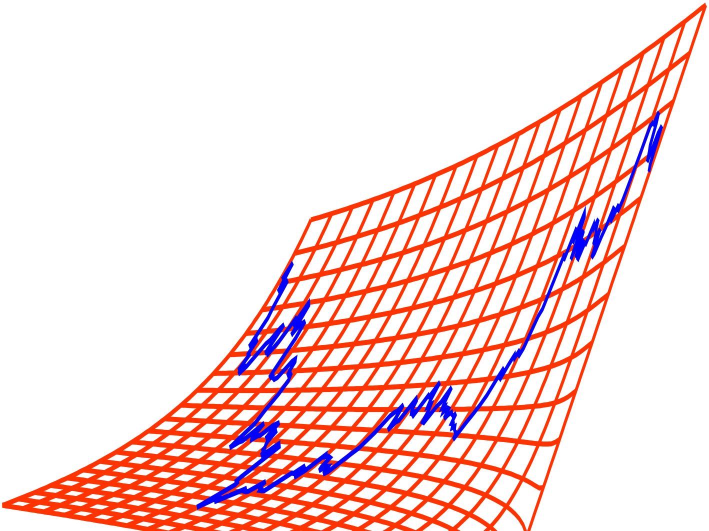
Nicolas Privault

Notes on Stochastic Finance



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Preface

This book is an introduction to a wide range of topics in financial mathematics, including Black-Scholes pricing, exotic and american options, term structure modeling and change of numéraire, stochastic volatility, as well as models with jumps. It presents the mathematics of pricing and hedging in discrete and continuous-time financial models, with an emphasis on the complementarity between analytical and probabilistic methods. The contents are mostly mathematical, and also aim at making the reader aware of both the power and limitations of mathematical models in finance, by taking into account their conditions of applicability. The text is targeted at the advanced undergraduate and graduate levels in applied mathematics, financial engineering, and economics.

The point of view adopted is that of mainstream mathematical finance, in which the computation of fair prices is based on the absence of arbitrage hypothesis, therefore excluding riskless profit based on arbitrage opportunities and basic (buying low/selling high) trading. Similarly, this document is not concerned with any "prediction" of stock price behaviors that belong to other domains such as technical analysis, which should not be confused with the statistical modeling of asset prices.

The descriptions of the asset model, self-financing portfolios, arbitrage, and market completeness, are first given in Chapter 1 in a simple two time-step setting. These notions are then reformulated in discrete time in Chapter 2. Here, the impossibility to access future information is formulated using the notion of adapted processes, which will play a central role in the construction of stochastic calculus in continuous time.

In order to trade efficiently, it would be useful to have a formula to estimate the "fair price" of a given risky asset, helping for example to determine whether the asset is undervalued or overvalued at a given time. Although such a formula is not available, we can instead derive formulas for the pricing of options that can act as insurance contracts to protect their holders against adverse changes in the prices of risky assets. The pricing and hedging of options in discrete time, particularly in the fundamental example of the Cox-Ross-Rubinstein model, are considered in Chapter 3, with a description of the passage from discrete to continuous time that prepares the transition to the subsequent chapters.

A simplified presentation of Brownian motion, stochastic integrals, and the associated Itô formula, is given in Chapter 4, with application to stochastic asset price modeling in Chapter 5. The Black-Scholes model is presented from the angle of partial differential equation (PDE) methods in Chapter 6, with the derivation of the Black-Scholes formula by transforming the Black-Scholes PDE into the standard heat equation, which is then solved by a heat kernel argument. The martingale approach to pricing and hedging is then presented in Chapter 7, and complements the PDE approach of Chapter 6 by recovering the Black-Scholes formula via a probabilistic argument. An introduction to stochastic volatility is given in Chapter 8, followed by a presentation of volatility estimation tools including historical, local, and implied volatilities, in Chapter 9. This chapter also contains a comparison of the prices obtained by the Black-Scholes formula with actual option price market data.

Exotic options such as barrier, lookback, and Asian options are treated in Chapters 11, 12, and 13, respectively, following an introduction to the properties of the maximum of Brownian motion given in Chapter 10. Optimal stopping and exercise, with application to the pricing of American options, are considered in Chapter 15, following the presentation of background material on filtrations and stopping times in Chapter 14. The construction of forward measures by change of numéraire is given in Chapter 16 and is applied to the pricing of interest rate derivatives such as caplets, caps, and swaptions in Chapter 19, after an introduction to bond pricing and to the modeling of forward rates in Chapters 17, and 18.

Stochastic calculus with jumps is dealt with in Chapter 20 and is restricted to compound Poisson processes, which only have a finite number of jumps on any bounded interval. Those processes are used for option pricing and hedging in jump models in Chapter 21, in which we mostly focus on risk-minimizing strategies as markets with jumps are generally incomplete. Chapter 22 contains an elementary introduction to finite difference methods for the numerical solution of PDEs and stochastic differential equations, dealing with the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milshtein schemes for SDEs. The text is completed with an appendix containing the needed probabilistic background.

