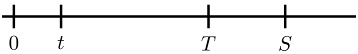
Chapter 18
Forward Rates

Forward rates are interest rates used in Forward Rate Agreements (FRA) for financial transactions, such as loans, that can take place at a future date. This chapter deals with the modeling of forward rates and swap rates in the Heath-Jarrow-Morton (HJM) and Brace-Gatarek-Musiela (BGM) models. It also includes a presentation of the Nelson and Siegel (1987) and Svensson (1994) curve parametrizations for yield curve fitting, and an introduction to two-factor interest rate models.

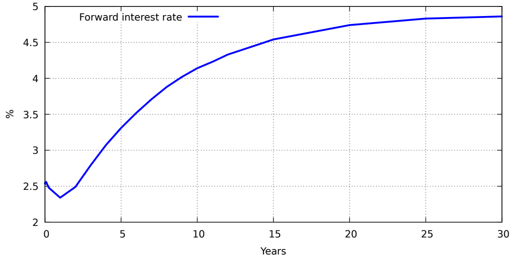
18.1 Construction of Forward Rates

A forward interest rate contract (or Forward Rate Agreement, FRA) gives to its holder the possibility to lock an interest rate denoted by $f(t,T,S)$ at present time t for a loan to be delivered over a future period of time $[T,S]$, with $t \leqslant T \leqslant S$.



The rate $f(t,T,S)$ is called a forward interest rate. When $T = t$, the *spot* forward rate $f(t,t,S)$ is also called the *yield*, see Relation (18.3) below.

Figure 18.1 presents a typical yield curve on the LIBOR (London Interbank Offered Rate) market with $t =$07 May 2003.

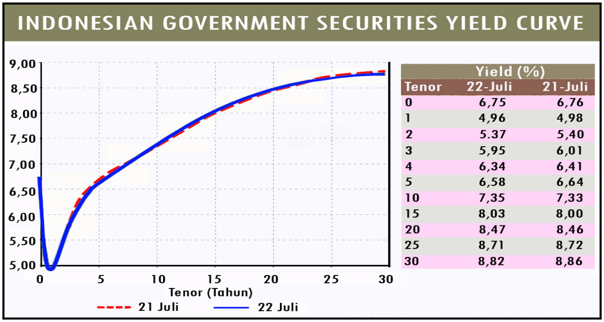


TimeSerieNb	505
AsOfDate	7-mai-03
2D	2,55
1W	2,53
1M	2,56
2M	2,52
3M	2,48
1Y	2,34
2Y	2,49
3Y	2,79
4Y	3,07
5Y	3,31
6Y	3,52
7Y	3,71
8Y	3,88
9Y	4,02
10Y	4,14
11Y	4,23
12Y	4,33
13Y	4,4
14Y	4,47
15Y	4,54
20Y	4,74
25Y	4,83
30Y	4,86

Fig. 18.1: Graph of the spot forward rate $S \mapsto f(t, t, S)$.

Maturity transformation, *i.e.*, the ability to transform short-term borrowing (debt with short maturities, such as deposits) into long term lending (credits with very long maturities, such as loans), is among the roles of banks. Profitability is then dependent on the difference between long rates and short rates.

Another example of market data is given in the next Figure 18.2, in which the red and blue curves refer respectively to July 21 and 22 of year 2011.



	Yield (%)	
Tenor	22-Juli	21-Juli
0	6,75	6,76
1	4,96	4,98
2	5.37	5,40
3	5,95	6,01
4	6,34	6,41
5	6,58	6,64
10	7,35	7,33
15	8,03	8,00
20	8,47	8,46
25	8,71	8,72
30	8,82	8,86

Fig. 18.2: Market example of yield curves, cf. (18.3).

Long maturities usually correspond to higher rates as they carry an increased risk. The dip observed with short maturities can correspond to a lower motivation to lend/invest in the short-term. However, yield curves can take diverse forms, see e.g. Figure 18.3.

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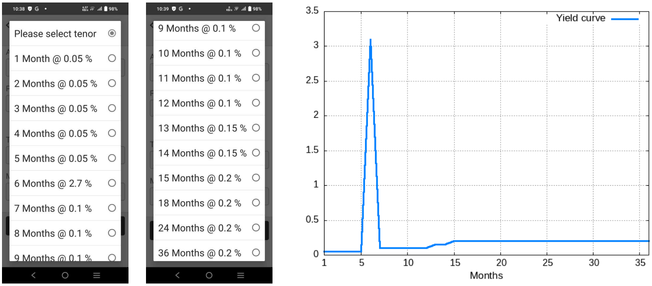

Fig. 18.3: Example of yield curve.

Forward rates from bond prices

Let us determine the arbitrage or "fair" value of the forward interest rate $f(t,T,S)$ by implementing the Forward Rate Agreement using the instruments available in the market, which are bonds priced at $P(t,T)$ for various maturity dates $T > t$.

The loan can be realized using the available instruments (here, bonds) on the market, by proceeding in two steps:

1) At time t, borrow the amount $P(t,S)$ by issuing (or short selling) one bond with maturity S, which means refunding \$1 at time S.

2) Since the money is only needed at time T, the rational investor will invest the amount $P(t,S)$ over the period $[t,T]$ by buying a (possibly fractional) quantity $P(t,S)/P(t,T)$ of a bond with maturity T priced $P(t,T)$ at time t. This will yield the amount

$$\$1 \times \frac{P(t,S)}{P(t,T)}$$

at time $T > 0$.

