

Nicolas Privault

Notes on
Stochastic Finance

This version: May 3, 2024

<https://personal.ntu.edu.sg/nprivault/index.html>

Preface

This book is an introduction to a wide range of topics in financial mathematics, including Black-Scholes pricing, exotic and american options, term structure modeling and change of numéraire, stochastic volatility, as well as models with jumps. It presents the mathematics of pricing and hedging in discrete and continuous-time financial models, with an emphasis on the complementarity between analytical and probabilistic methods. The contents are mostly mathematical, and also aim at making the reader aware of both the power and limitations of mathematical models in finance, by taking into account their conditions of applicability. The text is targeted at the advanced undergraduate and graduate levels in applied mathematics, financial engineering, and economics.

The point of view adopted is that of mainstream mathematical finance, in which the computation of fair prices is based on the absence of arbitrage hypothesis, therefore excluding riskless profit based on arbitrage opportunities and basic (buying low/selling high) trading. Similarly, this document is not concerned with any “prediction” of stock price behaviors that belong to other domains such as technical analysis, which should not be confused with the statistical modeling of asset prices.

The descriptions of the asset model, self-financing portfolios, arbitrage, and market completeness, are first given in Chapter 1 in a simple two time-step setting. These notions are then reformulated in discrete time in Chapter 2. Here, the impossibility to access future information is formulated using the notion of adapted processes, which will play a central role in the construction of stochastic calculus in continuous time.

In order to trade efficiently, it would be useful to have a formula to estimate the “fair price” of a given risky asset, helping for example to determine whether the asset is undervalued or overvalued at a given time. Although such a formula is not available, we can instead derive formulas for the pricing of options that can act as insurance contracts to protect their holders against adverse changes in the prices of risky assets. The pricing and hedging of options in discrete time, particularly in the fundamental example of the Cox-Ross-Rubinstein model, are considered in Chapter 3, with a description

of the passage from discrete to continuous time that prepares the transition to the subsequent chapters.

A simplified presentation of Brownian motion, stochastic integrals, and the associated Itô formula, is given in Chapter 4, with application to stochastic asset price modeling in Chapter 5. The Black-Scholes model is presented from the angle of partial differential equation (PDE) methods in Chapter 6, with the derivation of the Black-Scholes formula by transforming the Black-Scholes PDE into the standard heat equation, which is then solved by a heat kernel argument. The martingale approach to pricing and hedging is then presented in Chapter 7, and complements the PDE approach of Chapter 6 by recovering the Black-Scholes formula via a probabilistic argument. An introduction to stochastic volatility is given in Chapter 8, followed by a presentation of volatility estimation tools including historical, local, and implied volatilities, in Chapter 9. This chapter also contains a comparison of the prices obtained by the Black-Scholes formula with actual option price market data.


Exotic options such as barrier, lookback, and Asian options are treated in Chapters 11, 12, and 13, respectively, following an introduction to the properties of the maximum of Brownian motion given in Chapter 10. Optimal stopping and exercise, with application to the pricing of American options, are considered in Chapter 15, following the presentation of background material on filtrations and stopping times in Chapter 14. The construction of forward measures by change of numéraire is given in Chapter 16 and is applied to the pricing of interest rate derivatives such as caplets, caps, and swaptions in Chapter 19, after an introduction to bond pricing and to the modeling of forward rates in Chapters 17, and 18.

Stochastic calculus with jumps is dealt with in Chapter 20 and is restricted to compound Poisson processes, which only have a finite number of jumps on any bounded interval. Those processes are used for option pricing and hedging in jump models in Chapter 21, in which we mostly focus on risk-minimizing strategies as markets with jumps are generally incomplete. Chapter 22 contains an elementary introduction to finite difference methods for the numerical solution of PDEs and stochastic differential equations, dealing with the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milstein schemes for SDEs. The text is completed with an appendix containing the needed probabilistic background.

The material in this book has been used for teaching in the Masters of Science in Financial Engineering at City University of Hong Kong and at the Nanyang Technological University in Singapore. The author thanks Nicky van Foreest, Jinlong Guo, Kazuhiro Kojima, Sijian Lin, Panwar Samay, Sandu Ursu, and Ju-Yi Yen for corrections and improvements.

This text contains 277 exercises and 18 problems with complete solutions. Clicking on an exercise number inside the solution section will send to the

original problem text inside the file. Conversely, clicking on a problem number sends the reader to the corresponding solution, however this feature should not be misused. The cover graph represents the time evolution of the HSBC stock price from January to September 2009, plotted on the price surface of a European *put option* on that asset, expiring on October 05, 2009, see § 6.1.

This pdf file contains internal and external links, 29 tables and 381 figures, including 57 animated Figures 3.8, 3.10, 4.6, 4.7, 4.10, 4.11, 4.16, 5.5, 6.5, 10.1, 10.2, 10.3, 10.6, 11.11, 13.1, 12.1, 12.6, 12.14, 15.2, 17.16, 18.7, 18.10, 18.11, 18.18, 20.14, 20.16, 20.17, and S.18, 2 embedded videos in Figures 2 and 9.3, and 3 interacting 3D graphs in Figures 6.4, 6.11 and 11.1, that may require using Acrobat Reader for viewing on the complete pdf file. It also includes 30 Python codes e.g. on pages 75, 96, 100, 103, 145, 157, 236, 266, 363, 553 and 908, and 85  codes on pages 155, 157, 159, 163, 217, 213, 237, 239, 253, 245, 263, 266, 279, 363, 364, 379, 342, 421, 430, 652, 707, 731, 735, 749, 751, 825 and 828.

Nicolas Privault
May 2024



Contents

Preface	v
Introduction	1
1 Assets, Portfolios, and Arbitrage	21
1.1 Portfolio Allocation and Short Selling	21
1.2 Arbitrage	23
1.3 Risk-Neutral Probability Measures	28
1.4 Hedging of Contingent Claims	33
1.5 Market Completeness	36
1.6 Example: Binary Market	36
Exercises	44
2 Discrete-Time Market Model	53
2.1 Discrete-Time Compounding	53
2.2 Arbitrage and Self-Financing Portfolios	56
2.3 Contingent Claims	62
2.4 Martingales and Conditional Expectations	66
2.5 Market Completeness and Risk-Neutral Measures	72
2.6 The Cox-Ross-Rubinstein (CRR) Market Model	74
Exercises	78
3 Pricing and Hedging in Discrete Time	89
3.1 Pricing Contingent Claims	89
3.2 Pricing Vanilla Options in the CRR Model	95
3.3 Hedging Contingent Claims	101
3.4 Hedging Vanilla Options	103
3.5 Hedging Exotic Options	111
3.6 Convergence of the CRR Model	119
Exercises	126

4	Brownian Motion and Stochastic Calculus	149
	4.1 Brownian Motion	149
	4.2 Three Constructions of Brownian Motion	153
	4.3 Wiener Stochastic Integral	158
	4.4 Itô Stochastic Integral	168
	4.5 Stochastic Calculus	176
	Exercises	190
5	Continuous-Time Market Model	203
	5.1 Asset Price Modeling	203
	5.2 Arbitrage and Risk-Neutral Measures	205
	5.3 Self-Financing Portfolio Strategies	207
	5.4 Two-Asset Portfolio Model	210
	5.5 Geometric Brownian Motion	216
	Exercises	220
6	Black-Scholes Pricing and Hedging	229
	6.1 The Black-Scholes PDE	229
	6.2 European Call Options	235
	6.3 European Put Options	243
	6.4 Market Terms and Data	248
	6.5 The Heat Equation	253
	6.6 Solution of the Black-Scholes PDE	258
	Exercises	261
7	Martingale Approach to Pricing and Hedging	271
	7.1 Martingale Property of the Itô Integral	271
	7.2 Risk-Neutral Probability Measures	276
	7.3 Change of Measure and the Girsanov Theorem	280
	7.4 Pricing by the Martingale Method	283
	7.5 Hedging by the Martingale Method	290
	Exercises	297
8	Stochastic Volatility	325
	8.1 Stochastic Volatility Models	325
	8.2 Realized Variance Swaps	329
	8.3 Realized Variance Options	334
	8.4 European Options - PDE Method	343
	8.5 Perturbation Analysis	349
	Exercises	354
9	Volatility Estimation	359
	9.1 Historical Volatility	359
	9.2 Implied Volatility	362
	9.3 Local Volatility	369
	9.4 The VIX [®] Index	376

