Chapter 12
Lookback Options

Lookback call (resp. put) options are financial derivatives that allow their holders to exercise the option by setting the strike price at the minimum (resp. maximum) of the underlying asset price process $(S_t)_{t\in[0,T]}$ over the time interval $[0,T]$. Lookback options can be priced by PDE arguments or by computing the discounted expected values of their claim payoff C, namely $C = S_T - \min_{0\leqslant t\leqslant T} S_t$ in the case of call options, and $C = \operatorname{Max}_{0\leqslant t\leqslant T} S_t - S_T$ in the case of put options.

12.1 The Lookback Put Option

The standard lookback put option gives its holder the right to sell the underlying asset at its historically highest price. In this case, the floating strike price is M_0^T and the payoff is given by the terminal value

$$C = M_0^T - S_T$$

of the drawdown process $(M_0^t - S_t)_{t\in[0,T]}$. The following pricing formula for lookback put options is a direct consequence of Proposition 10.9.

Proposition 12.1. *The price at time $t \in [0,T]$ of the lookback put option with payoff $M_0^T - S_T$ is given by*

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$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T - S_T \,\big|\, \mathcal{F}_t\big] \\
&= M_0^t \,\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&\quad -S_t\,\mathrm{e}^{-(T-t)r}\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right) - S_t,
\end{aligned}
$$

where $\delta_\pm^T(s)$ is defined in (11.6).

Proof. We have

$$
\begin{aligned}
\mathbb{E}^*\big[M_0^T - S_T \,\big|\, \mathcal{F}_t\big] &= \mathbb{E}^*\big[M_0^T \,\big|\, \mathcal{F}_t\big] - \mathbb{E}^*[S_T \mid \mathcal{F}_t] \\
&= \mathbb{E}^*\big[M_0^T \,\big|\, \mathcal{F}_t\big] - \mathrm{e}^{(T-t)r}S_t,
\end{aligned}
$$

hence Proposition 10.9 shows that

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T - S_T \,\big|\, \mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T \,\big|\, \mathcal{F}_t\big] - \mathrm{e}^{-(T-t)r}\mathbb{E}^*[S_T \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T \,\big|\, M_0^t\big] - S_t \\
&= M_0^t\,\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - S_t\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&+ S_t\frac{\sigma^2}{2r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - S_t\frac{\sigma^2}{2r}\mathrm{e}^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).
\end{aligned}
$$

□

Figure 12.1 represents the lookback put option price as a function of S_t and M_0^t, for different values of the time to maturity $T-t$.

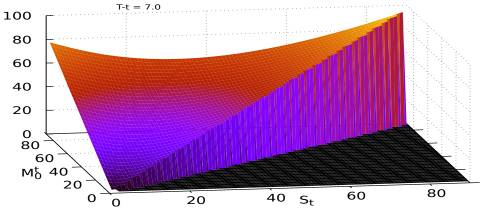
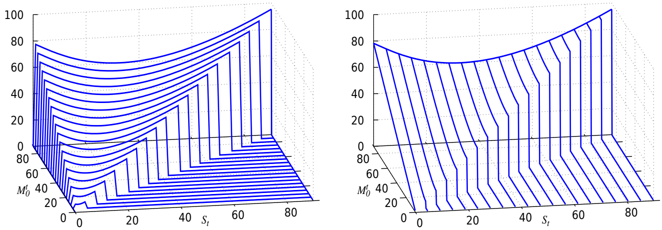


Fig. 12.1: Graph of the lookback put option price (3D).*

From Figures 12.1 and 12.2, we make the following observations.

i) Close to maturity, if the underlying asset price S_t is close to M_0^t then an increase in the value S_t can result into a higher put option price, as in this case a variation of S_t may increase the value of M_0^t.

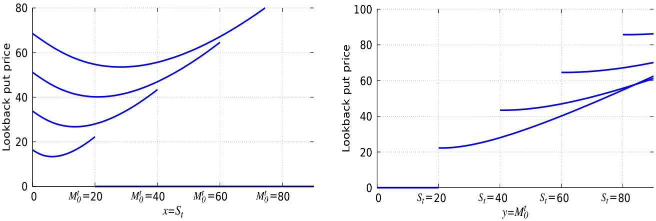
ii) When the underlying asset price S_t is far from M_0^t, an increase in S_t is less likely to affect the value of M_0^t when time t is close to maturity T, and this results into a lower option price.



(a) Put prices as functions of S. (b) Put prices as functions of M.

Fig. 12.2: Graph of lookback put option prices.

Figures 12.2 and 12.3 show accordingly that, from the Delta hedging strategy for lookback put options, see Proposition 12.2 below, one should short the underlying asset when S_t is far from M_0^t, and long this asset when S_t becomes closer to M_0^t.



(a) Lookback put prices for fixed M_0^t. (b) Lookback put prices for fixed S_t.

Fig. 12.3: Graph of lookback put option prices (2D).

* The animation works in Acrobat Reader on the entire pdf file.

12.2 PDE Method

Since the couple (S_t, M_0^t) is a Markov process, the price of the lookback put option at time $t \in [0, T]$ can be written as a function

$$f(t, S_t, M_0^t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T, M_0^T)\,\big|\,\mathcal{F}_t\big] \\ = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T, M_0^T)\,\big|\,S_t,\ M_0^t\big] \tag{12.1}$$

of S_t and M_0^t, $0 \leqslant t \leqslant T$.

Black-Scholes PDE for lookback put option prices

In the next proposition we derive the partial differential equation (PDE) for the pricing function $f(t, x, y)$ of a self-financing portfolio hedging a lookback put option. See Exercise 12.5 for the verification of the boundary conditions (12.3a)-(12.3c).

Proposition 12.2. *The function $f(t, x, y)$ defined by*

$$f(t, x, y) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T - S_T\,\big|\,S_t = x,\ M_0^T = y\big], \quad t \in [0, T],\ x, y > 0,$$

is $\mathcal{C}^2((0, T) \times (0, \infty)^2)$ and satisfies the Black-Scholes PDE

$$rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + rx\frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t, x, y), \tag{12.2}$$

$0 \leqslant t \leqslant T$, $x, y > 0$, *subject to the boundary conditions*

$$\begin{cases} f(t, 0^+, y) = \mathrm{e}^{-(T-t)r}y, \quad 0 \leqslant t \leqslant T, \quad y \geqslant 0, & (12.3a) \\ \dfrac{\partial f}{\partial y}(t, x, y)_{|y=x} = 0, \quad 0 \leqslant t \leqslant T, \quad y > 0, & (12.3b) \\ f(T, x, y) = y - x, \quad 0 \leqslant x \leqslant y. & (12.3c) \end{cases}$$