The material in this book has been used for teaching in the Masters of Science in Financial Engineering at City University of Hong Kong and at the Nanyang Technological University in Singapore. The author thanks Nicky van Foreest, Jinlong Guo, Kazuhiro Kojima, Sijian Lin, Panwar Samay, Sandu Ursu, and Ju-Yi Yen for corrections and improvements.

This text contains 277 exercises and 18 problems with complete solutions. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on a problem number sends the reader to the corresponding solution, however this feature should not be misused. The cover graph represents the time evolution of the HSBC stock price from January to September 2009, plotted on the price surface of a European *put option* on that asset, expiring on October 05, 2009, see § 6.1.

This pdf file contains internal and external links, 29 tables and 381 figures, including 57 animated Figures 3.8, 3.10, 4.6, 4.7, 4.10, 4.11, 4.16, 5.5, 6.5, 10.1, 10.2, 10.3, 10.6, 11.11, 13.1, 12.1, 12.6, 12.14, 15.2, 17.16, 18.7, 18.10, 18.11, 18.18, 20.14, 20.16, 20.17, and S.18, 2 embedded videos in Figures 2 and 9.3, and 3 interacting 3D graphs in Figures 6.4, 6.11 and 11.1, that may require using Acrobat Reader for viewing on the complete pdf file. It also includes 30 Python codes e.g. on pages 75, 96, 100, 103, 145, 157, 236, 266, 363, 553 and 908, and 85 R codes on pages 155, 157, 159, 163, 217, 213, 237, 239, 253, 245, 263, 266, 279, 363, 364, 379, 342, 421, 430, 652, 707, 731, 735, 749, 751, 825 and 828.

Nicolas Privault
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* Animated figures (work with Acrobat Reader).

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Introduction

Modern quantitative finance requires a strong background in fields such as stochastic calculus, optimization, partial differential equations (PDEs) and numerical methods, or even infinite dimensional analysis. In addition, the emergence of new complex financial instruments on the markets makes it necessary to rely on increasingly sophisticated mathematical tools. Not all readers of this book will eventually work in quantitative financial analysis, nevertheless they may have to interact with quantitative analysts, and becoming familiar with the tools they employ could be an advantage. In addition, despite the availability of ready made financial calculators it still makes sense to be able oneself to understand, design and implement such financial algorithms. This can be particularly useful under different types of conditions, including an eventual lack of trust in financial indicators, possible unreliability of expert advice such as buy/sell recommendations, or other factors such as market manipulation. Instead of relying on predictions of stock price movements based on various tools (*e.g.* technical analysis, charting, "cup & handle" figures), we acknowledge that predicting the future is a difficult task and we rely on the Efficient Market Hypothesis. In this framework, the time evolution of the prices of risky assets will be modeled by random walks and stochastic processes.

Historical sketch

We start with a description of some of the main steps, ideas and individuals that played an important role in the development of the field over the last century.

Robert Brown, botanist, 1828

Brown (1828) observed the movement of pollen particles as described in "A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies." Phil. Mag. 4, 161-173, 1828.

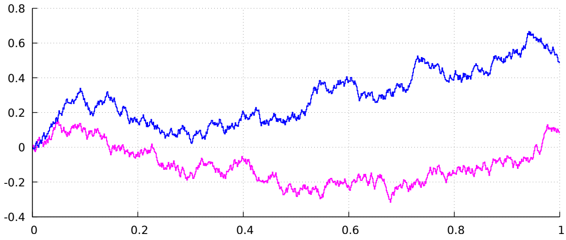

Fig. 1: Two sample paths of one-dimensional Brownian motion.

Philosophical Magazine, first published in 1798, is a journal that "publishes articles in the field of condensed matter describing original results, theories and concepts relating to the structure and properties of crystalline materials, ceramics, polymers, glasses, amorphous films, composites and soft matter."

Albert Einstein, physicist

Einstein received his 1921 Nobel Prize in part for investigations on the theory of Brownian motion: "... in 1905 Einstein founded a kinetic theory to account for this movement", presentation speech by S. Arrhenius, Chairman of the Nobel Committee, Dec. 10, 1922.

Einstein (1905) "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen", Annalen der Physik 17.

Louis Bachelier, mathematician, PhD 1900

Bachelier (1900) used Brownian motion for the modeling of stock prices in his PhD thesis "Théorie de la spéculation", Annales Scientifiques de l'Ecole Normale Supérieure 3 (17): 21-86, 1900.

Norbert Wiener, mathematician, founder of cybernetics

Wiener is credited, among other fundamental contributions, for the mathematical foundation of Brownian motion, published in 1923. In particular he constructed the Wiener space and Wiener measure on $\mathcal{C}_0([0,1])$ (the space of continuous functions from $[0,1]$ to $\mathbb{R}$ vanishing at 0).

Wiener (1923) "Differential space", Journal of Mathematics and Physics of the Massachusetts Institute of Technology, 2, 131-174, 1923.

