.6 (Exercise 10.5 continued). Consider an asset price $S_t$ given by $S_t = S_0 \mathrm{e}^{rt+\sigma B_t - \sigma^2 t/2}$, $t \geqslant 0$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r \geqslant 0$ and $\sigma > 0$.

a) Compute the average $\mathbb{E}^*\big[m_0^T\big]$ of the minimum $m_0^T := \min_{t\in[0,T]} S_t$ of $(S_t)_{t\in[0,T]}$ over $[0,T]$.

b) Compute the expected payoff $\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right]$ for $r > 0$. Using a finite expiration American put option pricer, compare the American put option price to the above expected payoff.

c) Compute the expected payoff $\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right]$ for $r = 0$.

Exercise 10.7 Recall that the maximum $X_0^t := \mathrm{Max}_{s\in[0,t]} W_s$ over $[0,t]$ of standard Brownian motion $(W_s)_{s\in[0,t]}$ has the probability density function

$$\varphi_{X_0^t}(x) = \sqrt{\frac{2}{\pi t}}\,\mathrm{e}^{-x^2/(2t)}, \qquad x \geqslant 0.$$

a) Let $\tau_a = \inf\{s \geqslant 0 \ : \ W_s = a\}$ denote the first hitting time of $a > 0$ by $(W_s)_{s\in\mathbb{R}_+}$. Using the relation between $\{\tau_a \leqslant t\}$ and $\{X_0^t \geqslant a\}$, write down the probability $\mathbb{P}(\tau_a \leqslant t)$ as an integral from $a$ to $\infty$.
b) Using integration by parts on $[a, \infty)$, compute the probability density function of $\tau_a$.

   Hint: the derivative of $\mathrm{e}^{-x^2/(2t)}$ with respect to $x$ is $-x\,\mathrm{e}^{-x^2/(2t)}/t$.
c) Compute the mean value $\mathbb{E}^*[(\tau_a)^{-2}]$ of $1/\tau_a^2$.

Exercise 10.8 From Relation (10.11) in Proposition 10.3 and the Jacobian change of variable formula, see *e.g.* https://online.stat.psu.edu/stat414/lesson/23/23.1, and assuming $S_0 > 0$, compute the joint probability density function of geometric Brownian motion $S_T := S_0\,\mathrm{e}^{\sigma W_T + (r-\sigma^2/2)T}$ and its maximum

$$M_0^T := \operatorname*{Max}_{t\in[0,T]} S_t = S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t + (r-\sigma^2/2)t}.$$