Exercises	381
10 Maximum of Brownian Motion	387
10.1 Running Maximum of Brownian Motion	387
10.2 The Reflection Principle	390
10.3 Maximum of Drifted Brownian Motion	394
10.4 Average of Geometric Brownian Extrema	405
Exercises	413
11 Barrier Options	417
11.1 Options on Extrema	417
11.2 Knock-Out Barrier	422
11.3 Knock-In Barrier	434
11.4 PDE Method	437
11.5 Hedging Barrier Options	441
Exercises	442
12 Lookback Options	449
12.1 The Lookback Put Option	449
12.2 PDE Method	452
12.3 The Lookback Call Option	458
12.4 Delta Hedging for Lookback Options	465
Exercises	471
13 Asian Options	473
13.1 Bounds on Asian Option Prices	473
13.2 Hartman-Watson Distribution	480
13.3 Laplace Transform Method	483
13.4 Moment Matching	484
13.5 PDE Method	490
Exercises	501
14 Optimal Stopping Theorem	507
14.1 Filtrations and Information Flow	507
14.2 Submartingales and Supermartingales	508
14.3 Optimal Stopping Theorem	512
14.4 Drifted Brownian Motion	519
Exercises	525
15 American Options	529
15.1 Perpetual American Put Options	529
15.2 PDE Method for Perpetual Put Options	535
15.3 Perpetual American Call Options	539
15.4 Finite Expiration American Options	544
15.5 PDE Method with Finite Expiration	548
Exercises	552

16	Change of Numéraire and Forward Measures	565
	16.1 Notion of Numéraire	565
	16.2 Change of Numéraire	568
	16.3 Foreign Exchange	578
	16.4 Pricing Exchange Options	587
	16.5 Hedging by Change of Numéraire	589
	Exercises	593
17	Short Rates and Bond Pricing	601
	17.1 Short-Term Mean-Reverting Models	601
	17.2 Calibration of the Vasicek Model	608
	17.3 Zero-Coupon and Coupon Bonds	613
	17.4 Bond Pricing PDE	618
	Exercises	633
18	Forward Rates	645
	18.1 Construction of Forward Rates	645
	18.2 LIBOR and SOFR Swap Rates	657
	18.3 The HJM Model	662
	18.4 Yield Curve Modeling	667
	18.5 Two-Factor Model	672
	18.6 The BGM Model	676
	Exercises	677
19	Pricing of Interest Rate Derivatives	685
	19.1 Forward Measures and Tenor Structure	685
	19.2 Bond Options	689
	19.3 Caplet Pricing	691
	19.4 Forward Swap Measures	698
	19.5 Swaption Pricing	699
	Exercises	709
20	Stochastic Calculus for Jump Processes	725
	20.1 The Poisson Process	725
	20.2 Compound Poisson Process	734
	20.3 Stochastic Integrals and Itô Formula with Jumps	739
	20.4 Stochastic Differential Equations with Jumps	752
	20.5 Girsanov Theorem for Jump Processes	757
	Exercises	765
21	Pricing and Hedging in Jump Models	771
	21.1 Fitting the Distribution of Market Returns	771
	21.2 Risk-Neutral Probability Measures	780
	21.3 Pricing in Jump Models	781
	21.4 Exponential Lévy Models	783
	21.5 Black-Scholes PDE with Jumps	787

21.6 Mean-Variance Hedging with Jumps	789
Exercises	793
22 Basic Numerical Methods	797
22.1 Euler Discretization	797
22.2 Milstein Discretization	798
22.3 Discretized Heat Equation	799
22.4 Discretized Black-Scholes PDE	802
Exercises	806
Appendix: Background on Probability Theory	807
A.1 Probability Sample Space and Events	807
A.2 Probability Measures	811
A.3 Conditional Probabilities and Independence	812
A.4 Random Variables	815
A.5 Probability Distributions	816
A.6 Expectation of Random Variables	824
A.7 Conditional Expectation	836
Exercises	841
Exercise Solutions	845
Chapter 1	845
Chapter 2	855
Chapter 3	868
Chapter 4	909
Chapter 5	940
Chapter 6	957
Chapter 7	973
Chapter 8	1027
Chapter 9	1033
Chapter 10	1047
Chapter 11	1064
Chapter 12	1087
Chapter 13	1097
Chapter 14	1110
Chapter 15	1119
Chapter 16	1147
Chapter 17	1159
Chapter 18	1180
Chapter 19	1193
Chapter 20	1217
Chapter 21	1230
Chapter 22	1236
Background on Probability Theory	1237
References	1241

Index	1257
Author index	1269



List of Figures

1	Two sample paths of one-dimensional Brownian motion	2
2	“As if a whole new world was laid out before me.”*	3
3	Comparison of WTI <i>vs.</i> Keppel price graphs	5
4	Hang Seng index	6
5	Two put option scenarios	7
6	Payoff function of a put option	7
7	Two call option scenarios	9
8	Payoff function of a call option	10
9	“Infogrames” stock price curve	11
10	Brent and WTI price graphs	11
11	Price map of a four-way collar option	12
12	Payoff function of a four-way call collar option	12
13	Four-way call collar payoff as a combination of call and put options*	13
14	Implied probabilities	17
15	Implied probabilities according to bookmakers	18
16	Implied probabilities according to polling	18
17	Fifty sample price paths used for the Monte Carlo method	19
18	Course plan	20
1.1	Triangular arbitrage	24
1.2	Arbitrage: Xbox Retail prices	26
1.3	Separation of convex sets	33
2.1	Illustration of the self-financing condition (2.7)	59
2.2	Why apply discounting?	61
2.3	Oil price graph	61
2.4	Take the quiz	66
2.5	Discrete-time asset price tree in the CRR model	76
2.6	Discrete-time asset price graphs in the CRR model	77
2.7	Function $x \mapsto ((1+x)^{21} - (1+x)^{10})/x$	80
2.8	Graph of the function $r \mapsto (1 - (1+r)^{-12})/r$	82

2.9	Transition probabilities in the recovery theorem	85
3.1	Graph of $120 = \binom{10}{7}$ paths with $n = 5$ and $k = 2^*$	97
3.2	Discrete-time call option pricing tree	101
3.3	Discrete-time call option hedging strategy (risky component)	105
3.4	Discrete-time call option hedging strategy (riskless component)	105
3.5	Tree of asset prices in the CRR model	110
3.6	Tree of option prices in the CRR model	111
3.7	Tree of hedging portfolio allocations in the CRR model	111
3.8	Galton board simulation*	121
3.9	A real-life Galton board	121
3.10	Multiplicative Galton board simulation*	122
3.11	Put spread collar price map	130
3.12	Call spread collar price map	131
3.13	Tree of market prices with $N = 2$	135
3.14	Trees of bid and ask prices with $N = 2$	136
3.15	BTC/USD order book example	138
3.16	Dividend detachment on Z74.SI	139
4.1	Sample paths of a one-dimensional Brownian motion	150
4.2	Evolution of the fortune of a poker player <i>vs.</i> number of games played	151
4.3	Web traffic ranking	151
4.4	Two sample paths of a two-dimensional Brownian motion	152
4.5	Sample path of a three-dimensional Brownian motion	153
4.6	Scaling property of Brownian motion*	153
4.7	Brownian motion as a random walk*	154
4.8	Statistics of one-dimensional Brownian paths <i>vs.</i> Gaussian distribution	156
4.9	Statistics of S&P 500 yearly return graphs from 1950 to 2022	157
4.10	Lévy's construction of Brownian motion*	157
4.11	Construction of Brownian motion by series expansions*	158
4.12	Step function	159
4.13	Area under the step function	160
4.14	Infinite <i>vs.</i> finite area under a curve	161
4.15	Squared step function	162
4.16	Step function approximation*	163
4.17	Adapted pair trading portfolio strategy	169
4.18	Step function approximation of Brownian motion*	171
4.19	Squared simple predictable process	173
4.20	NGram Viewer output for the term "stochastic calculus"	176
4.21	Wrong application of Itô's formula (sample)	185
4.22	Simulated path of (4.33) with $\alpha = 10$, $\sigma = 0.2$ and $X_0 = 0.5$	187
4.23	Simulated path of (4.37)	189
4.24	Simulated path of (4.38) with $\alpha = -5$ and $\sigma = 1$	190
5.1	Why apply discounting?	204