Kiyoshi Itô (伊藤清), mathematician, C.F. Gauss Prize 2006

Itô (1944) constructed the Itô integral with respect to Brownian motion, and the stochastic calculus with respect to Brownian motion, which laid the foundation for the development of calculus for random processes, see Itô (1951) "On stochastic differential equations", *in* Memoirs of the American Mathematical Society.

> *"Renowned math wiz Itô, 93, dies."* (The Japan Times, Saturday, Nov. 15, 2008)
>
> Kiyoshi Itô, an internationally renowned mathematician and professor emeritus at Kyoto University died Monday of respiratory failure at a Kyoto hospital, the university said Friday. He was 93. Itô was once dubbed "the most famous Japanese in Wall Street" thanks to his contribution to the founding of financial derivatives theory. He is known for his work on stochastic differential equations and the "Itô Formula", which laid the foundation for the Black and Scholes (1973) model, a key tool for financial engineering. His theory is also widely used in fields like physics and biology.

Paul Samuelson, economist, Nobel Prize 1970

Samuelson (1965) rediscovered Bachelier's ideas and proposed geometric Brownian motion as a model for stock prices. In an interview he stated "In the early 1950s I was able to locate by chance this unknown Bachelier (1900) book, rotting in the library of the University of Paris, and when I opened it up it was as if a whole new world was laid out before me." We refer to "Rational theory of warrant pricing" by Paul Samuelson, *Industrial Management Review*, p. 13-32, 1965.

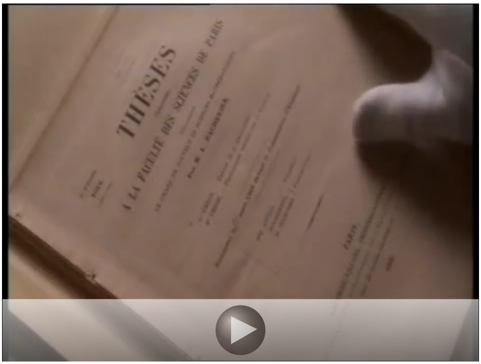

Fig. 2: Clark (2000) "As if a whole new world was laid out before me."*

* Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

In recognition of Bachelier's contribution, the Bachelier Finance Society was started in 1996 and now holds the World Bachelier Finance Congress every two years.

Robert Merton, Myron Scholes, economists

Robert Merton and Myron Scholes shared the 1997 Nobel Prize in economics: "In collaboration with Fisher Black, developed a pioneering formula for the valuation of stock options ... paved the way for economic valuations in many areas ... generated new types of financial instruments and facilitated more efficient risk management in society."*

Black and Scholes (1973) "The Pricing of Options and Corporate Liabilities". Journal of Political Economy 81 (3): 637-654.

The development of options pricing tools contributed greatly to the expansion of option markets and led to development several ventures such as the "Long Term Capital Management" (LTCM), founded in 1994. The fund yielded annualized returns of over 40% in its first years, but registered a loss of US$4.6 billion in less than four months in 1998, which resulted into its closure in early 2000.

Oldřich Vašíček, economist, 1977

Interest rates behave differently from stock prices, notably due to the phenomenon of mean reversion, and for this reason they are difficult to model using geometric Brownian motion. Vašíček (1977) was the first to suggest a mean-reverting model for stochastic interest rates, based on the Ornstein-Uhlenbeck process, in "An equilibrium characterization of the term structure", Journal of Financial Economics 5: 177-188.

David Heath, Robert Jarrow, Andrew Morton

These authors proposed in 1987 a general framework to model the evolution of (forward) interest rates, known as the Heath-Jarrow-Morton (HJM) model, see Heath et al. (1992) "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation", Econometrica, (January 1992), Vol. 60, No. 1, pp 77-105.

Alan Brace, Dariusz Gatarek, Marek Musiela (BGM)

The Brace et al. (1997) model is actually based on geometric Brownian motion, and it is especially useful for the pricing of interest rate derivatives such as interest rate caps and swaptions on the LIBOR market, see "The Market Model of Interest Rate Dynamics". Mathematical Finance Vol. 7, page 127. Blackwell 1997, by Alan Brace, Dariusz Gatarek, Marek Musiela. Although LIBOR rates are being phased out, we will still use this terminology when referring to simple or linear compounded forward rates.

* This has to be put in relation with the modern development of risk societies; "societies increasingly preoccupied with the future (and also with safety), which generates the notion of risk" (Wikipedia).

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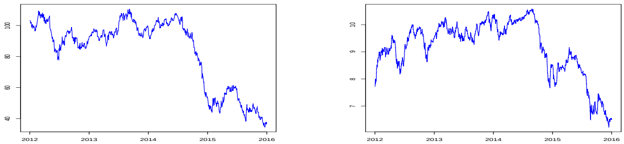
Financial derivatives

The following graphs exhibit a correlation between commodity (oil) prices and an oil-related asset price.



(a) WTI price graph.   (b) Graph of Keppel Corp. stock price

Fig. 3: Comparison of WTI *vs.* Keppel price graphs.