The replicating portfolio of the lookback put option is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, M_0^t), \qquad t \in [0, T]. \tag{12.4}$$

Proof. The existence of $f(t, x, y)$ follows from the Markov property, more precisely, from the time homogeneity of the asset price process $(S_t)_{t\in\mathbb{R}_+}$ the function $f(t, x, y)$ satisfies

$$f(t, x, y) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T, M_0^T)\,\big|\,S_t = x,\ M_0^t = y\big] \\ = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}, \mathrm{Max}\left(y, M_t^T\right)\right)\right]$$

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$$= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(x\frac{S_{T-t}}{S_0}, \mathrm{Max}\left(y, M_0^{T-t}\right)\right)\right], \quad t \in [0,T].$$

Applying the change of variable formula to the discounted portfolio value

$$\widetilde{f}(t,x,y) := \mathrm{e}^{-rt}f(t,x,y) = \mathrm{e}^{-rT}\mathbb{E}^*\left[\phi(S_T, M_0^T) \,\middle|\, S_t = x,\ M_0^t = y\right]$$

which is a martingale indexed by $t \in [0,T]$ under $\mathbb{P}^*$, we have

$$\begin{aligned}
&d\widetilde{f}\big(t,S_t,M_0^t\big) = -r\,\mathrm{e}^{-rt}f\big(t,S_t,M_0^t\big)dt + \mathrm{e}^{-rt}df\big(t,S_t,M_0^t\big)\\
&= -r\,\mathrm{e}^{-rt}f\big(t,S_t,M_0^t\big)dt + \mathrm{e}^{-rt}\frac{\partial f}{\partial t}\big(t,S_t,M_0^t\big)dt + r\,\mathrm{e}^{-rt}S_t\frac{\partial f}{\partial x}\big(t,S_t,M_0^t\big)dt\\
&\quad + \mathrm{e}^{-rt}\sigma S_t\frac{\partial f}{\partial x}\big(t,S_t,M_0^t\big)dB_t + \frac{1}{2}\,\mathrm{e}^{-rt}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}\big(t,S_t,M_0^t\big)dt\\
&\quad + \mathrm{e}^{-rt}\frac{\partial f}{\partial y}\big(t,S_t,M_0^t\big)dM_0^t, \qquad (12.5)
\end{aligned}$$

according to the following extension of the Itô multiplication table 12.1.

$\cdot$	dt	dB_t	dM_0^t
dt	0	0	0
dB_t	0	dt	0
dM_0^t	0	0	0

Table 12.1: Extended Itô multiplication table.

Since $\big(\widetilde{f}\big(t,S_t,M_0^t\big)\big)_{t\in[0,T]} = \big(\mathrm{e}^{-rT}\mathbb{E}^*\big[\phi(S_T,M_0^T)\,\big|\,\mathcal{F}_t\big]\big)_{t\in[0,T]}$ is a martingale under $\mathbb{P}$ and $\big(M_0^t\big)_{t\in[0,T]}$ has finite variation (it is in fact a non-decreasing process), (12.5) yields:

$$d\widetilde{f}\big(t,S_t,M_0^t\big) = \sigma S_t\frac{\partial \widetilde{f}}{\partial x}\big(t,S_t,M_0^t\big)dB_t, \qquad t \in [0,T], \qquad (12.6)$$

and the function $f(t,x,y)$ satisfies the equation

$$\begin{aligned}
&\frac{\partial f}{\partial t}\big(t,S_t,M_0^t\big)dt + rS_t\frac{\partial f}{\partial x}f\big(t,S_t,M_0^t\big)dt\\
&\quad + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}\big(t,S_t,M_0^t\big)dt + \frac{\partial f}{\partial y}\big(t,S_t,M_0^t\big)dM_0^t = rf\big(t,S_t,M_0^t\big)dt, \qquad (12.7)
\end{aligned}$$

which implies

$$\frac{\partial f}{\partial t}\big(t,S_t,M_0^t\big) + rS_t\frac{\partial f}{\partial x}\big(t,S_t,M_0^t\big) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}\big(t,S_t,M_0^t\big) = rf\big(t,S_t,M_0^t\big),$$

which is (12.2), and

$$\frac{\partial f}{\partial y}(t, S_t, M_0^t) dM_0^t = 0.$$

Indeed, M_0^t increases only on a set of zero Lebesgue measure (which has no isolated points), therefore the Lebesgue measure dt and the measure dM_0^t are mutually *singular*, hence by the Lebesgue decomposition theorem, both components in dt and dM_0^t should vanish in (12.7) if the sum vanishes, see also the Cantor function. This implies

$$\frac{\partial f}{\partial y}(t, S_t, M_0^t) = 0,$$

when $dM_0^t > 0$, hence since

$$\{S_t = M_0^t\} \quad \Longleftrightarrow dM_0^t > 0$$

and

$$\{S_t < M_0^t\} \quad \Longleftrightarrow dM_0^t = 0,$$

we have

$$\frac{\partial f}{\partial y}(t, S_t, S_t) = \frac{\partial f}{\partial y}(t, x, y)_{x=S_t,\ y=S_t} = 0,$$

since M_0^t hits S_t, *i.e.* $M_0^t = S_t$, only when M_0^t increases at time t, and this shows the boundary condition (12.3b).

On the other hand, (12.6) shows that

$$\phi(S_T, M_0^T) = \mathbb{E}^*[\phi(S_T, M_0^T)] + \sigma \int_0^T S_t \frac{\partial f}{\partial x}(t, x, M_0^t)_{|x=S_t} dB_t,$$

$0 \leqslant t \leqslant T$, which implies (12.4) as in the proof of Propositions 6.1 or 11.3. □

In other words, the price of the lookback put option takes the form

$$f(t, S_t, M_0^t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T - S_T \,\big|\, \mathcal{F}_t\big],$$

where the function $f(t, x, y)$ is given from Proposition 12.1 as

$$f(t,x,y) = y\,\mathrm{e}^{-(T-t)r}\Phi\big(-\delta_-^{T-t}(x/y)\big) + x\left(1+\frac{\sigma^2}{2r}\right)\Phi\big(\delta_+^{T-t}(x/y)\big) \\ -x\frac{\sigma^2}{2r}\,\mathrm{e}^{-(T-t)r}\left(\frac{y}{x}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(y/x)\right) - x. \tag{12.8}$$