5.2	Illustration of the self-financing condition (5.4)	208
5.3	Illustration of the self-financing condition (5.10)	211
5.4	Ten sample paths of geometric Brownian motion	214
5.5	Geometric Brownian motion started at $S_0 = 1^*$	217
5.6	Statistics of geometric Brownian paths <i>vs.</i> lognormal distribution	221
6.1	Underlying market prices	230
6.2	Simulated geometric Brownian motion	231
6.3	Graph of the Gaussian Cumulative Distribution Function (CDF)	236
6.4	Black-Scholes call price map*	237
6.5	Time-dependent solution of the Black-Scholes PDE (call option)*	238
6.6	Delta of a European call option	240
6.7	Gamma of European call and put options	241
6.8	HSBC Holdings stock price	241
6.9	Path of the Black-Scholes price for a call option on HSBC	242
6.10	Time evolution of a hedging portfolio for a call option on HSBC	243
6.11	Black-Scholes put price function*	244
6.12	Time-dependent solution of the Black-Scholes PDE (put option)*	245
6.13	Delta of a European put option	247
6.14	Path of the Black-Scholes price for a put option on HSBC	247
6.15	Time evolution of the hedging portfolio for a put option on HSBC	248
6.16	Time-dependent solutions of the Black-Scholes PDE with $r > 0^*$	249
6.17	Time-dependent solutions of the Black-Scholes PDE with $r < 0^*$	250
6.18	Time-dependent solutions of the Black-Scholes PDE with $r = 0^*$	250
6.19	Warrant terms and data	253
6.20	Time-dependent solution of the heat equation*	255
6.21	Time-dependent solution of the heat equation*	257
6.22	Short rate $t \mapsto r_t$ in the CIR model	262
6.23	Option price as a function of the volatility σ	265
7.1	Discretized drifted Brownian path	277
7.2	Drifted Brownian paths under a shifted Girsanov measure	279
7.3	Payoff functions of bull spread and bear spread options	298
7.4	Graphs of call/put payoff functions	299
7.5	Long call butterfly payoff function	301
7.6	Option price as a function of underlying asset price and time to maturity	319
7.7	Delta as a function of underlying asset price and time to maturity	320
7.8	Gamma as a function of underlying asset price and time to maturity	320
7.9	Option price as a function of underlying asset price and time to maturity	321
7.10	Delta as a function of underlying asset price and time to maturity	322
7.11	Gamma as a function of underlying asset price and time to maturity	322
8.1	Euro / SGD exchange rate	326
8.2	Fitting of a lognormal probability density function	335
8.3	Fitting of a gamma probability density function	340

8.4	Variance call option prices with $b = 0.15$	342
8.5	Variance call option prices with $b = -0.05$	342
8.6	Option price approximations plotted against v with $\rho = -0.5$	354
9.1	Underlying asset price <i>vs.</i> log-returns	361
9.2	Historical volatility graph	361
9.3	The <i>fugazi</i> : it's a wazy, it's a woozie. It's fairy dust*	362
9.4	Option price as a function of the volatility σ	363
9.5	Implied volatility of Asian options on light sweet crude oil futures	365
9.6	S&P500 option prices plotted against strike prices	366
9.7	Market stock price of Cheung Kong Holdings	367
9.8	Comparison of market call option prices <i>vs.</i> calibrated Black-Scholes prices	367
9.9	Market stock price of HSBC Holdings	368
9.10	Comparison of market call option prices <i>vs.</i> calibrated Black-Scholes prices	368
9.11	Comparison of market put option prices <i>vs.</i> calibrated Black-Scholes prices	369
9.12	Call option price <i>vs.</i> underlying asset price	369
9.13	Simulated path of (9.7) with $r = 0.5$ and $\sigma = 1.2$	370
9.14	Local volatility estimated from Boeing Co. option price data	375
9.15	VIX [®] Index <i>vs.</i> S&P 500	380
9.16	VIX [®] Index <i>vs.</i> historical volatility for the year 2011	380
9.17	Correlation estimates between GSPC and the VIX [®]	381
9.18	VIX [®] Index <i>vs.</i> 30 day historical volatility for the S&P 500	381
10.1	Brownian motion $(W_t)_{t \in \mathbb{R}_+}$ and its running maximum $(X_0^t)_{t \in \mathbb{R}_+}$ *	388
10.2	Running maximum of Brownian motion*	388
10.3	Zeroes of Brownian motion*	389
10.4	Graph of the Cantor function*	389
10.5	A function with no last point of increase before $t = 1$	390
10.6	Reflected Brownian motion with $a = 1.07$ *	391
10.7	Probability density of the maximum of Brownian motion	392
10.8	Probability density of the maximum of geometric Brownian motion	394
10.9	Probability computed as a volume integral	395
10.10	Reflected Brownian motion with $a = 1.07$ *	396
10.11	Joint probability density of Brownian motion and its maximum	397
10.12	Heat map of the joint density of W_1 and its maximum	398
10.13	Probability density of the maximum of drifted Brownian motion	401
11.1	Up-and-out barrier call option price with $B > K^*$	424
11.2	Up-and-out barrier put option price	429
11.3	Pricing data for an up-and-out barrier put option with $K = B = \$28$	430
11.4	Down-and-out barrier call option price	431
11.5	Down-and-out barrier call option price as a function of volatility	432

11.6	Down-and-out barrier put option price with $K > B$	433
11.7	Down-and-in barrier call option price with $K > B$	434
11.8	Up-and-in barrier call option price with $K > B$	435
11.9	Down-and-in barrier put option price with $K > B$	436
11.10	Up-and-in barrier put option price	437
11.11	Delta of the up-and-out barrier call option*	442
12.1	Lookback put option price (3D)*	450
12.2	Graph of lookback put option prices	451
12.3	Graph of lookback put option prices (2D)	451
12.4	Normalized lookback put option price	456
12.5	Normalized Black-Scholes put price and correction term	458
12.6	Lookback call option price*	459
12.7	Graph of lookback call option prices	459
12.8	Graphs of lookback call option prices (2D)	460
12.9	Normalized lookback call option price	462
12.10	Underlying asset prices	462
12.11	Running minimum of the underlying asset price	463
12.12	Lookback call option price	463
12.13	Normalized Black-Scholes call price and correction term	465
12.14	Delta of the lookback call option*	467
12.15	Rescaled portfolio strategy for the lookback call option	468
12.16	Delta of the lookback put option*	470
13.1	Brownian motion and its moving average*	474
13.2	Asian option price <i>vs.</i> European option price*	478
13.3	Asian call option prices	484
13.4	Lognormal approximation of probability density	486
13.5	Lognormal approximation to the Asian call option price	487
13.6	Dividend detachment graph on Z74.SI	506
14.1	Drifted Brownian path	509
14.2	Evolution of the fortune of a poker player <i>vs.</i> number of games played	510
14.3	Convex function example	510
14.4	Stopped process	514
14.5	Sample paths of a gambling process $(M_n)_{n \in \mathbb{N}}$	518
14.6	Brownian motion hitting a barrier	519
14.7	Drifted Brownian motion hitting a barrier	520
14.8	Hitting probabilities of drifted Brownian motion*	522
15.1	American put prices by exercising at τ_L for different values of L	533
15.2	Animated graph of American put prices $x \mapsto f_L(x)$ *	534
15.3	Option price as a function of L and of the underlying asset price	534
15.4	Path of the American put option price on the HSBC stock	535
15.5	American call prices by exercising at τ_L for different values of L	542
15.6	Animated graph of American call prices $x \mapsto f_L(x)$ *	542