As a consequence, the investor will actually receive $P(t,S)/P(t,T)$ at time T, to refund \$1 at time S.

The corresponding forward rate $f(t,T,S)$ is then given by the relation

$$\frac{P(t,S)}{P(t,T)} \exp\left((S-T)f(t,T,S)\right) = \$1, \qquad 0 \leqslant t \leqslant T \leqslant S, \tag{18.1}$$

where we used exponential compounding, which leads to the following definition (18.2).

Definition 18.1. *The forward rate $f(t,T,S)$ at time t for a loan on $[T,S]$ is given by*

$$f(t,T,S) = \frac{\log P(t,T) - \log P(t,S)}{S-T}. \tag{18.2}$$

The *spot* forward rate $f(t,t,S)$ coincides with the *yield* $y(t,S)$, with

$$f(t,t,S) = y(t,S) = -\frac{\log P(t,S)}{S-t}, \quad \text{or} \quad P(t,S) = \mathrm{e}^{-(S-t)f(t,t,S)}, \tag{18.3}$$

$0 \leqslant t \leqslant S$.

Instantaneous forward rates

Proposition 18.2. *The instantaneous forward rate $f(t,T) = f(t,T,T)$ is defined by taking the limit of $f(t,T,S)$ as $S \searrow T$, and satisfies*

$$f(t,T) := \lim_{S \searrow T} f(t,T,S) = -\frac{1}{P(t,T)} \frac{\partial P}{\partial T}(t,T). \tag{18.4}$$

Proof. We have

$$\begin{aligned} f(t,T) : &= \lim_{S \searrow T} f(t,T,S) \\ &= - \lim_{S \searrow T} \frac{\log P(t,S) - \log P(t,T)}{S-T} \\ &= - \lim_{\varepsilon \searrow 0} \frac{\log P(t,T+\varepsilon) - \log P(t,T)}{\varepsilon} \\ &= -\frac{\partial}{\partial T} \log P(t,T) \\ &= -\frac{1}{P(t,T)} \frac{\partial P}{\partial T}(t,T). \end{aligned}$$

□

The above equation (18.4) can be viewed as a differential equation to be solved for $\log P(t,T)$ under the initial condition $P(T,T) = 1$, which yields the following proposition.

Proposition 18.3. *The bond price $P(t,T)$ can be recovered from the instantaneous forward rate $f(t,s)$ as*

$$P(t,T) = \exp\left(- \int_t^T f(t,s) ds \right), \qquad 0 \leqslant t \leqslant T. \tag{18.5}$$

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Proof. We check that

$$\begin{aligned}\log P(t,T) &= \log P(t,T) - \log P(t,t)\\ &= \int_t^T \frac{\partial}{\partial s}\log P(t,s)ds\\ &= -\int_t^T f(t,s)ds.\end{aligned}$$

□

Proposition 18.3 also shows that

$$\begin{aligned}f(t,t,t) &= f(t,t)\\ &= \frac{\partial}{\partial T}\int_t^T f(t,s)ds_{|T=t}\\ &= -\frac{\partial}{\partial T}\log P(t,T)_{|T=t}\\ &= -\frac{1}{P(t,T)}_{|T=t}\frac{\partial P}{\partial T}(t,T)_{|T=t}\\ &= -\frac{1}{P(T,T)}\frac{\partial}{\partial T}\mathbb{E}^*\left[e^{-\int_t^T r_s ds} \;\Big|\; \mathcal{F}_t\right]_{|T=t}\\ &= \mathbb{E}^*\left[r_T\, e^{-\int_t^T r_s ds} \;\Big|\; \mathcal{F}_t\right]_{|T=t}\\ &= \mathbb{E}^*[r_t \mid \mathcal{F}_t]\\ &= r_t, \qquad (18.6)\end{aligned}$$

i.e. the short rate r_t can be recovered from the instantaneous forward rate as

$$r_t = f(t,t) = \lim_{T\searrow t} f(t,T).$$

As a consequence of (18.1) and (18.5) the forward rate $f(t,T,S)$ can be recovered from (18.2) and the instantaneous forward rate $f(t,s)$, as:

$$\begin{aligned}f(t,T,S) &= \frac{\log P(t,T) - \log P(t,S)}{S-T}\\ &= -\frac{1}{S-T}\left(\int_t^T f(t,s)ds - \int_t^S f(t,s)ds\right)\\ &= \frac{1}{S-T}\int_T^S f(t,s)ds, \qquad 0 \leqslant t \leqslant T < S. \qquad (18.7)\end{aligned}$$

Similarly, as a consequence of (18.3) and (18.5) we have the next proposition.

Proposition 18.4. *The* spot *forward rate or* yield $f(t,t,T)$ *can be written in terms of bond prices as*

$$f(t,t,T) = -\frac{\log P(t,T)}{T-t} = \frac{1}{T-t}\int_t^T f(t,s)ds, \qquad 0 \leqslant t < T. \quad (18.8)$$

Differentiation with respect to T of the above relation shows that the yield $f(t,t,T)$ and the instantaneous forward rate $f(t,s)$ are linked by the relation

$$\frac{\partial f}{\partial T}(t,t,T) = -\frac{1}{(T-t)^2}\int_t^T f(t,s)ds + \frac{1}{T-t}f(t,T), \quad 0 \leqslant t < T,$$

from which it follows that

$$\begin{aligned} f(t,T) &= \frac{1}{T-t}\int_t^T f(t,s)ds + (T-t)\frac{\partial f}{\partial T}(t,t,T) \\ &= f(t,t,T) + (T-t)\frac{\partial f}{\partial T}(t,t,T), \qquad 0 \leqslant t < T. \end{aligned}$$

Forward Vašíček (1977) rates

In this section we consider the Vasicek model, in which the short rate process is the solution (17.2) of (17.1) as illustrated in Figure 17.1.