Remark 12.3. *We have*

$$f(t,x,x) = xC(T-t),$$

with

$$\begin{aligned} C(\tau) &= \mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau(1)\right) + \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^\tau(1)\right) - \frac{\sigma^2}{2r}\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau(1)\right) - 1 \\ &= \mathrm{e}^{-r\tau}\Phi\left(-\frac{r-\sigma^2/2}{\sigma}\sqrt{\tau}\right) + \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{\tau}\right) \\ &\quad -\frac{\sigma^2}{2r}\mathrm{e}^{-r\tau}\Phi\left(-\frac{r-\sigma^2/2}{\sigma}\sqrt{\tau}\right) - 1, \qquad \tau > 0, \end{aligned}$$

hence

$$\frac{\partial f}{\partial x}(t,x,x) = C(T-t), \qquad t \in [0,T].$$

Scaling property of lookback put option prices

From (12.8) and the following argument we note the scaling property

$$\begin{aligned} f(t,x,y) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\operatorname{Max}\left(M_0^t, M_t^T\right) - S_T \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}x\mathbb{E}^*\left[\operatorname{Max}\left(\frac{M_0^t}{S_t}, \frac{M_t^T}{S_t}\right) - \frac{S_T}{S_t} \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}x\mathbb{E}^*\left[\operatorname{Max}\left(\frac{y}{x}, \frac{M_t^T}{x}\right) - \frac{S_T}{x} \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}x\mathbb{E}^*\left[\operatorname{Max}\left(M_0^t, M_t^T\right) - S_T \,\middle|\, S_t = 1,\ M_0^t = \frac{y}{x}\right] \\ &= \mathrm{e}^{-(T-t)r}x\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = 1,\ M_0^t = \frac{y}{x}\right] \\ &= xf(t,1,y/x) \\ &= xg(T-t,x/y), \end{aligned}$$

where we let

$$g(\tau,z) := \\ \frac{1}{z}\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau(z)\right) + \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^\tau(z)\right) - \frac{\sigma^2}{2r}\mathrm{e}^{-r\tau}\left(\frac{1}{z}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right) - 1,$$

with the boundary condition

$$\begin{cases} \frac{\partial g}{\partial z}(\tau,1) = 0, \qquad \tau > 0, & (12.9a) \\ g(0,z) = \frac{1}{z} - 1, \qquad z \in (0,1]. & (12.9b) \end{cases}$$

The next Figure 12.4 shows a graph of the function $g(\tau, z)$.

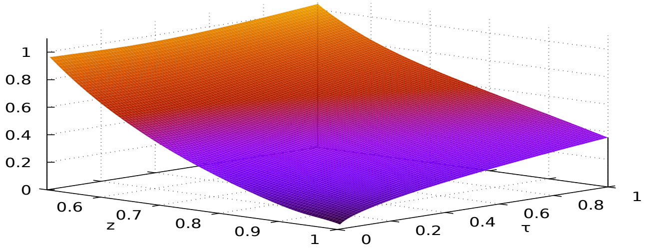


Fig. 12.4: Graph of the normalized lookback put option price.

Black-Scholes approximation of lookback put option prices

Letting

$$\mathrm{Bl_p}(x, K, r, \sigma, \tau) := K\,\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau\left(\frac{x}{K}\right)\right) - x\Phi\left(-\delta_+^\tau\left(\frac{x}{K}\right)\right)$$

denote the standard Black-Scholes formula for the price of the European put option.

Proposition 12.4. *The lookback put option price can be rewritten as*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T - S_T \,\big|\, \mathcal{F}_t\big] = \mathrm{Bl_p}(S_t, M_0^t, r, \sigma, T-t) \tag{12.10}$$
$$+S_t\frac{\sigma^2}{2r}\left(\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \mathrm{e}^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right).$$

In other words, we have

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[M_0^T - S_T \,\big|\, \mathcal{F}_t\big] = \mathrm{Bl_p}(S_t, M_0^t, r, \sigma, T-t) + S_t h_p\left(T-t, \frac{S_t}{M_0^t}\right)$$

where the function

$$h_p(\tau, z) = \frac{\sigma^2}{2r}\Phi\left(\delta_+^\tau(z)\right) - \frac{\sigma^2}{2r}\,\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right), \tag{12.11}$$

depends only on time τ and $z = S_t/M_0^t$. In other words, due to the relation

$$\mathrm{Bl_p}(x, y, r, \sigma, \tau) = y\,\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau\left(\frac{x}{y}\right)\right) - x\Phi\left(-\delta_+^\tau\left(\frac{x}{y}\right)\right)$$
$$= x\mathrm{Bl_p}(1, y/x, r, \sigma, \tau)$$

for the standard Black-Scholes put option price formula, we observe that $f(t, x, y)$ satisfies

$$f(t, x, y) = x\mathrm{Bl_p}\left(1, \frac{y}{x}, r, \sigma, T-t\right) + xh\left(T-t, \frac{x}{y}\right),$$

i.e.

$$f(t, x, y) = xg\left(T-t, \frac{x}{y}\right),$$

with

$$g(\tau, z) = \mathrm{Bl_p}\left(1, \frac{1}{z}, r, \sigma, \tau\right) + h_p(\tau, z), \tag{12.12}$$

where the function $h_p(\tau, z)$ is a correction term given by (12.11) which is small when $z = x/y$ or τ become small.

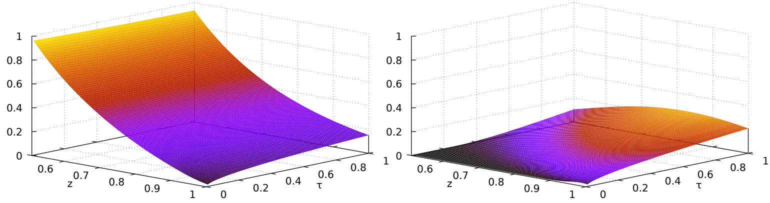
Note that $(x, y) \longmapsto xh_p(T-t, x/y)$ also satisfies the Black-Scholes PDE (12.2), in particular $(\tau, z) \longmapsto \mathrm{Bl_p}(1, 1/z, r, \sigma, \tau)$ and $h_p(\tau, z)$ both satisfy the PDE

$$\frac{\partial h_p}{\partial \tau}(\tau, z) = z\left(r + \sigma^2\right)\frac{\partial h_p}{\partial z}(\tau, z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 h_p}{\partial z^2}(\tau, z), \tag{12.13}$$

$\tau \in [0, T]$, $z \in [0, 1]$, subject to the boundary condition

$$h_p(0, z) = 0, \qquad 0 \leqslant z \leqslant 1.$$

The next Figure 12.5b illustrates the decomposition (12.12) of the normalized lookback put option price $g(\tau, z)$ in Figure 12.4 into the Black-Scholes put price function $\mathrm{Bl_p}(1, 1/z, r, \sigma, \tau)$ and $h_p(\tau, z)$.



(a) Lookback put price. (b) Correction term $h_p(\tau, z)$.

Fig. 12.5: Normalized Black-Scholes put price and correction term in (12.12).