The study of financial derivatives aims at finding functional relationships between the price of an underlying asset (a company stock price, a commodity price, etc.) and the price of a related financial contract (an option, a financial derivative, etc.).

Option contracts

Early accounts of option contracts can also be found in *The Politics* Aristotle (BCE) by Aristotle (384-322 BCE). Referring to the philosopher Thales of Miletus (c. 624 - c. 546 BCE), Aristotle writes:

> "He (Thales) knew by his skill in the stars while it was yet winter that there would be a great harvest of olives in the coming year; so, having a little money, he gave *deposits* for the use of all the olive-presses in Chios and Miletus, which he hired at a low price because no one bid against him. When the harvest-time came, and many were wanted all at once and of a sudden, he let them out at any rate which he pleased, and made a quantity of money".

In the above example, olive oil can be regarded as the underlying asset, while the oil press stands for the financial derivative. Option credit contracts appear to have been used as early as the $10th$ century by traders in the Mediterranean.

Next, we move to a description of (European) call and put options, which are at the basis of risk management.

European put option contracts

As previously mentioned, an important concern for the buyer of a stock at time t is whether its price S_T can decline at some future date T. The buyer of the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t. This contract is called a put option with strike price K and exercise date T.

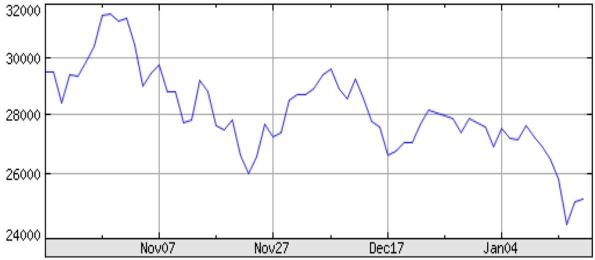


Fig. 4: Graph of the Hang Seng index - holding a put option might be useful here.

Definition 1. *A (European)* put *option is a contract that gives its holder the right (but not the obligation) to* sell *a quantity of assets at a predefined price K called the strike price (or exercise price) and at a predefined date T called the maturity.*

In case the price S_T falls down below the level K, exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option contract and have to sell the asset at the market price S_T. In turn, the issuer of the option contract will register a loss also equal to $K - S_T$ (in the absence of transaction costs and other fees).

If S_T is above K, then the holder of the option contract will not exercise the option as he may choose to sell at the price S_T. In this case the profit derived from the option contract is 0. Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 5.

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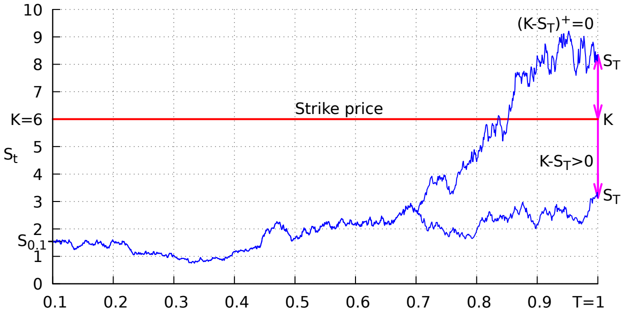


Fig. 5: Two put option scenarios.

Cash settlement *vs.* physical delivery

Physical delivery. In the case of physical delivery, the put option contract issuer will pay the strike price $\$K$ to the option contract holder in exchange for one unit of the risky asset priced S_T.

Cash settlement. In the case of a cash settlement, the put option issuer will satisfy the contract by transferring the amount $C = (K - S_T)^+$ to the option contract holder.

In general, the payoff of a (so-called *European*) put option contract can be written as

$$\phi(S_T) = (K - S_T)^+ := \begin{cases} K - S_T & \text{if } S_T \leqslant K, \\ 0, & \text{if } S_T \geqslant K. \end{cases}$$

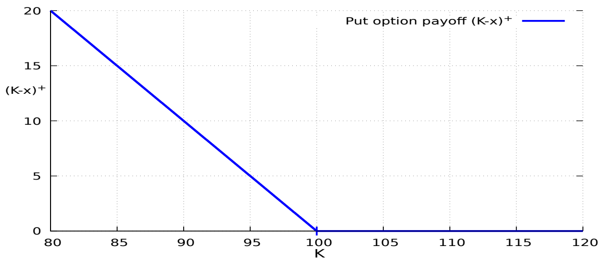


Fig. 6: Payoff function of a put option with strike price $K = 100$.

See *e.g.* https://optioncreator.com/stwwxvz.

Put option examples

a) The **buy back guarantee*** in currency exchange;

b) the **price drop protection** in online ticket booking

are common examples of European put options.

The derivatives market

As of year 2015, the size of the financial derivatives market is estimated at over \$1.2 quadrillion† USD, which is more than 10 times the Gross World Product (GWP). See **here** or **here** for up-to-date data on outstanding notional amounts and gross market value from the Bank for International Settlements (BIS).