In the Vasicek model, the forward rate is given by

$$\begin{aligned} f(t,T,S) &= -\frac{\log P(t,S) - \log P(t,T)}{S-T} \\ &= -\frac{r_t(C(S-t) - C(T-t)) + A(S-t) - A(T-t))}{S-T} \\ &= -\frac{\sigma^2 - 2ab}{2b^2} \\ &-\frac{1}{S-T}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(\mathrm{e}^{-(S-t)b} - \mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{4b^3}\left(\mathrm{e}^{-2(S-t)b} - \mathrm{e}^{-2(T-t)b}\right)\right), \end{aligned}$$

and the spot forward rate, or yield, satisfies

$$\begin{aligned} f(t,t,T) &= -\frac{\log P(t,T)}{T-t} = -\frac{r_t C(T-t) + A(T-t)}{T-t} \\ &= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(1 - \mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{4b^3}\left(1 - \mathrm{e}^{-2(T-t)b}\right)\right), \end{aligned}$$

with the mean

$$\mathbb{E}[f(t,t,T)]$$

$$= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\left(\frac{\mathbb{E}[r_t]}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(1 - e^{-(T-t)b}\right) - \frac{\sigma^2}{4b^3}\left(1 - e^{-2(T-t)b}\right)\right)$$
$$= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\frac{r_0}{b}\, e^{-bt} + \frac{a}{b^2}\left(1 - e^{-bt}\right) + \frac{\sigma^2 - ab}{b^3}\right)\left(1 - e^{-(T-t)b}\right)$$
$$-\frac{\sigma^2}{4b^3(T-t)}\left(1 - e^{-2(T-t)b}\right).$$

In this model, the forward rate $t \mapsto f(t,t,T)$ can be represented as in the following Figure 18.4, with $a = 0.06$, $b = 0.1$, $\sigma = 0.1$, $r_0 = \%1$ and $T = 50$.

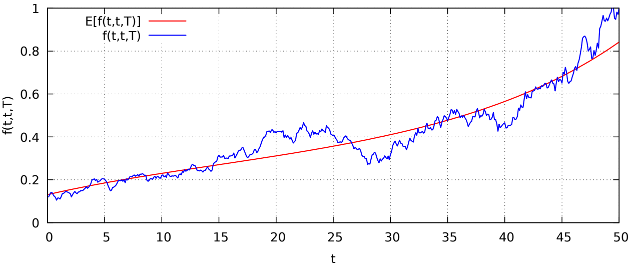

Fig. 18.4: Forward rate process $t \mapsto f(t,t,T)$.

We note that the Vasicek forward rate curve $t \mapsto f(t,t,T)$ appears flat for small values of t, *i.e.* longer rates are more stable, while shorter rates show higher volatility or risk. Similar features can be observed in Figure 18.5 for the instantaneous short rate given by

$$f(t,T) := -\frac{\partial}{\partial T} \log P(t,T) \tag{18.9}$$
$$= r_t\, e^{-(T-t)b} + \frac{a}{b}\left(1 - e^{-(T-t)b}\right) - \frac{\sigma^2}{2b^2}\left(1 - e^{-(T-t)b}\right)^2,$$

from which the relation $\lim_{T \searrow t} f(t,T) = r_t$ can be easily recovered. We can also evaluate the mean

$$\mathbb{E}[f(t,T)] = \mathbb{E}[r_t]\, e^{-(T-t)b} + \frac{a}{b}\left(1 - e^{-(T-t)b}\right) - \frac{\sigma^2}{2b^2}\left(1 - e^{-(T-t)b}\right)^2$$
$$= r_0\, e^{-bT} + \frac{a}{b}\left(1 - e^{-bT}\right) - \frac{\sigma^2}{2b^2}\left(1 - e^{-(T-t)b}\right)^2.$$

The instantaneous forward rate $t \mapsto f(t,T)$ can be represented as in the following Figure 18.5, with $a = 0.06$, $b = 0.1$, $\sigma = 0.1$ and $r_0 = \%1$.

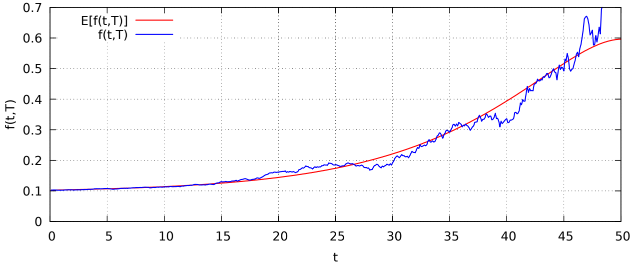

Fig. 18.5: Instantaneous forward rate process $t \mapsto f(t,T)$.