Note that in Figure 12.5b the condition $h_p(0, z) = 0$ is not fully respected as z tends to 1, due to numerical instabilities in the approximation of the function Φ.

12.3 The Lookback Call Option

The standard Lookback call option gives the right to buy the underlying asset at its historically lowest price. In this case, the floating strike price is m_0^T and the payoff is

$$C = S_T - m_0^T.$$

The following result gives the price of the lookback call option, cf. *e.g.* Proposition 9.5.1, page 270 of Dana and Jeanblanc (2007).

Proposition 12.5. *The price at time $t \in [0, T]$ of the lookback call option with payoff $S_T - m_0^T$ is given by*

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T - m_0^T \,\middle|\, \mathcal{F}_t\right] \\
&= S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - m_0^t\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\
&\quad + \mathrm{e}^{-(T-t)r}S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - S_t\frac{\sigma^2}{2r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)
\end{aligned}$$

Proof. By Proposition 10.10 we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T - m_0^T \,\middle|\, \mathcal{F}_t\right] = S_t - \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[m_0^T \,\middle|\, \mathcal{F}_t\right] \\
&= S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \mathrm{e}^{-(T-t)r}m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\
&+ \mathrm{e}^{-(T-t)r}\frac{S_t\sigma^2}{2r}\left(\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - \mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\right).
\end{aligned}$$

□

Figure 12.6 represents the price of the lookback call option as a function of m_0^t and S_t for different values of the time to maturity $T-t$.

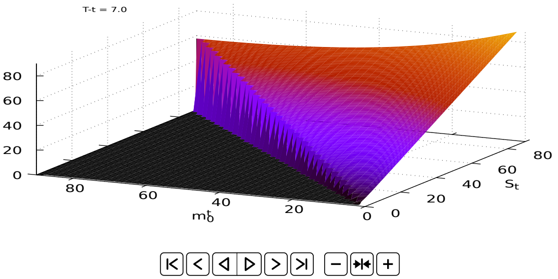
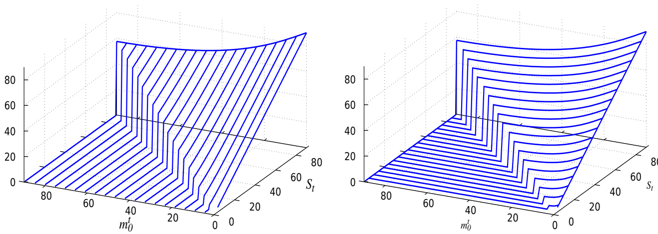


Fig. 12.6: Graph of the lookback call option price.*

From Figures 12.6 and 12.7, we note the following.

i) When the underlying asset price S_t is far from m_0^t, an increase in the value S_t clearly results into a higher call option price.
ii) When the underlying asset price S_t is close to m_0^t, a decrease in S_t could lead to a decrease in the value of m_0^t, however on average this appears insufficient to increase the average option payoff.

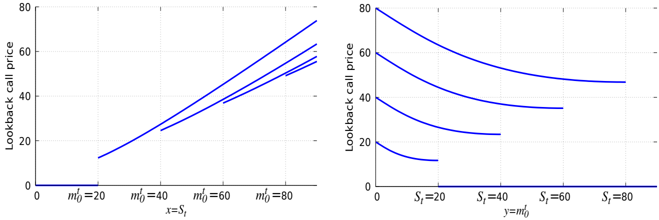


(a) Call prices as functions of S. (b) Call prices as functions of m.

Fig. 12.7: Graph of lookback call option prices.

Figures 12.7 and 12.8 show accordingly that, from the Delta hedging strategy for lookback call options, see Propositions 12.6 and 12.8, one should long the underlying asset in order to hedge a lookback call option.

* The animation works in Acrobat Reader on the entire pdf file.



(a) Lookback call prices for fixed m_0^t. (b) Lookback call prices for fixed S_t.

Fig. 12.8: Graphs of lookback call option prices (2D).

Black-Scholes PDE for lookback call option prices

Since the couple (S_t, m_0^t) is also a Markov process, the price of the lookback call option at time $t \in [0, T]$ can be written as a function

$$\begin{aligned} f(t, S_t, m_0^t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T, m_0^T) \,\big|\, \mathcal{F}_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi(S_T, m_0^T) \,\big|\, S_t,\ m_0^t\big] \end{aligned}$$

of S_t and m_0^t, $0 \leqslant t \leqslant T$. By the same argument as in the proof of Proposition 12.2, we obtain the following result.

Proposition 12.6. *The function $f(t, x, y)$ defined by*

$$f(t, x, y) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[S_T - m_0^T \,\big|\, S_t = x,\ m_0^t = y\big], \quad t \in [0, T],\ x, y > 0,$$

is $\mathcal{C}^2((0, T) \times (0, \infty)^2)$ and satisfies the Black-Scholes PDE

$$rf(t, x, y) = \frac{\partial f}{\partial t}(t, x, y) + rx\frac{\partial f}{\partial x}(t, x, y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t, x, y),$$

$0 \leqslant t \leqslant T$, $x > 0$, *subject to the boundary conditions*

$$\begin{cases} \lim_{y \searrow 0} f(t, x, y) = x, & 0 \leqslant t \leqslant T, \quad x > 0, & (12.14a) \\ \dfrac{\partial f}{\partial y}(t, x, y)_{y=x} = 0, & 0 \leqslant t \leqslant T, \quad y > 0, & (12.14b) \\ f(T, x, y) = x - y, & 0 < y \leqslant x, & (12.14c) \end{cases}$$

and the corresponding self-financing hedging strategy is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, m_0^t), \qquad t \in [0, T], \tag{12.15}$$

which represents the quantity of the risky asset S_t to be held at time t in the hedging portfolio.