European call option contracts

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t.

Definition 2. *A (European)* call *option is a contract that gives its holder the right (but not the obligation) to* purchase *a quantity of assets at a predefined price K called the strike price, and at a predefined date T called the maturity.*

Here, in the event that S_T goes above K, the buyer of the option contract will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 7.

* Right-click to open or save the attachment.

† One thousand trillion, or one million billion, or 10^{15}.

Fig. 7: Two call option scenarios.

Cash settlement *vs.* physical delivery

Physical delivery. In the case of physical delivery, the call option contract issuer will transfer one unit of the risky asset priced S_T to the option contract holder in exchange for the strike price $\$K$. Physical delivery may include physical goods, commodities or assets such as coffee, airline fuel or live cattle, see Schroeder and Coffey (2018).

Cash settlement. In the case of a cash settlement, the call option issuer will fulfill the contract by transferring the amount $C = (S_T - K)^+$ to the option contract holder.

In general, the payoff of a (so-called European) call option contract can be written as

$$\phi(S_T) = (S_T - K)^+ := \begin{cases} S_T - K & \text{if } S_T \geqslant K, \\ 0, & \text{if } S_T \leqslant K. \end{cases}$$

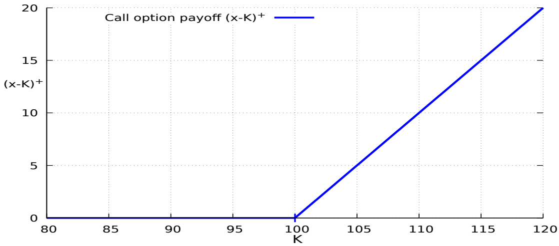


Fig. 8: Payoff function of a call option with strike price $K = 100$.

See *e.g.* https://optioncreator.com/stqhbgn.

Call option example: The **price lock guarantee*** in online ticket booking is a common example of a European *call* option.

According to market practice, options are often divided into a certain number n of *warrants*, the (possibly fractional) quantity n being called the *entitlement ratio*.

Option pricing

In order for an option contract to be fair, the buyer of the option contract should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, and is known as option *pricing*.

Option hedging

The second important issue is that of *hedging*, *i.e.* how to manage a given portfolio in such a way that it contains the required random payoff $(K - S_T)^+$ (for a put option) or $(S_T - K)^+$ (for a call option) at the maturity date T.

The next Figure 9 illustrates a sharp increase and sharp drop in asset price, making it valuable to hold a call option contract during the first half of the graph, whereas holding a put option contract would be recommended during the second half.

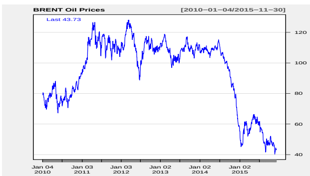
* Right-click to open or save the attachment.

Fig. 9: “Infogrames” stock price curve.

Example: Fuel hedging and the four-way zero-collar option

```
install.packages("Quandl")
library(Quandl);library(quantmod)
getSymbols("DCOILBRENTEU", src="FRED")
chartSeries(DCOILBRENTEU,up.col="blue",theme="white",name = "BRENT Oil
      Prices",lwd=5)
BRENT = Quandl("FRED/DCOILBRENTEU",start_date="2010-01-01",
      end_date="2015-11-30",type="xts")
chartSeries(BRENT,up.col="blue",theme="white",name = "BRENT Oil Prices",lwd=5)
getSymbols("WTI", from="2010-01-01", to="2015-11-30")
WTI <- Ad(`WTI`)
chartSeries(WTI,up.col="blue",theme="white",name = "WTI Oil Prices",lwd=5)
```

(a) WTI price graph.



(b) Brent price graph

Fig. 10: Brent and WTI price graphs.

(April 2011)
Fuel hedge promises Kenya Airways smooth ride in volatile oil market.*

(November 2015)
A close look at the role of fuel hedging in Kenya Airways $259 million loss.*

* Right-click to open or save the attachment.

The four-way call collar call option requires its holder to purchase the underlying asset (here, airline fuel) at a price specified by the blue curve in Figure 11, when the underlying asset price is represented by the red line.

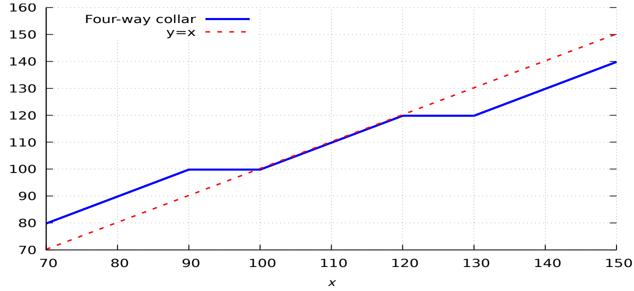


Fig. 11: Price map of a four-way collar option.

The four-way call collar option contract will result into a positive or negative payoff depending on current fuel prices, as illustrated in Figure 12.