15.7	American call prices for different values of L	543
15.8	Black-Scholes call option price <i>vs.</i> $(x, t) \mapsto (x - K)^+$	545
15.9	Black-Scholes put option price <i>vs.</i> $(x, t) \mapsto (K - x)^+$	546
15.10	Optimal frontier for the exercise of a put option	547
15.11	PDE estimates of finite expiration American put option prices	549
15.12	Longstaff-Schwartz estimates of finite expiration American put prices	550
15.13	Comparison between Longstaff-Schwartz and finite differences	550
15.14	Butterfly payoff function	554
16.1	Why change of numéraire?	567
16.2	Overseas investment opportunity	578
16.3	Evolution of exchange rate <i>vs.</i> interest rate	581
17.1	Short rate $t \mapsto r_t$ in the Vasicek model	603
17.2	CBOE 10 Year Treasury Note (TNX) yield	605
17.3	Short rate $t \mapsto r_t$ in the CIR model	606
17.4	Calibrated Vasicek simulation <i>vs.</i> market data	613
17.5	Five-dollar 1875 Louisiana bond with 7.5% biannual coupons	613
17.6	Discrete-time coupon bond pricing	616
17.7	Continuous-time coupon bond pricing	617
17.8	Comparison of Monte Carlo and PDE solutions	624
17.9	Bond price $t \mapsto P(t, T)$ <i>vs.</i> $t \mapsto e^{-r_0(T-t)}$	625
17.10	Bond price $t \mapsto P_c(t, T)$ with a 5% coupon rate	625
17.11	Bond price with coupon rate 6.25%	626
17.12	Orange Cnty Calif bond prices	627
17.13	Orange Cnty Calif bond yields	627
17.14	Fitting of a gamma probability density function	631
17.15	Approximation of Dothan bond prices	631
17.16	Brownian bridge	633
18.1	Graph of the spot forward rate $S \mapsto f(t, t, S)$	646
18.2	Indonesian government securities yield curve	646
18.3	Example of yield curve	647
18.4	Forward rate process $t \mapsto f(t, t, T)$	651
18.5	Instantaneous forward rate process $t \mapsto f(t, T)$	652
18.6	Federal Reserve yield curves from 1982 to 2012	653
18.7	European Central Bank yield curves*	653
18.8	August 2019 Federal Reserve yield curve inversion*	654
18.9	Stochastic process of forward curves	662
18.10	Forward instantaneous curve in the Vasicek model*	666
18.11	Forward instantaneous curve $x \mapsto f(0, x)$ in the Vasicek model*	666
18.12	Short-term interest rate curve $t \mapsto r_t$ in the Vasicek model	667
18.13	Nelson-Siegel graph	668
18.14	Svensson graph	668
18.15	Fitting of a Svensson curve to market data	669

18.16	Graphs of forward rates	670
18.17	Forward instantaneous curve in the Vasicek model	670
18.18	ECB data <i>vs.</i> fitted yield curve*	672
18.19	Bond prices $t \mapsto P(t, T_2), P(t, T_2), P(t, T_3)$	672
18.20	Forward rates in a two-factor model	675
18.21	Evolution of instantaneous forward rates in a two-factor model	675
18.22	Roadmap of stochastic interest rate modeling	677
19.1	Implied swaption volatilities	705
20.1	Sample path of a counting process $(N_t)_{t \in \mathbb{R}_+}$	726
20.2	Sample path of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$	729
20.3	Sample path of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$	732
20.4	Sample path of the compensated Poisson process $(N_t - \lambda t)_{t \in \mathbb{R}_+}$	733
20.5	Probability density function	734
20.6	Sample path of a compound Poisson process $(Y_t)_{t \in \mathbb{R}_+}$	735
20.7	Sample path of a compensated compound Poisson process	739
20.8	Sample trajectories of a gamma process	749
20.9	Sample trajectories of a variance gamma process	750
20.10	Sample trajectories of an inverse Gaussian process	750
20.11	Sample trajectories of a negative inverse Gaussian process	750
20.12	Sample trajectories of a stable process	751
20.13	USD/CNY Exchange rate data	751
20.14	Geometric Poisson process*	754
20.15	Ranking data	755
20.16	Geometric compound Poisson process*	755
20.17	Geometric Brownian motion with compound Poisson jumps*	756
20.18	Share price with jumps	757
21.1	Market returns <i>vs.</i> normalized Gaussian returns	772
21.2	Empirical <i>vs.</i> Gaussian CDF	773
21.3	Quantile-Quantile plot	773
21.4	Empirical density <i>vs.</i> normalized Gaussian density	774
21.5	Empirical density <i>vs.</i> normalized lognormal density	775
21.6	Empirical density <i>vs.</i> power density	775
21.7	Gram-Charlier expansions	780
22.1	Divergence of the explicit finite difference method	804
22.2	Stability of the implicit finite difference method	805
A.1	Probability density function	817
A.2	Exponential CDF and PDF	819
A.3	Probability computed as a volume integral	820
S.1	Strike price as a function of risk-free rate	855
S.2	Investment graph (1)	856

S.3	Investment graph (2)	857
S.4	Histogram of replies to Question <i>c</i>).	860
S.5	Phone screenshots	861
S.6	Range forward contract payoff as a combination of call and put option payoffs*	876
S.7	Put spread collar price map	882
S.8	Put spread collar payoff function	882
S.9	Put spread collar payoff as a combination of call and put payoffs*	883
S.10	Call spread collar price map	883
S.11	Call spread collar payoff function	884
S.12	Call spread collar payoff as a combination of call and put payoffs*	884
S.13	Tree of market prices with $N = 2$	894
S.14	Put option prices in the trinomial model	908
S.15	Function $x \mapsto f_\varepsilon(x)$	927
S.16	Derivative $x \mapsto f'_\varepsilon(x)$	928
S.17	Samples of linear interpolations	931
S.18	Brownian crossings of level 1*	950
S.19	Brownian path	952
S.20	Risk-neutral pricing of a foreign exchange option	952
S.21	Delta hedging of a foreign exchange option	953
S.22	Bitcoin XBT/USD order book	953
S.23	Time spent by Brownian motion within a given range	954
S.24	Market data for the warrant #01897 on the MTR Corporation	964
S.25	Lower bound <i>vs.</i> Black-Scholes call price	977
S.26	Lower bound <i>vs.</i> Black-Scholes put option price	977
S.27	Bull spread option as a combination of call and put options*	978
S.28	Bear spread option as a combination of call and put options*	979
S.29	Butterfly option as a combination of call options*	982
S.30	Delta of a butterfly option	983
S.31	Gaussian approximation of spread probability density function	994
S.32	Gaussian approximation of spread option prices	995
S.33	Price of a binary call option	998
S.34	Risky hedging portfolio value for a binary call option	999
S.35	Risk-free hedging portfolio value for a binary call option	999
S.36	Black-Scholes price of the maximum chooser option	1002
S.37	Delta of the maximum chooser option	1002
S.38	Black-Scholes price of the minimum chooser option	1003
S.39	Delta of the minimum chooser option	1004
S.40	Sample path of $dS_t = S_t dB_t / \sqrt{1-t}$	1016
S.41	Sample path of $dS_t = S_t^2 dB_t$	1017
S.42	Sample path of $dS_t = S_t^2 dB_t$	1017
S.43	“Infogrames” stock price curve	1020
S.44	Butterfly option payoff as a combination of call and put options*	1036
S.45	Implied <i>vs.</i> local volatility	1041
S.46	Average return by selling at the maximum <i>vs.</i> selling at maturity	1053

S.47	Ratios of average returns	1055
S.48	Cumulative distribution function of the time of maximum of Brownian motion	1056
S.49	Black-Scholes call price upper bound.	1058
S.50	Black-Scholes put price upper bound.	1060
S.51	“Optimal exercise” put price upper bound.	1062
S.52	Down-and-out barrier call option price with rebate as a function of volatility	1068
S.53	Price of the up-and-in long forward contract	1069
S.54	Delta of the up-and-in long forward contract	1070
S.55	Price of the up-and-out long forward contract	1071
S.56	Delta of up-and-out long forward contract price	1072
S.57	Price of the down-and-in long forward contract	1073
S.58	Delta of down-and-in long forward contract	1073
S.59	Price of the down-and-out long forward contract	1074
S.60	Delta of down-and-out long forward contract	1075
S.61	Payoff function of the European knock-out call option	1081
S.62	Price map of the European knock-out call option	1082
S.63	Payoff function of the European knock-in put option	1083
S.64	Price map of the European knock-in put option	1083
S.65	Payoff function of the European knock-in call option	1084
S.66	Price map of the European knock-in call option	1085
S.67	Payoff function of the European knock-out put option	1085
S.68	Price map of the European knock-out put option	1086
S.69	Expected minimum of geometric Brownian motion	1088
S.70	Black-Scholes put price upper bound.	1089
S.71	Time derivative of the expected minimum	1089
S.72	Expected maximum of geometric Brownian motion	1091
S.73	Black-Scholes call price upper bound.	1092
S.74	Time derivative of the expected maximum	1092
S.75	Lookback call option price as a function of maturity time T	1095
S.76	Lookback put option price (2D) as a function of M_0^t	1096
S.77	Hitting times of a straight line started at $\alpha < 0$	1116
S.78	Hitting times of a straight line started at $\alpha < 0$	1117
S.80	Sample path of the random walk $(S_n)_{n \in \mathbb{N}}$	1118
S.81	American butterfly payoff and price functions	1121
S.82	Perpetual <i>vs.</i> finite expiration American put option price	1122
S.83	American put price approximation.	1123
S.84	Perpetual American binary put price map	1136
S.85	Perpetual American binary call price map	1136
S.86	Finite expiration American binary call price map	1139
S.87	Finite expiration American binary put price map	1140
S.88	Log bond prices correlation graph in the two-factor model	1191

* Animated figures (work with Acrobat Reader).