In other words, the price of the lookback call option takes the form

$$f(t, S_t, m_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \big[S_T - m_0^T \,\big|\, \mathcal{F}_t \big],$$

where the function $f(t, x, y)$ is given by

$$\begin{aligned} f(t,x,y) &= x\Phi\left(\delta_+^{T-t}\left(\frac{x}{y}\right)\right) - \mathrm{e}^{-(T-t)r} y \Phi\left(\delta_-^{T-t}\left(\frac{x}{y}\right)\right) \\ &\quad + \mathrm{e}^{-(T-t)r} x \frac{\sigma^2}{2r}\left(\left(\frac{y}{x}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{y}{x}\right)\right) - \mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right)\right) \\ &= x - y\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{x}{y}\right)\right) - x\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) \\ &\quad + x\,\mathrm{e}^{-(T-t)r}\frac{\sigma^2}{2r}\left(\frac{y}{x}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{y}{x}\right)\right). \end{aligned} \tag{12.16}$$

Scaling property of lookback call option prices

We note the scaling property

$$\begin{aligned} f(t,x,y) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[S_T - m_0^T \,\big|\, S_t = x,\ m_0^t = y\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[S_T - \min\big(m_0^t, m_t^T\big) \,\big|\, S_t = x,\ m_0^t = y\big] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^*\left[\frac{S_T}{S_t} - \min\left(\frac{m_0^t}{S_t}, \frac{m_t^T}{S_t}\right) \,\middle|\, S_t = x,\ m_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^*\left[\frac{S_T}{x} - \min\left(\frac{y}{x}, \frac{m_t^T}{x}\right) \,\middle|\, S_t = x,\ m_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^*\left[S_T - \min\big(m_0^t, m_t^T\big) \,\middle|\, S_t = 1,\ m_0^t = \frac{y}{x}\right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^*\left[S_T - m_0^T \,\middle|\, S_t = 1,\ m_0^t = \frac{y}{x}\right] \\ &= x f(t, 1, y/x) \\ &= x g\left(T-t, \frac{1}{z}\right), \end{aligned}$$

where

$g(\tau, z) :=$

$$1-\frac{1}{z}\mathrm{e}^{-r\tau}\Phi\left(\delta_{-}^{\tau}(z)\right)-\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_{+}^{\tau}(z)\right)+\frac{\sigma^2}{2r}\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(\delta_{-}^{\tau}\left(\frac{1}{z}\right)\right),$$

with $g(\tau,1)=C(T-t)$, and

$$f(t,x,y)=xg\left(T-t,\frac{x}{y}\right)$$

and the boundary condition

$$\begin{cases}\dfrac{\partial g}{\partial z}(\tau,1)=0, & \tau>0, \quad (12.17a)\\ g(0,z)=1-\dfrac{1}{z}, & z\geqslant 1. \quad (12.17b)\end{cases}$$

The next Figure 12.9 shows a graph of the function $g(\tau,z)$.

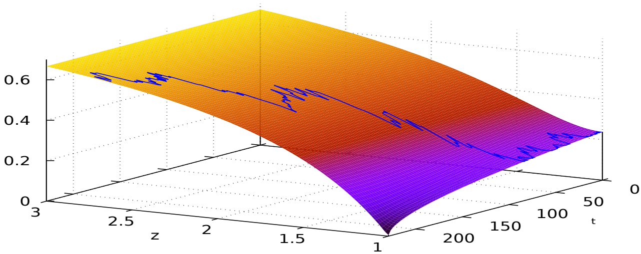


Fig. 12.9: Normalized lookback call option price.

The next Figure 12.10 represents the path of the underlying asset price used in Figure 12.9.

Fig. 12.10: Graph of underlying asset prices.

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The next Figure 12.11 represents the corresponding underlying asset price and its running minimum.

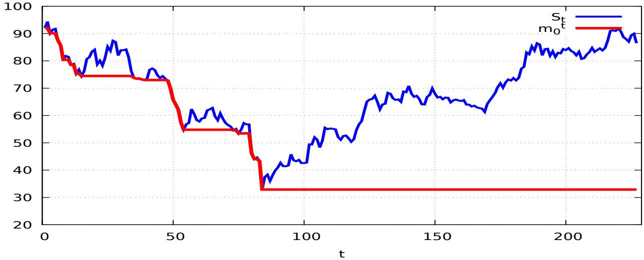

Fig. 12.11: Running minimum of the underlying asset price.

Next, we represent the option price as a function of time, together with the process $\left(S_t - m_0^t\right)_{t\in\mathbb{R}_+}$.

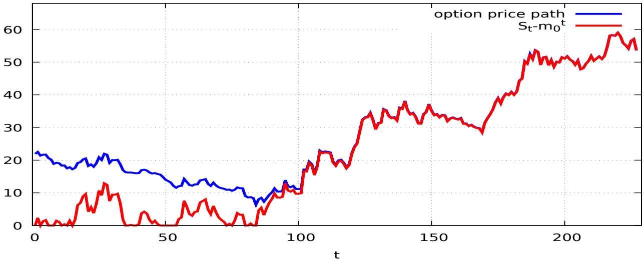

Fig. 12.12: Graph of the lookback call option price.

Black-Scholes approximation of lookback call option prices

Let

$$\mathrm{Bl}_c(S, K, r, \sigma, \tau) = S\Phi\left(\delta_+^\tau\left(\frac{S}{K}\right)\right) - K\,\mathrm{e}^{-r\tau}\Phi\left(\delta_-^\tau\left(\frac{S}{K}\right)\right)$$

denote the standard Black-Scholes formula for the price of the European call option.

Proposition 12.7. *The lookback call option price can be rewritten as*

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T - m_0^T \,\middle|\, \mathcal{F}_t\right] = \mathrm{Bl}_c(S_t, m_0^t, r, \sigma, T-t) \qquad (12.18)$$

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$$-S_t\frac{\sigma^2}{2r}\left(\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)-\mathrm{e}^{-(T-t)r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)\right).$$

In other words, we have

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T-m_0^T\,\middle|\,\mathcal{F}_t\right]:=\mathrm{Bl_c}(S_t,m_0^t,r,\sigma,T-t)+S_t h_c\left(T-t,\frac{S_t}{m_0^t}\right)$$

where the correction term

$$h_c(\tau,z)=-\frac{\sigma^2}{2r}\left(\Phi\left(-\delta_+^\tau(z)\right)-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right),\qquad(12.19)$$

is small when $z=S_t/m_0^t$ becomes large or τ becomes small. In addition, $h_p(\tau,z)$ is linked to $h_c(\tau,z)$ by the relation

$$h_c(\tau,z)=h_p(\tau,z)-\frac{\sigma^2}{2r}\left(1-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\right),\qquad\tau\geqslant 0,\quad z\geqslant 0,$$

where $(z,\tau)\longmapsto\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}$ also solves the PDE (12.13). Due to the relation

$$\begin{aligned}\mathrm{Bl_c}(x,y,r,\sigma,\tau)&=x\Phi\left(\delta_+^\tau\left(\frac{x}{y}\right)\right)-y\,\mathrm{e}^{-r\tau}\Phi\left(\delta_-^\tau\left(\frac{x}{y}\right)\right)\\&=x\mathrm{Bl_c}\left(1,\frac{y}{x},r,\sigma,\tau\right)\end{aligned}$$

for the standard Black-Scholes call price formula, recall that from Proposition 12.7, $f(t,x,y)$ can be decomposed as

$$f(t,x,y)=x\mathrm{Bl_c}\left(1,\frac{y}{x},r,\sigma,T-t\right)+xh_c\left(T-t,\frac{x}{y}\right),$$