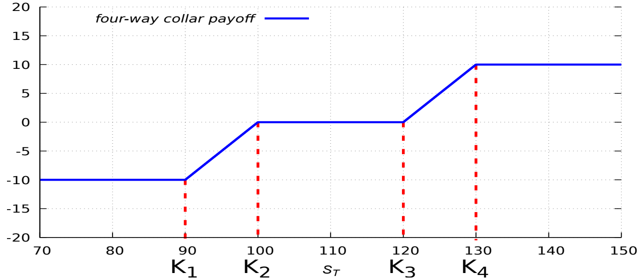


Fig. 12: Payoff function of a four-way call collar option.

The four-way call collar payoff can be written as a linear combination

$$\phi(S_T) = (K_1 - S_T)^+ - (K_2 - S_T)^+ + (S_T - K_3)^+ - (S_T - K_4)^+$$

of call and put option payoffs with respective strike prices

$$K_1 = 90, \quad K_2 = 100, \quad K_3 = 120, \quad K_4 = 130,$$

see *e.g.* https://optioncreator.com/st5rf51.

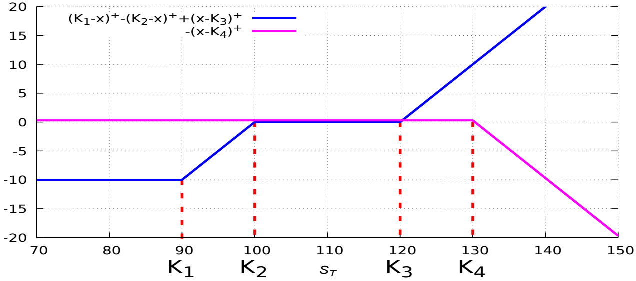


Fig. 13: Four-way call collar payoff as a combination of call and put options.*

Therefore, the four-way call collar option contract can be *synthesized* by:

1. purchasing a *put option* with strike price $K_1 = \$90$, and
2. selling (or issuing) a *put option* with strike price $K_2 = \$100$, and
3. purchasing a *call option* with strike price $K_3 = \$120$, and
4. selling (or issuing) a *call option* with strike price $K_4 = \$130$.

Moreover, the call collar option contract can be made *costless* by adjusting the boundaries K_1, K_2, K_3, K_4, in which case it becomes a *zero-collar* option.

Example - The one-step 4-5-2 model

We close this introduction with a simplified example of the pricing and hedging technique in a binary model. Consider:

i) A risky underlying stock valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values

$$S_1 = \begin{cases} \$5 \\ \$2 \end{cases}$$

at time $t = 1$.

ii) An option contract that promises a claim payoff C whose values are defined contingent to the market data of S_1 as:

$$C := \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \$0 & \text{if } S_1 = \$2. \end{cases}$$

* The animation works in Acrobat Reader on the entire pdf file.

Exercise: Does C represent the payoff of a put option contract? Of a call option contract? If yes, with which strike price K?

Quiz: Using this **form**, submit your own intuitive estimate for the price of the claim C.

At time $t = 0$ the option contract issuer (or writer) chooses to invest ξ units in the risky asset S, while keeping $\$\eta$ on our bank account, meaning that we invest a total amount

$$\xi S_0 + \$\eta \qquad \text{at time} \quad t = 0.$$

Here, the amount $\$\eta$ may be positive or negative, depending on whether it is corresponds to savings or to debt, and is interpreted as a *liability*.

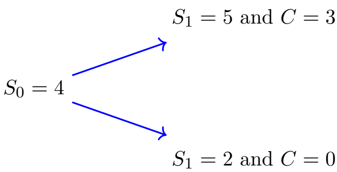
The following issues can be addressed:

a) *Hedging*: How to choose the portfolio allocation $(\xi, \$\eta)$ so that the *value*

$$\xi S_1 + \$\eta$$

of the portfolio matches the future payoff C at time $t = 1$?

b) *Pricing*: How to determine the initial *cost* $\xi S_0 + \$\eta$ of the portfolio built by the option contract issuer at time $t = 0$?



$S_0 = 4$	$S_1 = 5$ and $C = 3$
	$S_1 = 2$ and $C = 0$

Hedging or *replicating* the contract means that at time $t = 1$ the portfolio *value* matches the future payoff C, *i.e.*

$$\xi S_1 + \$\eta = C.$$

Hedge, then price. This condition can be rewritten as

$$C = \begin{cases} \$3 = \xi \times \$5 + \$\eta & \text{if } S_1 = \$5, \\ \$0 = \xi \times \$2 + \$\eta & \text{if } S_1 = \$2, \end{cases}$$

i.e.

$$\begin{cases} 5\xi + \eta = 3, \\ 2\xi + \eta = 0, \end{cases} \qquad \text{which yields} \qquad \begin{cases} \xi = 1 \text{ stock}, \\ \$\eta = -\$2. \end{cases}$$

In other words, the option contract issuer purchases 1 (one) unit of the stock S at the price $S_0 = \$4$, and borrows \$2 from the bank. The price of the option contract is then given by the portfolio value

$$\xi S_0 + \$\eta = 1 \times \$4 - \$2 = \$2.$$

at time $t = 0$.