Yield curve data

We refer to Chapter III-12 of Charpentier (2014) on the R package "Yield-Curve" Guirreri (2015) for the following R code and further details on yield curve and interest rate modeling using R.

```
install.packages("YieldCurve");require(YieldCurve);data(FedYieldCurve)
first(FedYieldCurve,'3 month');last(FedYieldCurve,'3 month')
mat.Fed=c(0.25,0.5,1,2,3,5,7,10);n=50
plot(mat.Fed, FedYieldCurve[n,], type="o",xlab="Maturities structure in years", ylab="Interest
      rates values", col = "blue", lwd=3)
title(main=paste("Federal Reserve yield curve observed at",time(FedYieldCurve[n], sep=" ")))
grid()
```

The next Figure 18.6 is plotted using this code* which is adapted from https://www.quantmod.com/examples/chartSeries3d/chartSeries3d.alpha.R

* Click to open or download.

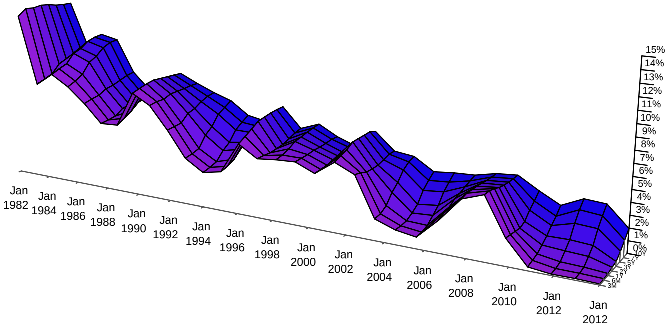

Fig. 18.6: Federal Reserve yield curves from 1982 to 2012.

European Central Bank (ECB) data can be similarly obtained by the next R code.

```
data(ECBYieldCurve);first(ECBYieldCurve,'3 month');last(ECBYieldCurve,'3 month')
mat.ECB<-c(3/12,0.5,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23, 24,25,26,27,28,
      29,30)
dev.new(width=16,height=7)
for (n in 200:400) {
plot(mat.ECB, ECBYieldCurve[n,], type="o",xlab="Maturity structure in years",
      ylab="Interest rates values",ylim=c(3.1,5.1),col="blue",lwd=2,cex.axis=1.5,cex.lab=1.5)
title(main=paste("European Central Bank yield curve observed at",time(ECBYieldCurve[n],
      sep=" ")))
grid();Sys.sleep(0.5)}
```

The next Figure 18.7 represents the output of the above script.

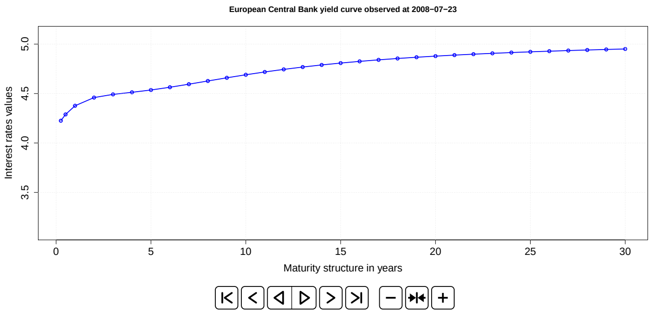

Fig. 18.7: European Central Bank yield curves.*

* The animation works in Acrobat Reader on the entire pdf file.

Yield curve inversion

Increasing yield curves are typical of economic expansion phases. Decreasing yield curves can occur when central banks attempt to limit inflation by tightening interest rates, such as in the case of an economic recession, see **here**.*
In this case, uncertainty triggers increased investment in long bonds whose rates tend to drop as a consequence, while reluctance to lend in the short term can lead to higher short rates.

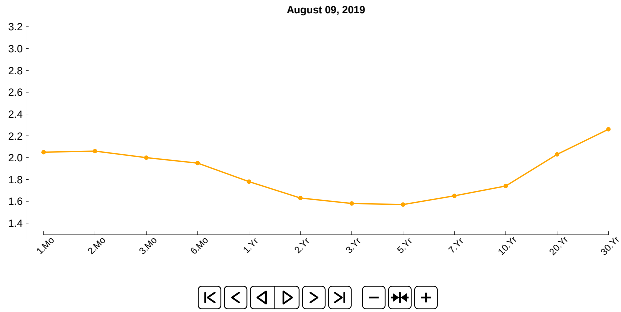

Fig. 18.8: August 2019 Federal Reserve yield curve inversion.†

The above Figure 18.8 illustrates a Federal Reserve (FED) yield curve inversions occurring in February and August 2019.

LIBOR (London Interbank Offered) Rates

Recall that the forward rate $f(t,T,S)$, $0 \leqslant t \leqslant T \leqslant S$, is defined using exponential compounding, from the relation

$$f(t,T,S) = \frac{\log P(t,T) - \log P(t,S)}{S-T}. \tag{18.10}$$

In order to compute swaption prices one prefers to use forward rates as defined on the London InterBank Offered Rates (LIBOR) market instead of the standard forward rates given by (18.10). Other types of LIBOR rates include EURIBOR (European Interbank Offered Rates), HIBOR (Hong Kong Interbank Offered Rates), SHIBOR (Shanghai Interbank Offered Rates), SIBOR (Singapore Interbank Offered Rates), TIBOR (Tokyo Interbank Offered Rates), etc. Most LIBOR rates have been replaced by alternatives such as

* Right-click to open or save the attachment.