List of Tables

1.1	Mark Six “Investment Table”	26
2.1	Self-financing portfolio value process	60
2.2	NTRC Input investment plan	79
2.3	Avenda Insurance investment plan	79
4.1	Itô multiplication table	182
6.1	Black-Scholes Greeks	248
6.2	Variations of Black-Scholes prices	249
7.1	Call and put options on the Hang Seng Index (HSI)	300
7.2	Contract summary	300
11.1	Barrier option types	421
11.2	Boundary conditions for barrier option prices	441
12.1	Extended Itô multiplication table	453
14.1	Martingales and stopping times	518
14.2	List of martingales	524
15.1	Optimal exercise strategies	552
16.1	Local <i>vs.</i> foreign exchange options	586
18.1	Stochastic interest rate models	676
19.1	Forward rates arranged according to a tenor structure	685
19.2	A list of numéraire processes and their applications	708
20.1	Itô multiplication table with jumps	747

21.1	Market models and their properties	793
S.1	Fixed deposit returns	861
S.2	CRR pricing and hedging table	870
S.3	CRR pricing tree	872
S.4	CRR pricing and hedging tree	873
S.5	CRR pricing tree	878
S.6	CRR pricing and hedging tree	878
S.7	CRR pricing tree	880
S.8	CRR pricing and hedging tree	880
S.9	Call and put options on the Hang Seng Index (HSI)	980
S.10	Original call/put options on the Hang Seng Index (HSI)	981

Introduction

Modern quantitative finance requires a strong background in fields such as stochastic calculus, optimization, partial differential equations (PDEs) and numerical methods, or even infinite dimensional analysis. In addition, the emergence of new complex financial instruments on the markets makes it necessary to rely on increasingly sophisticated mathematical tools. Not all readers of this book will eventually work in quantitative financial analysis, nevertheless they may have to interact with quantitative analysts, and becoming familiar with the tools they employ could be an advantage. In addition, despite the availability of ready made financial calculators it still makes sense to be able oneself to understand, design and implement such financial algorithms. This can be particularly useful under different types of conditions, including an eventual lack of trust in financial indicators, possible unreliability of expert advice such as buy/sell recommendations, or other factors such as market manipulation. Instead of relying on predictions of stock price movements based on various tools (*e.g.* technical analysis, charting, “cup & handle” figures), we acknowledge that predicting the future is a difficult task and we rely on the [Efficient Market Hypothesis](#). In this framework, the time evolution of the prices of risky assets will be modeled by random walks and stochastic processes.

Historical sketch

We start with a description of some of the main steps, ideas and individuals that played an important role in the development of the field over the last century.

Robert Brown, botanist, 1828

[Brown \(1828\)](#) observed the movement of pollen particles as described in “A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on

the general existence of active molecules in organic and inorganic bodies.”
Phil. Mag. 4, 161-173, 1828.

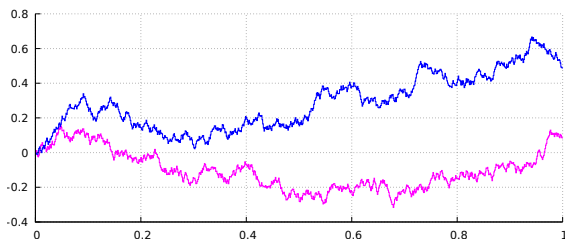


Fig. 1: Two sample paths of one-dimensional Brownian motion.

Philosophical Magazine, first published in 1798, is a journal that “publishes articles in the field of condensed matter describing original results, theories and concepts relating to the structure and properties of crystalline materials, ceramics, polymers, glasses, amorphous films, composites and soft matter.”

Albert Einstein, physicist

Einstein received his 1921 Nobel Prize in part for investigations on the theory of Brownian motion: “... in 1905 Einstein founded a kinetic theory to account for this movement”, presentation speech by S. Arrhenius, Chairman of the Nobel Committee, Dec. 10, 1922.

[Einstein \(1905\)](#) “Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen”, *Annalen der Physik* 17.

Louis Bachelier, mathematician, PhD 1900

[Bachelier \(1900\)](#) used Brownian motion for the modeling of stock prices in his PhD thesis “Théorie de la spéculation”, *Annales Scientifiques de l’Ecole Normale Supérieure* 3 (17): 21-86, 1900.

Norbert Wiener, mathematician, founder of cybernetics

Wiener is credited, among other fundamental contributions, for the mathematical foundation of Brownian motion, published in 1923. In particular he constructed the Wiener space and Wiener measure on $C_0([0, 1])$ (the space of continuous functions from $[0, 1]$ to \mathbb{R} vanishing at 0).

[Wiener \(1923\)](#) “Differential space”, *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, 2, 131-174, 1923.

Kiyoshi Itô (伊藤清), mathematician, C.F. Gauss Prize 2006

[Itô \(1944\)](#) constructed the Itô integral with respect to Brownian motion, and the stochastic calculus with respect to Brownian motion, which laid the

foundation for the development of calculus for random processes, see [Itô \(1951\)](#) “On stochastic differential equations”, in *Memoirs of the American Mathematical Society*.

“*Renowned math wiz Itô, 93, dies.*” (The Japan Times, Saturday, Nov. 15, 2008)

Kiyoshi Itô, an internationally renowned mathematician and professor emeritus at Kyoto University died Monday of respiratory failure at a Kyoto hospital, the university said Friday. He was 93. Itô was once dubbed “the most famous Japanese in Wall Street” thanks to his contribution to the founding of financial derivatives theory. He is known for his work on stochastic differential equations and the “Itô Formula”, which laid the foundation for the [Black and Scholes \(1973\)](#) model, a key tool for financial engineering. His theory is also widely used in fields like physics and biology.

Paul Samuelson, economist, Nobel Prize 1970

[Samuelson \(1965\)](#) rediscovered Bachelier’s ideas and proposed geometric Brownian motion as a model for stock prices. In an interview he stated “In the early 1950s I was able to locate by chance this unknown [Bachelier \(1900\)](#) book, rotting in the library of the University of Paris, and when I opened it up it was as if a whole new world was laid out before me.” We refer to “Rational theory of warrant pricing” by Paul Samuelson, *Industrial Management Review*, p. 13-32, 1965.

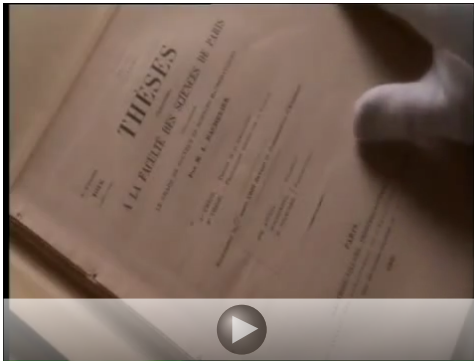


Fig. 2: [Clark \(2000\)](#) “As if a [whole new world](#) was laid out before me.”*

* Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

In recognition of Bachelier's contribution, the Bachelier Finance Society was started in 1996 and now holds the World Bachelier Finance Congress every two years.

Robert Merton, Myron Scholes, economists

Robert Merton and Myron Scholes shared the 1997 Nobel Prize in economics: "In collaboration with Fisher Black, developed a pioneering formula for the valuation of stock options ... paved the way for economic valuations in many areas ... generated new types of financial instruments and facilitated more efficient risk management in society."*

[Black and Scholes \(1973\)](#) "The Pricing of Options and Corporate Liabilities". *Journal of Political Economy* 81 (3): 637-654.