The above computation is implemented in the attached **IPython notebook***
that can be run **here** or **here**. This algorithm is scalable and can be extended to recombining binary trees over multiple time steps.

Definition 3. *The* arbitrage-free price *of the option contract is defined as the initial cost $\xi S_0 + \$\eta$ of the portfolio hedging the claim payoff C.*

Conclusion: in order to deliver the random payoff $C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \$0 & \text{if } S_1 = \$2. \end{cases}$
to the option contract holder at time $t = 1$, the option contract issuer (or writer) will:

1. charge $\xi S_0 + \$\eta = \$2$ (the option contract price) at time $t = 0$,
2. borrow $-\$\eta = \$2$ from the bank,
3. invest those $\$2 + \$2 = \$4$ into the purchase of $\xi = 1$ unit of stock valued at $S_0 = \$4$ at time $t = 0$,
4. wait until time $t = 1$ to find that the portfolio value has evolved into

$$C = \begin{cases} \xi \times \$5 + \$\eta = 1 \times \$5 - \$2 = \$3 & \text{if } S_1 = \$5, \\ \xi \times \$2 + \$\eta = 1 \times \$2 - \$2 = 0 & \text{if } S_1 = \$2, \end{cases}$$

   so that the option contract and the equality $C = \xi S_1 + \$\eta$ can be fulfilled, allowing the option issuer to break even whatever the evolution of the risky asset price S.

   In a *cash settlement*, the stock is sold at the price $S_1 = \$5$ or $S_1 = \$2$, the payoff $C = (S_1 - K)^+ = \$3$ or \$0 is issued to the option contract holder, and the loan is refunded with the remaining \$2.

   In the case of *physical delivery*, $\xi = 1$ share of stock is handed in to the option holder in exchange for the strike price $K = \$2$ which is used to refund the initial \$2 loan subscribed by the issuer.

Here, the option contract price $\xi S_0 + \$\eta = \$2$ is interpreted as the cost of hedging the option. In Chapters 2 and 3 we will see that this model is scalable and extends to discrete time.

* Right-click to save as attachment (may not work on 🍎).

We note that the initial option contract price of \$2 can be turned to $C = \$3$ (%50 profit) ... or into $C = \$0$ (total ruin).

Thinking further

1) The expected claim payoff at time $t = 1$ is

$$\begin{aligned}\mathbb{E}[C] &= \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0)\\ &= \$3 \times \mathbb{P}(S_1 = \$5).\end{aligned}$$

In absence of arbitrage opportunities ("fair market"), this expected payoff $\mathbb{E}[C]$ should equal the initial amount \$2 invested in the option. In that case we should have

$$\begin{cases}\mathbb{E}[C] = \$3 \times \mathbb{P}(S_1 = \$5) = \$2\\ \mathbb{P}(S_1 = \$5) + \mathbb{P}(S_1 = \$2) = 1.\end{cases}$$

from which we can *infer* the probabilities

$$\begin{cases}\mathbb{P}(S_1 = \$5) = \dfrac{2}{3}\\[2mm] \mathbb{P}(S_1 = \$2) = \dfrac{1}{3},\end{cases} \tag{1}$$

which are called *risk-neutral* probabilities. We see that under the risk-neutral probabilities, the stock S has twice more chances to go up than to go down in a "fair" market.

2) Based on the probabilities (1) we can also compute the expected value $\mathbb{E}[S_1]$ of the stock at time $t = 1$. We find

$$\begin{aligned}\mathbb{E}[S_1] &= \$5 \times \mathbb{P}(S_1 = \$5) + \$2 \times \mathbb{P}(S_1 = \$2)\\ &= \$5 \times \frac{2}{3} + \$2 \times \frac{1}{3}\\ &= \$4\\ &= S_0.\end{aligned}$$

Here, this means that, on average, no extra profit or loss can be made from an investment on the risky stock, and the probabilities $(2/3, 1/3)$ are termed *risk-neutral* probabilities. In a more realistic model we can assume that the riskless bank account yields an interest rate equal to r, in which case the above analysis is modified by letting $\$\eta$ become $\$(1 + r)\eta$ at time $t = 1$, nevertheless the main conclusions remain unchanged.

Market-implied probabilities

By matching the theoretical price $\mathbb{E}[C]$ to an actual market price data $\$M$ as

$$\$M = \mathbb{E}[C] = \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) = \$3 \times \mathbb{P}(S_1 = \$5)$$

we can infer the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \dfrac{\$M}{3} \\ \mathbb{P}(S_1 = \$2) = \dfrac{3 - \$M}{3}, \end{cases} \tag{2}$$

which are *implied probabilities* estimated from market data, as illustrated in Figure 14. We note that the conditions

$$0 < \mathbb{P}(S_1 = \$5) < 1, \qquad 0 < \mathbb{P}(S_1 = \$2) < 1$$

are equivalent to $0 < \$M < 3$, which is consistent with financial intuition in a non-deterministic market. Figure 14 shows the time evolution of probabilities $p(t)$, $q(t)$ of two opposite outcomes.