† The animation works in Acrobat Reader on the entire pdf file.

the Secured Overnight Financing Rate (SOFR) starting with the end of year 2021, see below, page 656.

The forward LIBOR rate $L(t,T,S)$ for a loan on $[T,S]$ is defined by replacing exponential compounding with linear compounding in the argument leading to (18.1), *i.e.* by replacing (18.10) with the relation

$$1+(S-T)L(t,T,S)=\frac{P(t,T)}{P(t,S)},\qquad t\geqslant T, \tag{18.11}$$

which yields the following definition.

Definition 18.5. *The forward LIBOR rate $L(t,T,S)$ at time t for a loan on $[T,S]$ is given by*

$$L(t,T,S)=\frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)}-1\right),\qquad 0\leqslant t\leqslant T<S. \tag{18.12}$$

Note that (18.12) above yields the same formula for the (LIBOR) instantaneous forward rate

$$\begin{aligned}
L(t,T) &:= \lim_{S\searrow T} L(t,T,S)\\
&= \lim_{S\searrow T}\frac{P(t,T)-P(t,S)}{(S-T)P(t,S)}\\
&= \lim_{\varepsilon\searrow 0}\frac{P(t,T)-P(t,T+\varepsilon)}{\varepsilon P(t,T+\varepsilon)}\\
&= \frac{1}{P(t,T)}\lim_{\varepsilon\searrow 0}\frac{P(t,T)-P(t,T+\varepsilon)}{\varepsilon}\\
&= -\frac{1}{P(t,T)}\frac{\partial P}{\partial T}(t,T)\\
&= -\frac{\partial}{\partial T}\log P(t,T)\\
&= f(t,T),
\end{aligned}$$

as in (18.4). In addition, Relation (18.12) shows that the LIBOR rate can be viewed as a forward price $\widehat{X}_t=X_t/N_t$ with numéraire $N_t=(S-T)P(t,S)$ and $X_t=P(t,T)-P(t,S)$, according to Relation (16.4) of Chapter 16. As a consequence, from Proposition 16.4 we have the following result, which uses the forward measure $\widehat{\mathbb{P}}_S$ defined by its Radon-Nikodym density

$$\frac{d\widehat{\mathbb{P}}_S}{d\mathbb{P}^*}:=\frac{1}{P(0,S)}\,\mathrm{e}^{-\int_0^S r_t dt}, \tag{18.13}$$

from the numéraire process $N_t:=P(t,S)$, $t\in[0,S]$, see Definition 16.1.

Proposition 18.6. *The (simply compounded) LIBOR forward rate* $(L(t,T,S))_{t\in[0,T]}$ *is a martingale under* $\widehat{\mathbb{P}}_S$*, i.e. we have*

$$L(t,T,S) = \widehat{\mathbb{E}}_S[L(T,T,S) \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T.$$

SOFR (Secured Overnight Financing) Rates

The repurchase agreement ("repo") market is a market where government treasury securities can be borrowed on the short term. The SOFR rate is a measure of the cost of borrowing which is estimated using overnight activity on the repo market. In that sense, the SOFR, which is transaction-based, differs from LIBOR which is relied on a survey of a panel of banks and subject to manipulation. On the other hand, an important difference is that LIBOR rates are *forward-looking* using a term structure, whereas SOFR rates are *backward-looking*.

The next definition uses the integral convention $\int_a^b = -\int_b^a$, $a < b$.

Definition 18.7. *The* backward-looking *bond price is defined for* $t \geqslant T$ *as*

$$P(t,T) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_u du} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\mathrm{e}^{\int_T^t r_u du} \,\middle|\, \mathcal{F}_t\right] = \mathrm{e}^{\int_T^t r_u du}, \quad t \geqslant T.$$

The forward SOFR rate $R(t,T,S)$ for a loan on $[T,S]$ is defined using linear compounding by the same absence of arbitrage argument leading to (18.11), as

$$1 + (S-T)R(t,T,S) = \frac{P(t,T)}{P(t,S)}, \qquad 0 \leqslant T \leqslant t,$$

which yields the following definition.

Definition 18.8. *The forward SOFR rate* $R(t,T,S)$ *at time* $t \in [T,S]$ *for a loan on the time interval* $[T,S]$ *is given by*

$$R(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right), \qquad 0 \leqslant T \leqslant t \leqslant S. \qquad (18.14)$$

We have

$$R(t,T,S) = \frac{1}{S-T}\left(\frac{\mathrm{e}^{\int_T^t r_u du}}{P(t,S)} - 1\right), \qquad 0 \leqslant T \leqslant t \leqslant S,$$

and in particular, the spot Effective Federal Funds Rate (EFFR) is given for $t = S$ as

$$R(S,T,S) = \frac{1}{S-T}\left(\mathrm{e}^{\int_T^S r_u du} - 1\right).$$

The following proposition, see Rutkowski and Bickersteth (2021), uses the forward S-measure $\widehat{\mathbb{P}}_S$ defined by its Radon-Nikodym density (18.13).

Proposition 18.9. *The SOFR forward rate* $(R(t,T,S))_{t\in[T,S]}$ *is a martingale under* $\widehat{\mathbb{P}}_S$*, i.e. we have*

$$R(t,T,S) = \widehat{\mathbb{E}}_S[R(S,T,S) \mid \mathcal{F}_t] = \widehat{\mathbb{E}}_S\left[\frac{1}{S-T}\left(\mathrm{e}^{\int_T^S r_u du} - 1\right) \Big| \mathcal{F}_t\right],$$

$T \leqslant t \leqslant S$.