where $h_c(\tau,z)$ is the function given by (12.19), *i.e.*

$$f(t,x,y)=xg\left(T-t,\frac{x}{y}\right),$$

with

$$g(\tau,z)=\mathrm{Bl_c}\left(1,\frac{1}{z},r,\sigma,\tau\right)+h_c(\tau,z),\qquad(12.20)$$

where $(x,y)\longmapsto xh_c(T-t,x/y)$ also satisfies the Black-Scholes PDE (12.2), *i.e.* $(\tau,z)\longmapsto\mathrm{Bl_c}(1,1/z,r,\sigma,\tau)$ and $h_c(\tau,z)$ both satisfy the PDE (12.13) subject to the boundary condition

$$h_c(0,z)=0,\qquad z\geqslant 1.$$

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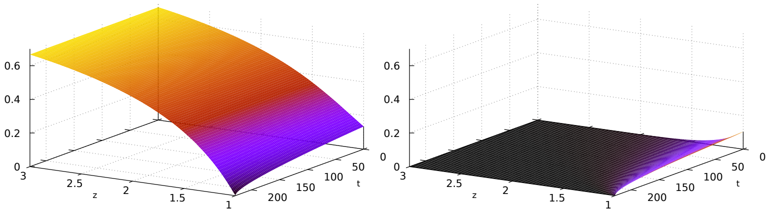
The next Figures 12.13a and 12.13b show the decomposition of $g(t,z)$ in (12.20) and Figures 12.9-12.10 into the sum of the Black-Scholes call price function $\mathrm{Bl_c}(1,1/z,r,\sigma,\tau)$ and $h(t,z)$.



(a) Lookback call price $g(\tau,z)$. (b) Correction term $h_c(\tau,z)$.

Fig. 12.13: Normalized Black-Scholes call price and correction term in (12.20).

We also note that

$$\begin{aligned}
\mathbb{E}^*\big[M_0^T - m_0^T \,\big|\, S_0 = x\big] &= x - x\,\mathrm{e}^{-(T-t)r}\Phi\big(\delta_-^{T-t}(1)\big) \\
&\quad -x\left(1+\frac{\sigma^2}{2r}\right)\Phi\big(-\delta_+^{T-t}(1)\big) + x\,\mathrm{e}^{-(T-t)r}\frac{\sigma^2}{2r}\Phi\big(\delta_-^{T-t}(1)\big) \\
&\quad +x\,\mathrm{e}^{-(T-t)r}\Phi\big(-\delta_-^{T-t}(1)\big) + x\left(1+\frac{\sigma^2}{2r}\right)\Phi\big(\delta_+^{T-t}(1)\big) \\
&\quad -x\frac{\sigma^2}{2r}\mathrm{e}^{-(T-t)r}\Phi\big(-\delta_-^{T-t}(1)\big) - x \\
&= x\left(1+\frac{\sigma^2}{2r}\right)\Big(\Phi\big(\delta_+^{T-t}(1)\big) - \Phi\big(-\delta_+^{T-t}(1)\big)\Big) \\
&\quad +x\,\mathrm{e}^{-(T-t)r}\left(\frac{\sigma^2}{2r}-1\right)\Big(\Phi\big(\delta_-^{T-t}(1)\big) - \Phi\big(-\delta_-^{T-t}(1)\big)\Big).
\end{aligned}$$

12.4 Delta Hedging for Lookback Options

In this section we compute hedging strategies for lookback call and put options by application of the Delta hedging formula (12.15). See Bermin (1998), § 2.6.1, page 29, for another approach to the following result using the Clark-Ocone formula. Here we use (12.15) instead, cf. Proposition 4.6 of El Khatib and Privault (2003).

Proposition 12.8. *The Delta hedging strategy of the lookback call option is given by*

$$\xi_t = 1 - \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \tag{12.21}$$

$$+\,\mathrm{e}^{-(T-t)r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\left(\frac{\sigma^2}{2r}-1\right)\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right),\qquad 0\leqslant t\leqslant T.$$

Proof. By (12.15) and (12.18), we need to differentiate

$$f(t,x,y)=\mathrm{Bl}_{\mathrm{c}}(x,y,r,\sigma,T-t)+xh_c\left(T-t,\frac{x}{y}\right)$$

with respect to the variable x, where

$$h_c(\tau,z)=-\frac{\sigma^2}{2r}\left(\Phi\left(-\delta_+^\tau(z)\right)-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right)$$

is given by (12.19) First, we note that the relation

$$\frac{\partial}{\partial x}\mathrm{Bl}_c(x,y,r,\sigma,\tau)=\Phi\left(\delta_+^\tau\left(\frac{x}{y}\right)\right)$$

is known, cf. Propositions 6.4 and 7.13. Next, we have

$$\frac{\partial}{\partial x}\left(xh_c\left(\tau,\frac{x}{y}\right)\right)=h_c\left(\tau,\frac{x}{y}\right)+\frac{x}{y}\frac{\partial h_c}{\partial z}\left(\tau,\frac{x}{y}\right),$$

and

$$\begin{aligned}
\frac{\partial h_c}{\partial z}(\tau,z)&=-\frac{\sigma^2}{2r}\left(\frac{\partial}{\partial z}\left(\Phi\left(-\delta_+^\tau(z)\right)\right)-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\frac{\partial}{\partial z}\left(\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right)\right)\\
&-\frac{\sigma^2}{2r}\left(\frac{2r}{\sigma^2}\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right)\\
&=\frac{\sigma}{2rz\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}(\delta_+^\tau(z))^2\right)\\
&-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\frac{\sigma}{2rz\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)^2\right)-\frac{2r}{\sigma^2}\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right).
\end{aligned}$$

Next, we note that

$$\begin{aligned}
\mathrm{e}^{-(\delta_-^\tau(1/z))^2/2}&=\exp\left(-\frac{1}{2}\left(\delta_+^\tau(z)\right)^2-\frac{1}{2}\left(\frac{4r^2}{\sigma^2}\tau-\frac{4r}{\sigma}\delta_+^\tau(z)\sqrt{\tau}\right)\right)\\
&=\mathrm{e}^{-\left(\delta_+^\tau(z)\right)^2/2}\exp\left(-\frac{1}{2}\left(\frac{4r^2}{\sigma^2}\tau-\frac{4r}{\sigma^2}\left(\log z+\left(r+\frac{1}{2}\sigma^2\right)\tau\right)\right)\right)\\
&=\mathrm{e}^{-\left(\delta_+^\tau(z)\right)^2/2}\exp\left(\frac{-2r^2}{\sigma^2}\tau+\frac{2r}{\sigma^2}\log z+\frac{2r^2}{\sigma^2}\tau+r\tau\right)\\
&=\mathrm{e}^{r\tau}z^{2r/\sigma^2}\mathrm{e}^{-(\delta_+^\tau(z))^2/2}
\end{aligned}\qquad(12.22)$$