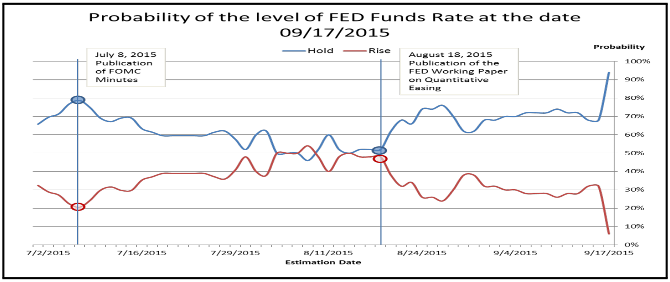


Fig. 14: Implied probabilities.

Note that implied probabilities should also be used with caution, as shown in Figures 15-16.

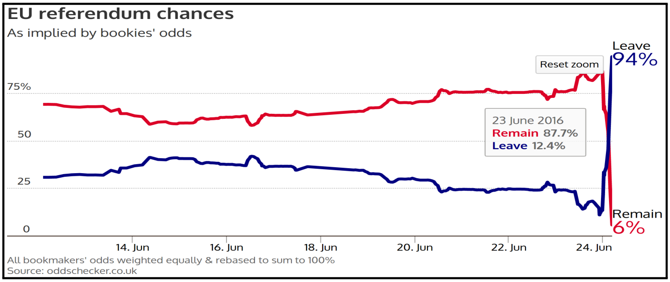


Fig. 15: Implied probabilities according to bookmakers.

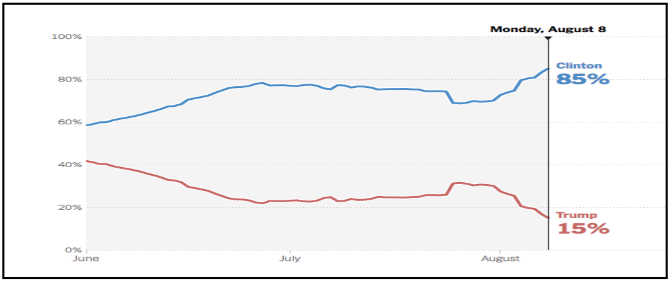


Fig. 16: Implied probabilities according to polling.

Implied probabilities can be estimated using *e.g.* binary options, see for example Exercise 3.11.

The *Practitioner* expects a good model to be:

- *Robust* with respect to missing, spurious or noisy data,
- *Fast* - prices have to be delivered daily in the morning,
- *Easy* to calibrate - parameter estimation,
- *Stable* with respect to re-calibration and the use of new data sets.

Typically, a medium size bank manages 5,000 options and 10,000 deals daily over 1,000 possible scenarios and dozens of time steps. This can mean a hundred million computations of $\mathbb{E}[C]$ daily, or close to a billion such computations for a large bank.

The *mathematician* tends to focus on more theoretical features, such as:

- *Elegance*,

- *Sophistication*,
- *Existence* of analytical (closed-form) solutions / error bounds,
- *Significance* to mathematical finance.

This includes:

- *Creating* new payoff functions and structured products,
- *Defining* new models for underlying asset prices,
- *Finding* new ways to compute expectations $\mathbb{E}[C]$ and hedging strategies.

The methods involved include:

- Monte Carlo methods (60%),

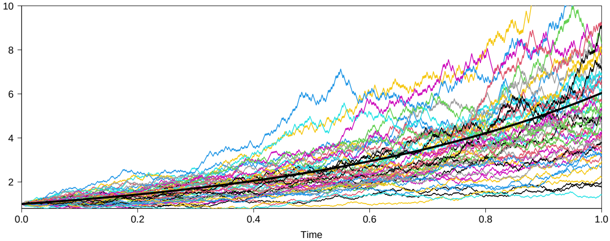


Fig. 17: Fifty sample price paths used for the Monte Carlo method.

- PDEs and finite differences methods (30%),
- Other analytic methods and approximation methods (10%),

+ AI and Machine Learning techniques.

Course plan

The course plan from Chapter 1 to Chapter 7 is structured in layers that repeat the main concepts (arbitrage, pricing, hedging, risk-neutral measures) in different time scale settings (one-step, discrete-time, continuous-time).

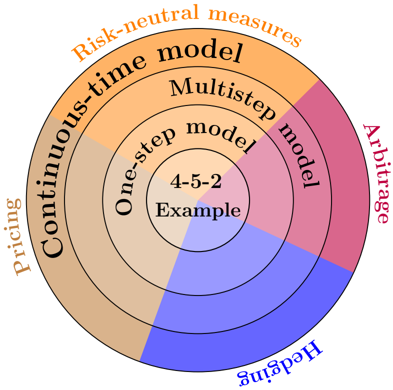


Fig. 18: Course plan.