Proof. We have

$$\begin{aligned}
R(t,T,S) &= \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \\
&= \frac{1}{S-T}\left(\frac{\mathrm{e}^{\int_T^t r_u du}}{P(t,S)} - 1\right) \\
&= \frac{1}{S-T}\left(\frac{1}{P(t,S)}\mathbb{E}\left[\mathrm{e}^{\int_T^t r_u du} \Big| \mathcal{F}_t\right] - 1\right) \\
&= \frac{1}{S-T}\left(\frac{1}{P(t,S)}\mathbb{E}\left[\mathrm{e}^{-\int_t^S r_u du}\,\mathrm{e}^{\int_T^S r_u du} \Big| \mathcal{F}_t\right] - 1\right) \\
&= \frac{1}{S-T}\left(\widehat{\mathbb{E}}_S\left[\mathrm{e}^{\int_T^S r_u du} \Big| \mathcal{F}_t\right] - 1\right) \\
&= \frac{1}{S-T}\left(\widehat{\mathbb{E}}_S[P(S,T) \mid \mathcal{F}_t] - 1\right) \\
&= \widehat{\mathbb{E}}_S[R(S,T,S) \mid \mathcal{F}_t], \qquad T \leqslant t \leqslant S.
\end{aligned}$$

□

18.2 LIBOR and SOFR Swap Rates

The first interest rate swap occurred in 1981 between the World Bank, which was interested in borrowing German Marks and Swiss Francs, and IBM, which already had large amounts of those currencies but needed to borrow U.S. dollars.

The vanilla interest rate swap makes it possible to exchange a sequence of variable LIBOR rates $L(t,T_k,T_{k+1})$, $k = 1,2,\ldots,n-1$, against a fixed rate κ over a succession of time intervals $[T_i,T_{i+1}),\ldots,[T_{j-1},T_j]$ defining a *tenor structure*, see Section 19.1 for details.

Making the agreement fair results into an exchange of cashflows

$$\underbrace{(T_{k+1}-T_k)L(t,T_k,T_{k+1})}_{\text{floating leg}} - \underbrace{(T_{k+1}-T_k)\kappa}_{\text{fixed leg}},$$

at the dates $T_{i+1}, \ldots, T_j$ between the two parties, therefore generating a cumulative discounted cash flow

$$\sum_{k=i}^{j-1} \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} (T_{k+1} - T_k)(L(t, T_k, T_{k+1}) - \kappa),$$

at time $t = T_0$, in which we used simple (or linear) interest rate compounding. This corresponds to a *payer swap* in which the swap holder receives the *floating leg* and pays the *fixed leg* κ, whereas the holder of a *seller swap* receives the *fixed leg* κ and pays the *floating leg*.

The above cash flow is used to make the contract fair, and it can be priced *at time* t as

$$\begin{aligned}
\mathbb{E}^* & \left[\sum_{k=i}^{j-1} (T_{k+1} - T_k) \, \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} (L(t, T_k, T_{k+1}) - \kappa) \;\Big|\; \mathcal{F}_t \right] \\
&= \sum_{k=i}^{j-1} (T_{k+1} - T_k)(L(t, T_k, T_{k+1}) - \kappa) \mathbb{E}^* \left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} \;\Big|\; \mathcal{F}_t \right] \\
&= \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \big(L(t, T_k, T_{k+1}) - \kappa \big). \qquad (18.15)
\end{aligned}$$

The swap rate $S(t, T_i, T_j)$ is by definition the value of the rate κ that makes the contract fair by making the above cash flow $\mathcal{C}(t)$ vanish.

Definition 18.10. *The LIBOR swap rate $S(t, T_i, T_j)$ is the value of the break-even rate κ that makes the contract fair by making the cash flow* (18.15) *vanish, i.e.*

$$\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \big(L(t, T_k, T_{k+1}) - \kappa \big) = 0. \qquad (18.16)$$

The next Proposition 18.11 makes use of the annuity numéraire

$$\begin{aligned}
P(t, T_i, T_j) &:= \mathbb{E}^* \left[\sum_{k=i}^{j-1} (T_{k+1} - T_k) \, \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} \;\Big|\; \mathcal{F}_t \right] \qquad (18.17) \\
&= \sum_{k=i}^{j-1} (T_{k+1} - T_k) \mathbb{E}^* \left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} \;\Big|\; \mathcal{F}_t \right] \\
&= \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}), \qquad 0 \leqslant t \leqslant T_2,
\end{aligned}$$

which represents the present value at time t of future \$1 receipts at times $T_i, \ldots, T_j$, weighted by the lengths $T_{k+1} - T_k$ of the time intervals $(T_k, T_{k+1}]$, $k = i, \ldots, j-1$.