as in the proof of Proposition 6.4, hence

$$\frac{\partial h_c}{\partial z}\left(\tau, \frac{x}{y}\right) = -\,\mathrm{e}^{-r\tau} z^{-1-2r/\sigma^2} \Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right),$$

and

$$\frac{\partial}{\partial x}\left(x h_c\left(\tau, \frac{x}{y}\right)\right) = h_c\left(\tau, \frac{x}{y}\right) - \mathrm{e}^{-r\tau}\left(\frac{y}{x}\right)^{2r/\sigma^2} \Phi\left(\delta_-^\tau\left(\frac{y}{x}\right)\right),$$

which concludes the proof. □

We note that $\xi_t = 1 > 0$ as T tends to infinity, and that at maturity $t = T$, the delta hedging strategy satisfies

$$\xi_T = \begin{cases} 1 & \text{if } m_0^T < S_T, \\ 1 - \frac{1}{2}\left(1 + \frac{\sigma^2}{2r}\right) + \frac{1}{2}\left(\frac{\sigma^2}{2r} - 1\right) = 0 & \text{if } m_0^T = S_T. \end{cases}$$

In Figure 12.14 we represent the Delta of the lookback call option, as given by (12.21).

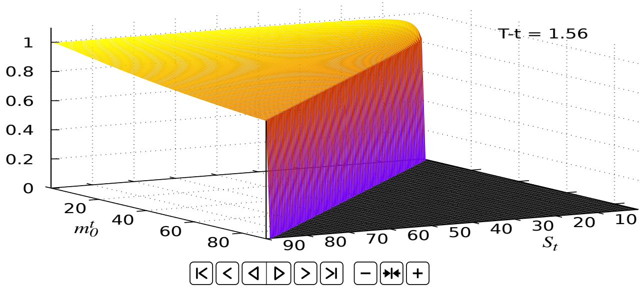


Fig. 12.14: Delta of the lookback call option with $r = 2\%$ and $\sigma = 0.41$.*

The above scaling procedure can be applied to the Delta of lookback call options by noting that ξ_t can be written as

$$\xi_t = \zeta\left(t, \frac{S_t}{m_0^t}\right),$$

where the function $\zeta(t, z)$ is given by

$$\zeta(t, z) = \Phi\left(\delta_+^{T-t}(z)\right) - \frac{\sigma^2}{2r}\Phi\left(-\delta_+^{T-t}(z)\right) \qquad (12.23)$$

* The animation works in Acrobat Reader on the entire pdf file.

$$+\mathrm{e}^{-(T-t)r} z^{-2r/\sigma^2}\left(\frac{\sigma^2}{2r}-1\right)\Phi\left(\delta_-^{T-t}\left(\frac{1}{z}\right)\right),$$

$t \in [0,T]$, $z \in [0,1]$. The graph of the function $(t,z) \mapsto \zeta(t,z)$ is given in Figure 12.15.

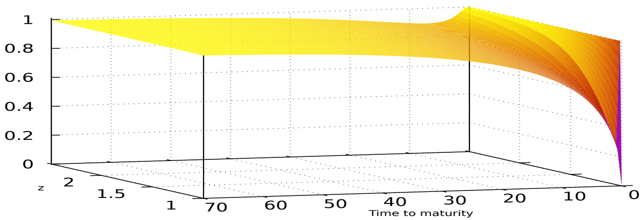


Fig. 12.15: Rescaled portfolio strategy for the lookback call option.

Similar calculations using (12.4) can be carried out for other types of lookback options, such as options on extrema and partial lookback options, cf. El Khatib (2003). As a consequence of Propositions 12.5 and 12.8, we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T - m_0^T \mid \mathcal{F}_t\right]\\
&= S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - m_0^t\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\\
&\quad + \mathrm{e}^{-(T-t)r}S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - S_t\frac{\sigma^2}{2r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\\
&= \xi_t S_t + m_0^t\,\mathrm{e}^{-(T-t)r}\left(\left(\frac{S_t}{m_0^t}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\right),
\end{aligned}$$

and the quantity of the riskless asset e^{rt} in the portfolio is given by

$$\eta_t = m_0^t\,\mathrm{e}^{-rT}\left(\left(\frac{S_t}{m_0^t}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\right),$$

so that the portfolio value V_t at time t satisfies

$$V_t = \xi_t S_t + \eta_t\,\mathrm{e}^{rt}, \qquad t \geqslant 0.$$

Proposition 12.9. *The Delta hedging strategy of the lookback put option is given by*

$$\xi_t = \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \tag{12.24}$$

$$+\mathrm{e}^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)-1, \quad 0\leqslant t\leqslant T.$$

Proof. By (12.15) and (12.10), we need to differentiate

$$f(t,x,y)=\mathrm{Bl}_{\mathrm{p}}(x,y,r,\sigma,T-t)+xh_p\left(T-t,\frac{x}{y}\right)$$

where

$$h_p(\tau,z)=\frac{\sigma^2}{2r}\Phi\left(\delta_+^\tau(z)\right)-\mathrm{e}^{-r\tau}\frac{\sigma^2}{2r}z^{-2r/\sigma^2}\Phi\big(-\delta_-^\tau(1/z)\big),$$

and

$$\delta_\pm^\tau(z):=\frac{1}{\sigma\sqrt{\tau}}\left(\log z+\left(r\pm\frac{1}{2}\sigma^2\right)\tau\right), \qquad z>0.$$

We have

$$\begin{aligned}\frac{\partial h_p}{\partial z}(\tau,z) &= \frac{\sigma^2}{2r}\delta_+'^\tau(z)\,\varphi\left(\delta_+^\tau(z)\right)+\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right)\\ &\quad+\frac{\sigma^2}{2rz^2}\delta_-'^\tau\left(\frac{1}{z}\right)\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\varphi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\\ &= \mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right)\\ &\quad+\frac{\sigma}{2rz\sqrt{\tau}}\left(\varphi\big(\delta_+^\tau(z)\big)-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\varphi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right).\end{aligned}$$

From the relation

$$\begin{aligned}\left(\delta_+^{T-t}(z)\right)^2-\left(\delta_-^{T-t}\left(\frac{1}{z}\right)\right)^2 &= \left(\delta_+^{T-t}(z)+\delta_-^{T-t}\left(\frac{1}{z}\right)\right)\left(\delta_+^{T-t}(z)-\delta_-^{T-t}\left(\frac{1}{z}\right)\right)\\ &= \frac{2r}{\sigma^2}\log z+2r(T-t),\end{aligned}$$

we have

$$\varphi\big(\delta_+^{T-t}(z)\big)=z^{-2r/\sigma^2}\,\mathrm{e}^{-r(T-t)}\varphi\left(\delta_-^{T-t}\left(\frac{1}{z}\right)\right),$$

hence

$$\frac{\partial h_p}{\partial z}(\tau,z)=\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right).$$