The development of options pricing tools contributed greatly to the expansion of option markets and led to development several ventures such as the "Long Term Capital Management" (LTCM), founded in 1994. The fund yielded annualized returns of over 40% in its first years, but registered a loss of US\$4.6 billion in less than four months in 1998, which resulted into its closure in early 2000.

Oldřich Vašíček, economist, 1977

Interest rates behave differently from stock prices, notably due to the phenomenon of mean reversion, and for this reason they are difficult to model using geometric Brownian motion. [Vašíček \(1977\)](#) was the first to suggest a mean-reverting model for stochastic interest rates, based on the Ornstein-Uhlenbeck process, in "An equilibrium characterization of the term structure", *Journal of Financial Economics* 5: 177-188.

David Heath, Robert Jarrow, Andrew Morton

These authors proposed in 1987 a general framework to model the evolution of (forward) interest rates, known as the Heath-Jarrow-Morton (HJM) model, see [Heath et al. \(1992\)](#) "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation", *Econometrica*, (January 1992), Vol. 60, No. 1, pp 77-105.

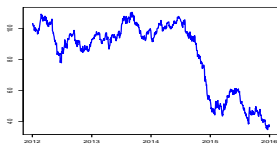
Alan Brace, Dariusz Gatarek, Marek Musiela (BGM)

The [Brace et al. \(1997\)](#) model is actually based on geometric Brownian motion, and it is especially useful for the pricing of interest rate derivatives such as interest rate caps and swaptions on the LIBOR market, see "The Market Model of Interest Rate Dynamics". *Mathematical Finance* Vol. 7, page 127. Blackwell 1997, by Alan Brace, Dariusz Gatarek, Marek Musiela. Although LIBOR rates are being phased out, we will still use this terminology when referring to simple or linear compounded forward rates.

* This has to be put in relation with the modern development of [risk societies](#); "societies increasingly preoccupied with the future (and also with safety), which generates the notion of risk" (Wikipedia).

Financial derivatives

The following graphs exhibit a correlation between commodity (oil) prices and an oil-related asset price.



(a) WTI price graph.



(b) Graph of Keppel Corp. stock price

Fig. 3: Comparison of WTI *vs.* Keppel price graphs.

The study of financial derivatives aims at finding functional relationships between the price of an underlying asset (a company stock price, a commodity price, etc.) and the price of a related financial contract (an option, a financial derivative, etc.).

Option contracts

Early accounts of option contracts can also be found in *The Politics* Aristotle (BCE) by Aristotle (384-322 BCE). Referring to the philosopher Thales of Miletus (c. 624 - c. 546 BCE), Aristotle writes:

“He (Thales) knew by his skill in the stars while it was yet winter that there would be a great harvest of olives in the coming year; so, having a little money, he gave *deposits* for the use of all the olive-presses in Chios and Miletus, which he hired at a low price because no one bid against him. When the harvest-time came, and many were wanted all at once and of a sudden, he let them out at any rate which he pleased, and made a quantity of money”.

In the above example, olive oil can be regarded as the underlying asset, while the oil press stands for the financial derivative. Option credit contracts appear to have been used as early as the 10th century by traders in the Mediterranean.

Next, we move to a description of (European) call and put options, which are at the basis of risk management.

European put option contracts

As previously mentioned, an important concern for the buyer of a stock at time t is whether its price S_T can decline at some future date T . The buyer of

the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t . This contract is called a put option with strike price K and exercise date T .

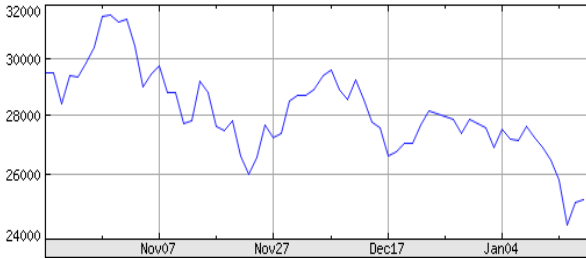


Fig. 4: Graph of the Hang Seng index - holding a put option might be useful here.

Definition 1. A (European) put option is a contract that gives its holder the right (but not the obligation) to sell a quantity of assets at a predefined price K called the strike price (or exercise price) and at a predefined date T called the maturity.

In case the price S_T falls down below the level K , exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option contract and have to sell the asset at the market price S_T . In turn, the issuer of the option contract will register a loss also equal to $K - S_T$ (in the absence of transaction costs and other fees).

If S_T is above K , then the holder of the option contract will not exercise the option as he may choose to sell at the price S_T . In this case the profit derived from the option contract is 0. Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 5.



Fig. 5: Two put option scenarios.

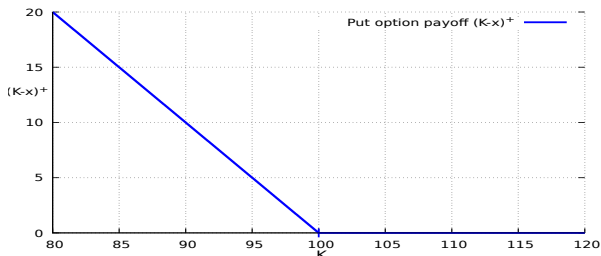
Cash settlement *vs.* physical delivery

Physical delivery. In the case of physical delivery, the put option contract issuer will pay the strike price $\$K$ to the option contract holder in exchange for one unit of the risky asset priced S_T .

Cash settlement. In the case of a cash settlement, the put option issuer will satisfy the contract by transferring the amount $C = (K - S_T)^+$ to the option contract holder.

In general, the payoff of a (so-called *European*) put option contract can be written as

$$\phi(S_T) = (K - S_T)^+ := \begin{cases} K - S_T & \text{if } S_T \leq K, \\ 0, & \text{if } S_T \geq K. \end{cases}$$

Fig. 6: Payoff function of a put option with strike price $K = 100$.

See e.g. <https://optioncreator.com/stwwxvz>.

Put option examples

- a) The **buy back guarantee*** in currency exchange;
 - b) the **price drop protection** in online ticket booking
- are common examples of European put options.

The derivatives market

As of year 2015, the size of the financial derivatives market is estimated at over \$1.2 quadrillion[†] USD, which is more than 10 times the Gross World Product (GWP). See [here](#) or [here](#) for up-to-date data on outstanding notional amounts and gross market value from the Bank for International Settlements (BIS).

European call option contracts

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t .

Definition 2. *A (European) call option is a contract that gives its holder the right (but not the obligation) to purchase a quantity of assets at a predefined price K called the strike price, and at a predefined date T called the maturity.*

Here, in the event that S_T goes above K , the buyer of the option contract will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 7.

* Right-click to open or save the attachment.

[†] One thousand trillion, or one million billion, or 10^{15} .



Fig. 7: Two call option scenarios.

Cash settlement *vs.* physical delivery

Physical delivery. In the case of physical delivery, the call option contract issuer will transfer one unit of the risky asset priced S_T to the option contract holder in exchange for the strike price $\$K$. Physical delivery may include physical goods, commodities or assets such as coffee, airline fuel or live cattle, see [Schroeder and Coffey \(2018\)](#).

Cash settlement. In the case of a cash settlement, the call option issuer will fulfill the contract by transferring the amount $C = (S_T - K)^+$ to the option contract holder.

In general, the payoff of a (so-called European) call option contract can be written as

$$\phi(S_T) = (S_T - K)^+ := \begin{cases} S_T - K & \text{if } S_T \geq K, \\ 0, & \text{if } S_T \leq K. \end{cases}$$

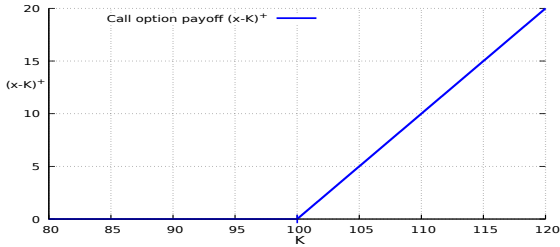


Fig. 8: Payoff function of a call option with strike price $K = 100$.

See e.g. <https://optioncreator.com/stqhbgn>.

Call option example: The **price lock guarantee*** in online ticket booking is a common example of a European *call* option.

According to market practice, options are often divided into a certain number n of *warrants*, the (possibly fractional) quantity n being called the *entitlement ratio*.

Option pricing

In order for an option contract to be fair, the buyer of the option contract should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, and is known as *option pricing*.