The time intervals $(T_{k+1} - T_k)_{k=i,\ldots,j-1}$ in the definition (18.17) of the annuity numéraire can be replaced by coupon payments $(c_{k+1})_{k=i,\ldots,j-1}$ occurring at times $(T_{k+1})_{k=i,\ldots,j-1}$, in which case the annuity numéraire becomes

$$\begin{aligned} P(t, T_i, T_j) &:= \mathbb{E}^*\left[\sum_{k=i}^{j-1} c_{k+1} \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} \,\Big|\, \mathcal{F}_t \right] \\ &= \sum_{k=i}^{j-1} c_{k+1} \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} \,\Big|\, \mathcal{F}_t \right] \\ &= \sum_{k=i}^{j-1} c_{k+1} P(t, T_{k+1}), \qquad 0 \leqslant t \leqslant T_i, \end{aligned} \tag{18.18}$$

which represents the value at time t of the future coupon payments discounted according to the bond prices $(P(t, T_{k+1}))_{k=i,\ldots,j-1}$. This expression can also be used to define *amortizing swaps* in which the value of the notional decreases over time, or *accreting swaps* in which the value of the notional increases over time.

LIBOR Swap rates

The LIBOR *swap rate* $S(t, T_i, T_j)$ is defined by solving Relation (18.16) for the forward rate $S(t, T_k, T_{k+1})$, *i.e.*

$$\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \big(L(t, T_k, T_{k+1}) - S(t, T_i, T_j) \big) = 0. \tag{18.19}$$

Proposition 18.11. *The LIBOR swap rate $S(t, T_i, T_j)$ is given by*

$$S(t, T_i, T_j) = \frac{1}{P(t, T_i, T_j)} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) L(t, T_k, T_{k+1}), \tag{18.20}$$

$0 \leqslant t \leqslant T_i$.

Proof. By definition, $S(t, T_i, T_j)$ is the (fixed) break-even rate over $[T_i, T_j]$ that will be agreed in exchange for the family of forward rates $L(t, T_k, T_{k+1})$, $k = i, \ldots, j-1$, and it solves (18.19), *i.e.* we have

$$\begin{aligned}
&\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})-P(t,T_i,T_j)S(t,T_i,T_j)\\
&=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})\\
&\quad -S(t,T_i,T_j)\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\\
&=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})-S(t,T_i,T_j)P(t,T_i,T_j)\\
&=0,
\end{aligned}$$

which shows (18.20) by solving the above equation for $S(t,T_i,T_j)$. □

The LIBOR swap rate $S(t,T_i,T_j)$ is defined by the same relation as (18.16), with the forward rate $L(t,T_k,T_{k+1})$ replaced with the LIBOR rate $L(t,T_k,T_{k+1})$. In this case, using the Definition 18.12 of LIBOR rates we obtain the next corollary.

Corollary 18.12. *The LIBOR swap rate $S(t,T_i,T_j)$ is given by*

$$S(t,T_i,T_j)=\frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)},\qquad 0\leqslant t\leqslant T_i. \tag{18.21}$$

Proof. By (18.20), (18.12) and a telescoping summation argument we have

$$\begin{aligned}
S(t,T_i,T_j)&=\frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})\\
&=\frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}P(t,T_{k+1})\left(\frac{P(t,T_k)}{P(t,T_{k+1})}-1\right)\\
&=\frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(P(t,T_k)-P(t,T_{k+1}))\\
&=\frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)}. \qquad (18.22)
\end{aligned}$$

□

By (18.21), the bond prices $P(t,T_i)$ can be recovered from the values of the forward swap rates $S(t,T_i,T_j)$.

Clearly, a simple expression for the swap rate such as that of Corollary 18.12 cannot be obtained using the standard (*i.e.* non-LIBOR) rates defined in (18.10). Similarly, it will not be available for amortizing or accreting swaps because the telescoping summation argument does not apply to the expression (18.18) of the annuity numéraire.

When $n = 2$, the LIBOR swap rate $S(t, T_1, T_2)$ coincides with the LIBOR rate $L(t, T_1, T_2)$, as from (18.18) we have

$$\begin{aligned} S(t, T_1, T_2) &= \frac{P(t, T_1) - P(t, T_2)}{P(t, T_1, T_2)} \\ &= \frac{P(t, T_1) - P(t, T_2)}{(T_2 - T_1) P(t, T_2)} \\ &= L(t, T_1, T_2). \end{aligned} \tag{18.23}$$

Similarly to the case of LIBOR rates, Relation (18.21) shows that the LIBOR swap rate can be viewed as a forward price with (annuity) numéraire $N_t = P(t, T_i, T_j)$ and $X_t = P(t, T_i) - P(t, T_j)$. Consequently the LIBOR swap rate $(S(t, T_i, T_j)_{t \in [T,S]}$ is a martingale under the forward measure $\widehat{\mathbb{P}}$ defined from (16.1) by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \frac{P(T_i, T_i, T_j)}{P(0, T_i, T_j)} \mathrm{e}^{-\int_0^{T_i} r_t dt}.$$

SOFR Swap rate

The expressions

$$S(t, T_i, T_j) = \frac{1}{P(t, T_i, T_j)} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) R(t, T_k, T_{k+1}) \tag{18.24}$$

and

$$S(t, T_i, T_j) = \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)}, \qquad T_i \leqslant t \leqslant T_j, \tag{18.25}$$

defining the SOFR swap rate $S(t, T_i, T_j)$ are identical to the ones defining the LIBOR swap rate in (18.20) and (18.21) by taking $t \geqslant T_i$ in the case of the SOFR swap rate.

18.3 The HJM Model

In this section we turn to the modeling of instantaneous forward rate curves in the HJM Model. From the beginning of this chapter we have started with the modeling of the short rate $(r_t)_{t\in\mathbb{R}_+}$, followed by its consequences on the pricing of bonds $P(t,T)$ and on the expressions of the forward rates $f(t,T,S)$ and $L(t,T,S)$.