Therefore, knowing that the Black-Scholes put Delta is

$$-\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right)=-1+\Phi\left(\delta_+^{T-t}\left(\frac{x}{y}\right)\right),$$

see *e.g.* Proposition 6.7, we have

$$
\begin{aligned}
\frac{\partial f}{\partial x}(t,x,y) &= -\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) + h_p\left(T-t,\frac{x}{y}\right) + \frac{x}{y}\frac{\partial h_p}{\partial z}\left(T-t,\frac{x}{y}\right) \\
&= -\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) + \frac{\sigma^2}{2r}\Phi\left(\delta_+^{T-t}\left(\frac{x}{y}\right)\right) \\
&\quad + \mathrm{e}^{-(T-t)r}\left(\frac{y}{x}\right)^{2r/\sigma^2}\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_-^{T-t}\left(\frac{y}{x}\right)\right),
\end{aligned}
$$

which yields (12.24). □

Note that we have $\xi_t = \sigma^2/(2r) > 0$ as T tends to infinity. At maturity $t = T$, the delta hedging strategy satisfies

$$
\xi_T = \begin{cases} -1 & \text{if } M_0^T > S_T, \\ \frac{1}{2} + \frac{\sigma^2}{4r} + \frac{1}{2}\left(1 - \frac{\sigma^2}{2r}\right) - 1 = 0 & \text{if } M_0^T = S_T. \end{cases}
$$

In Figure 12.16 we represent the Delta of the lookback put option, as given by (12.24).

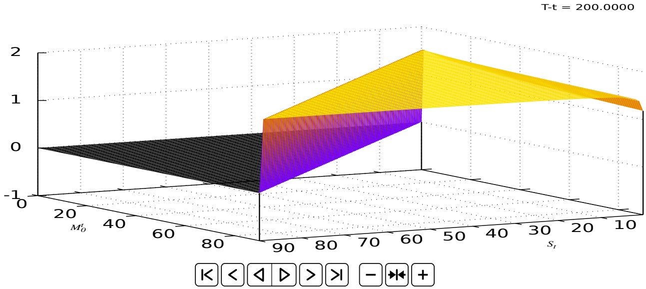


Fig. 12.16: Delta of the lookback put option with $r = 2\%$ and $\sigma = 0.25$.*

As a consequence of Propositions 12.1 and 12.9, we have

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, \mathcal{F}_t\right] \\
&= M_0^t\,\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&\quad - S_t\,\mathrm{e}^{-(T-t)r}\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right) - S_t \\
&= \xi_t S_t + M_0^t\,\mathrm{e}^{-(T-t)r}\left(\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{S_t}{M_0^t}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right),
\end{aligned}
$$

* The animation works in Acrobat Reader on the entire pdf file.

and the quantity of the riskless asset e^{rt} in the portfolio is given by

$$\eta_t = M_0^t \mathrm{e}^{-rT} \left(\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t} \right) \right) - \left(\frac{S_t}{M_0^t} \right)^{1-2r/\sigma^2} \Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t} \right) \right) \right)$$

so that the portfolio value V_t at time t satisfies

$$V_t = \xi_t S_t + \eta_t \mathrm{e}^{rt}, \qquad t \geqslant 0.$$

Exercises

Exercise 12.1

a) Give the probability density function of the maximum of drifted Brownian motion $\operatorname*{Max}_{t\in[0,1]}(B_t + \sigma t/2)$.

b) Taking $S_t := \mathrm{e}^{\sigma B_t - \sigma^2 t/2}$, compute the expected value

$$\mathbb{E}\Big[\min_{t\in[0,1]} S_t \Big] = \mathbb{E}\left[\min_{t\in[0,1]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} \right] = \mathbb{E}\left[\mathrm{e}^{-\sigma \operatorname{Max}_{t\in[0,1]}(B_t + \sigma t/2)} \right].$$

c) Compute the "optimal exercise" price $\mathbb{E}\left[\left(K - S_0 \min_{t\in[0,1]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} \right)^+ \right]$ of a finite expiration American put option with $S_0 \leqslant K$.

Exercise 12.2 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion.

a) Compute the expected value

$$\mathbb{E}\Big[\operatorname*{Max}_{t\in[0,1]} S_t \Big] = \mathbb{E}\left[\mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,1]}(B_t - \sigma t/2)} \right].$$

b) Compute the "optimal exercise" price $\mathbb{E}\left[\left(S_0 \operatorname*{Max}_{t\in[0,1]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} - K \right)^+ \right]$ of a finite expiration American call option with $S_0 \geqslant K$.

Exercise 12.3 Consider a risky asset whose price S_t is given by

$$dS_t = \sigma S_t dB_t + \sigma^2 S_t dt/2,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Compute the cumulative distribution function and the probability density function of the minimum $\min_{t\in[0,T]} B_t$ over the interval $[0,T]$?
b) Compute the price value

$$e^{-\sigma^2 T/2}\mathbb{E}^*\left[S_T - \min_{t\in[0,T]} S_t\right]$$

of a lookback call option on S_T with maturity T.

Exercise 12.4 (Dassios and Lim (2019)) The digital drawdown call option with qualifying period pays a unit amount when the drawdown period reaches one unit of time, if this happens before fixed maturity T, but only if the size of drawdown at this stopping time is larger than a prespecified K. This provides an insurance against a prolonged drawdown, if the drawdown amount is large. Specifically, the digital drawdown call option is priced as

$$\mathbb{E}^*\big[e^{-r\tau}\mathbb{1}_{\{\tau\leqslant T\}}\mathbb{1}_{\{M_0^\tau - S_\tau\geqslant K\}}\big],$$

where $M_0^t := \mathrm{Max}_{u\in[0,t]} S_u$, $U_t := t - \mathrm{Sup}\{0\leqslant u\leqslant t \ : \ M_0^t = S_u\}$, and $\tau := \inf\{t\in\mathbb{R}_+ \ : \ U_t = 1\}$. Write the price of the drawdown option as a triple integral using the joint probability density function $f_{(\tau,S_\tau,M_\tau)}(t,x,y)$ of (τ, S_τ, M_τ) under the risk-neutral probability measure $\mathbb{P}^*$.

Exercise 12.5

a) Check explicitly that the boundary conditions (12.3a)-(12.3c) are satisfied.
b) Check explicitly that the boundary conditions (12.14a)-(12.14b) are satisfied.

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