Option hedging

The second important issue is that of *hedging*, i.e. how to manage a given portfolio in such a way that it contains the required random payoff $(K - S_T)^+$ (for a put option) or $(S_T - K)^+$ (for a call option) at the maturity date T .

The next Figure 9 illustrates a sharp increase and sharp drop in asset price, making it valuable to hold a call option contract during the first half of the graph, whereas holding a put option contract would be recommended during the second half.

* Right-click to open or save the attachment.



Fig. 9: “Infogrames” stock price curve.

Example: Fuel hedging and the four-way zero-collar option

```

1  install.packages("Quandl")
2  library(Quandl);library(quantmod)
3  getSymbols("DCOILBRENTU", src="FRED")
4  chartSeries(DCOILBRENTU,up.col="blue",theme="white",name = "BRENT Oil
   Prices",lwd=5)
5  BRENT = Quandl("FRED/DCOILBRENTU",start_date="2010-01-01",
6  end_date="2015-11-30",type="xts")
7  chartSeries(BRENT,up.col="blue",theme="white",name = "BRENT Oil Prices",lwd=5)
8  getSymbols("WTI", from="2010-01-01", to="2015-11-30")
9  WTI <- Ad(`WTI`)
10 chartSeries(WTI,up.col="blue",theme="white",name = "WTI Oil Prices",lwd=5)

```



(a) WTI price graph.



(b) Brent price graph

Fig. 10: Brent and WTI price graphs.

(April 2011)

Fuel hedge promises Kenya Airways smooth ride in volatile oil market.*

(November 2015)

A close look at the role of fuel hedging in Kenya Airways \$259 million loss.*

* Right-click to open or save the attachment.

The four-way call collar call option requires its holder to purchase the underlying asset (here, airline fuel) at a price specified by the blue curve in Figure 11, when the underlying asset price is represented by the red line.

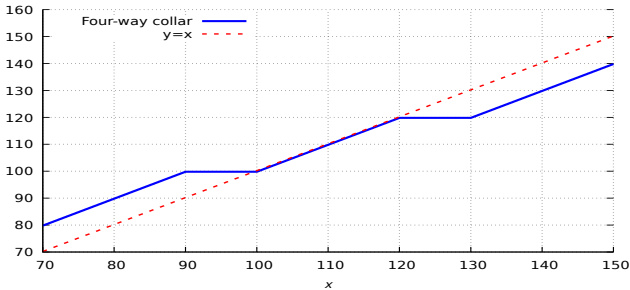


Fig. 11: Price map of a four-way call option.

The four-way call collar option contract will result into a positive or negative payoff depending on current fuel prices, as illustrated in Figure 12.

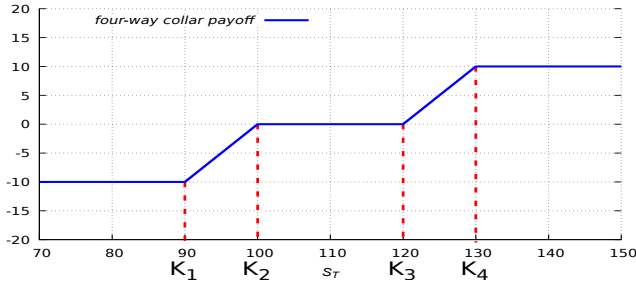


Fig. 12: Payoff function of a four-way call collar option.

The four-way call collar payoff can be written as a linear combination

$$\phi(S_T) = (K_1 - S_T)^+ - (K_2 - S_T)^+ + (S_T - K_3)^+ - (S_T - K_4)^+$$

of call and put option payoffs with respective strike prices

$$K_1 = 90, \quad K_2 = 100, \quad K_3 = 120, \quad K_4 = 130,$$

see *e.g.* <https://optioncreator.com/st5rf51>.

Fig. 13: Four-way call collar payoff as a combination of call and put options.*

Therefore, the four-way call collar option contract can be *synthesized* by:

1. purchasing a *put option* with strike price $K_1 = \$90$, and
2. selling (or issuing) a *put option* with strike price $K_2 = \$100$, and
3. purchasing a *call option* with strike price $K_3 = \$120$, and
4. selling (or issuing) a *call option* with strike price $K_4 = \$130$.

Moreover, the call collar option contract can be made *costless* by adjusting the boundaries K_1, K_2, K_3, K_4 , in which case it becomes a *zero-collar* option.

Example - The one-step 4-5-2 model

We close this introduction with a simplified example of the pricing and hedging technique in a binary model. Consider:

- i) A risky underlying stock valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values

$$S_1 = \begin{cases} \$5 \\ \$2 \end{cases}$$

at time $t = 1$.

- ii) An option contract that promises a claim payoff C whose values are defined contingent to the market data of S_1 as:

$$C := \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \$0 & \text{if } S_1 = \$2. \end{cases}$$

* The animation works in Acrobat Reader on the entire pdf file.

Exercise: Does C represent the payoff of a put option contract? Of a call option contract? If yes, with which strike price K ?

Quiz: Using this **form**, submit your own intuitive estimate for the price of the claim C .

At time $t = 0$ the option contract issuer (or writer) chooses to invest ξ units in the risky asset S , while keeping $\$ \eta$ on our bank account, meaning that we invest a total amount

$$\xi S_0 + \$ \eta \quad \text{at time } t = 0.$$

Here, the amount $\$ \eta$ may be positive or negative, depending on whether it corresponds to savings or to debt, and is interpreted as a *liability*.

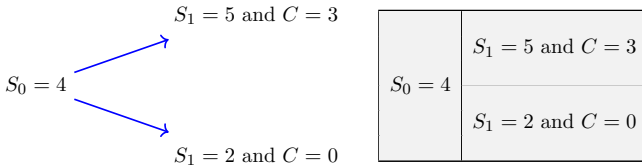
The following issues can be addressed:

- a) *Hedging:* How to choose the portfolio allocation $(\xi, \$ \eta)$ so that the *value*

$$\xi S_1 + \$ \eta$$

of the portfolio matches the future payoff C at time $t = 1$?

- b) *Pricing:* How to determine the initial *cost* $\xi S_0 + \$ \eta$ of the portfolio built by the option contract issuer at time $t = 0$?



Hedging or *replicating* the contract means that at time $t = 1$ the portfolio *value* matches the future payoff C , *i.e.*

$$\xi S_1 + \$ \eta = C.$$

Hedge, then price. This condition can be rewritten as

$$C = \begin{cases} \$3 = \xi \times \$5 + \$ \eta & \text{if } S_1 = \$5, \\ \$0 = \xi \times \$2 + \$ \eta & \text{if } S_1 = \$2, \end{cases}$$

i.e.

$$\begin{cases} 5\xi + \eta = 3, \\ 2\xi + \eta = 0, \end{cases} \quad \text{which yields} \quad \begin{cases} \xi = 1 \text{ stock,} \\ \$ \eta = -\$2. \end{cases}$$

In other words, the option contract issuer purchases 1 (one) unit of the stock S at the price $S_0 = \$4$, and borrows $\$2$ from the bank. The price of the option contract is then given by the portfolio value

$$\xi S_0 + \$\eta = 1 \times \$4 - \$2 = \$2.$$

at time $t = 0$.

The above computation is implemented in the attached [IPython notebook*](#) that can be run [here](#) or [here](#). This algorithm is scalable and can be extended to recombining binary trees over multiple time steps.

Definition 3. *The arbitrage-free price of the option contract is defined as the initial cost $\xi S_0 + \$\eta$ of the portfolio hedging the claim payoff C .*

Conclusion: in order to deliver the random payoff $C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \$0 & \text{if } S_1 = \$2. \end{cases}$ to the option contract holder at time $t = 1$, the option contract issuer (or writer) will:

1. charge $\xi S_0 + \$\eta = \2 (the option contract price) at time $t = 0$,
2. borrow $-\$2 = \2 from the bank,
3. invest those $\$2 + \$2 = \$4$ into the purchase of $\xi = 1$ unit of stock valued at $S_0 = \$4$ at time $t = 0$,
4. wait until time $t = 1$ to find that the portfolio value has evolved into


$$C = \begin{cases} \xi \times \$5 + \$\eta = 1 \times \$5 - \$2 = \$3 & \text{if } S_1 = \$5, \\ \xi \times \$2 + \$\eta = 1 \times \$2 - \$2 = 0 & \text{if } S_1 = \$2, \end{cases}$$

so that the option contract and the equality $C = \xi S_1 + \$\eta$ can be fulfilled, allowing the option issuer to break even whatever the evolution of the risky asset price S .

In a *cash settlement*, the stock is sold at the price $S_1 = \$5$ or $S_1 = \$2$, the payoff $C = (S_1 - K)^+ = \$3$ or $\$0$ is issued to the option contract holder, and the loan is refunded with the remaining $\$2$.

In the case of *physical delivery*, $\xi = 1$ share of stock is handed in to the option holder in exchange for the strike price $K = \$2$ which is used to refund the initial $\$2$ loan subscribed by the issuer.

Here, the option contract price $\xi S_0 + \$\eta = \2 is interpreted as the cost of hedging the option. In Chapters 2 and 3 we will see that this model is scalable and extends to discrete time.

* Right-click to save as attachment (may not work on .

We note that the initial option contract price of \$2 can be turned to $C = \$3$ (%50 profit) ... or into $C = \$0$ (total ruin).

Thinking further

1) The expected claim payoff at time $t = 1$ is

$$\begin{aligned}\mathbb{E}[C] &= \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) \\ &= \$3 \times \mathbb{P}(S_1 = \$5).\end{aligned}$$

In absence of arbitrage opportunities (“fair market”), this expected payoff $\mathbb{E}[C]$ should equal the initial amount \$2 invested in the option. In that case we should have

$$\begin{cases} \mathbb{E}[C] = \$3 \times \mathbb{P}(S_1 = \$5) = \$2 \\ \mathbb{P}(S_1 = \$5) + \mathbb{P}(S_1 = \$2) = 1. \end{cases}$$

from which we can *infer* the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \frac{2}{3} \\ \mathbb{P}(S_1 = \$2) = \frac{1}{3}, \end{cases} \quad (1)$$

which are called *risk-neutral* probabilities. We see that under the risk-neutral probabilities, the stock S has twice more chances to go up than to go down in a “fair” market.

2) Based on the probabilities (1) we can also compute the expected value $\mathbb{E}[S_1]$ of the stock at time $t = 1$. We find

$$\begin{aligned}\mathbb{E}[S_1] &= \$5 \times \mathbb{P}(S_1 = \$5) + \$2 \times \mathbb{P}(S_1 = \$2) \\ &= \$5 \times \frac{2}{3} + \$2 \times \frac{1}{3} \\ &= \$4 \\ &= S_0.\end{aligned}$$

Here, this means that, on average, no extra profit or loss can be made from an investment on the risky stock, and the probabilities $(2/3, 1/3)$ are termed *risk-neutral* probabilities. In a more realistic model we can assume that the riskless bank account yields an interest rate equal to r , in which case the above analysis is modified by letting $\$ \eta$ become $\$(1+r)\eta$ at time $t = 1$, nevertheless the main conclusions remain unchanged.

Market-implied probabilities

By matching the theoretical price $\mathbb{E}[C]$ to an actual market price data $\$M$ as

$$\$M = \mathbb{E}[C] = \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) = \$3 \times \mathbb{P}(S_1 = \$5)$$

we can infer the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \frac{\$M}{3} \\ \mathbb{P}(S_1 = \$2) = \frac{3 - \$M}{3}, \end{cases} \quad (2)$$

which are *implied probabilities* estimated from market data, as illustrated in Figure 14. We note that the conditions

$$0 < \mathbb{P}(S_1 = \$5) < 1, \quad 0 < \mathbb{P}(S_1 = \$2) < 1$$

are equivalent to $0 < \$M < 3$, which is consistent with financial intuition in a non-deterministic market. Figure 14 shows the time evolution of probabilities $p(t)$, $q(t)$ of two opposite outcomes.

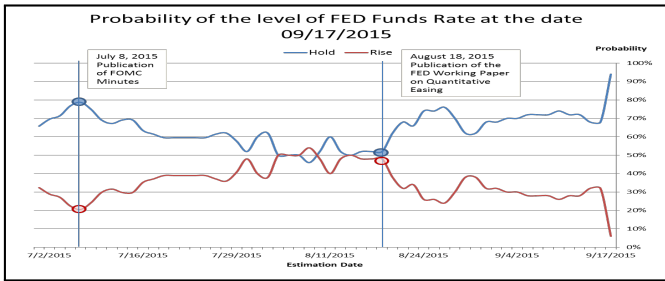


Fig. 14: Implied probabilities.

Note that implied probabilities should also be used with caution, as shown in Figures 15-16.

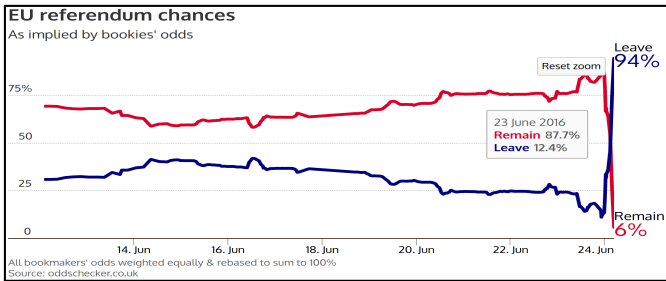


Fig. 15: Implied probabilities according to bookmakers.

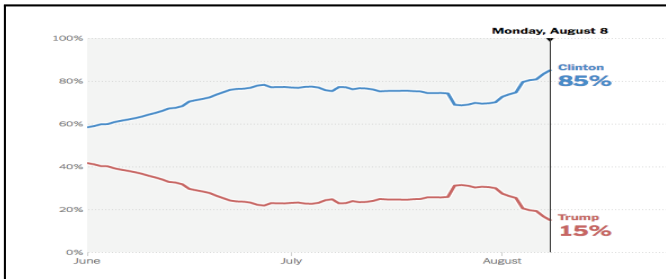


Fig. 16: Implied probabilities according to polling.

Implied probabilities can be estimated using *e.g.* binary options, see for example Exercise 3.11.

The *Practitioner* expects a good model to be:

- *Robust* with respect to missing, spurious or noisy data,
- *Fast* - prices have to be delivered daily in the morning,
- *Easy* to calibrate - parameter estimation,
- *Stable* with respect to re-calibration and the use of new data sets.

Typically, a medium size bank manages 5,000 options and 10,000 deals daily over 1,000 possible scenarios and dozens of time steps. This can mean a hundred million computations of $\mathbb{E}[C]$ daily, or close to a billion such computations for a large bank.

The *mathematician* tends to focus on more theoretical features, such as:

- *Elegance*,

- *Sophistication*,
- *Existence* of analytical (closed-form) solutions / error bounds,
- *Significance* to mathematical finance.

This includes:

- *Creating* new payoff functions and structured products,
- *Defining* new models for underlying asset prices,
- *Finding* new ways to compute expectations $\mathbb{E}[C]$ and hedging strategies.

The methods involved include:

- Monte Carlo methods (60%),

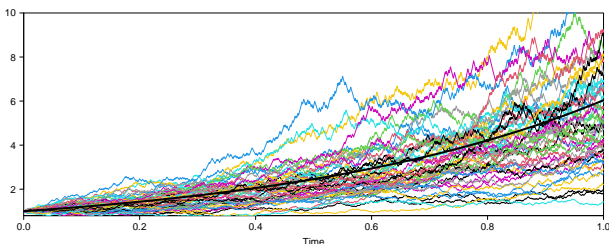


Fig. 17: Fifty sample price paths used for the Monte Carlo method.

- PDEs and finite differences methods (30%),
 - Other analytic methods and approximation methods (10%),
- + AI and Machine Learning techniques.

Course plan

The course plan from Chapter 1 to Chapter 7 is structured in layers that repeat the main concepts (arbitrage, pricing, hedging, risk-neutral measures) in different time scale settings (one-step, discrete-time, continuous-time).

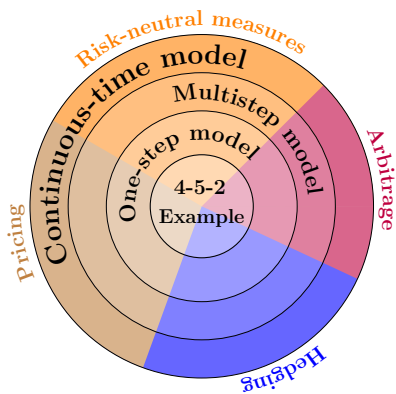


Fig. 18: Course plan.