In this section we choose a different starting point and consider the problem of directly modeling the instantaneous forward rate $f(t,T)$. The graph given in Figure 18.9 presents a possible random evolution of a forward interest rate curve using the Musiela convention, *i.e.* we will write

$$g(x) = f(t,t+x) = f(t,T), \tag{18.26}$$

under the substitution $x = T - t$, $x \geqslant 0$, and represent a sample of the instantaneous forward curve $x \mapsto f(t,t+x)$ for each $t \geqslant 0$.

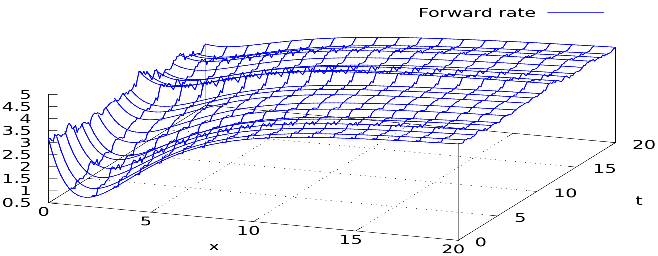

Fig. 18.9: Stochastic process of forward curves.

Definition 18.13. *In the Heath-Jarrow-Morton (HJM) model, the instantaneous forward rate $f(t,T)$ is modeled under $\mathbb{P}^*$ by a stochastic differential equation of the form*

$$d_t f(t,T) = \alpha(t,T)dt + \sigma(t,T)dB_t, \qquad 0 \leqslant t \leqslant T, \tag{18.27}$$

where $t \mapsto \alpha(t,T)$ and $t \mapsto \sigma(t,T)$, $0 \leqslant t \leqslant T$, are allowed to be random, $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, processes.

In the above equation, the date T is fixed and the differential d_t is with respect to the time variable t.

Under basic Markovianity assumptions, a HJM model with deterministic coefficients $\alpha(t,T)$ and $\sigma(t,T)$ will yield a short rate process $(r_t)_{t\in\mathbb{R}_+}$ of the form

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dB_t,$$

see § 7.4 in Privault (2021b), which is the Hull and White (1990) model, with the explicit solution

$$r_t = r_s \,\mathrm{e}^{-\int_s^t b(\tau)d\tau} + \int_s^t \mathrm{e}^{-\int_u^t b(\tau)d\tau} a(u)du + \int_s^t \sigma(u)\,\mathrm{e}^{-\int_u^t b(\tau)d\tau} dB_u,$$

$0 \leqslant s \leqslant t$.

The HJM condition

How to "encode" absence of arbitrage in the defining HJM Equation (18.27) is an important question. Recall that under absence of arbitrage, the bond price $P(t,T)$ has been constructed as

$$P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \,\Big|\, \mathcal{F}_t\right] = \exp\left(-\int_t^T f(t,s)ds\right), \qquad (18.28)$$

cf. Proposition 18.3, hence the discounted bond price process is given by

$$t \mapsto \exp\left(-\int_0^t r_s ds\right) P(t,T) = \exp\left(-\int_0^t r_s ds - \int_t^T f(t,s)ds\right) \qquad (18.29)$$

is a martingale under $\mathbb{P}^*$ by Proposition 17.1 and Relation (18.5) in Proposition 18.3. This shows that $\mathbb{P}^*$ is a risk-neutral probability measure, and by the first fundamental theorem of asset pricing Theorem 5.7 we conclude that the market is without arbitrage opportunities.

Proposition 18.14. *(HJM Condition, Heath et al. (1992)). Under the condition*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds, \qquad 0 \leqslant t \leqslant T, \qquad (18.30)$$

which is known as the HJM absence of arbitrage condition*, the discounted bond price process* (18.29) *is a martingale, and the probability measure* $\mathbb{P}^*$ *is risk-neutral.*

Proof. Using the process $(X_t)_{t\in[0,T]}$ defined as

$$X_t := \int_t^T f(t,s)ds = -\log P(t,T), \qquad 0 \leqslant t \leqslant T,$$

such that $P(t,T) = \mathrm{e}^{-X_t}$, we rewrite the spot forward rate, or yield

$$f(t,t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds,$$

see (18.8), as

$$f(t,t,T)=\frac{1}{T-t}\int_t^T f(t,s)ds=\frac{X_t}{T-t}, \qquad 0\leqslant t\leqslant T,$$

where the dynamics of $t\mapsto f(t,s)$ is given by (18.27). We also use the extended Leibniz integral rule

$$d_t\int_t^T f(t,s)ds=-f(t,t)dt+\int_t^T d_tf(t,s)ds=-r_tdt+\int_t^T d_tf(t,s)ds,$$

see (18.6). This identity can be checked in the particular case where $f(t,s)=g(t)h(s)$ is a smooth function that satisfies the separation of variables property, as

$$\begin{aligned}d_t\left(\int_t^T g(t)h(s)ds\right)&=d_t\left(g(t)\int_t^T h(s)ds\right)\\&=\int_t^T h(s)dsdg(t)+g(t)d_t\int_t^T h(s)ds\\&=g'(t)\left(\int_t^T h(s)ds\right)dt-g(t)h(t)dt.\end{aligned}$$

We have

$$\begin{aligned}d_tX_t&=d_t\int_t^T f(t,s)ds\\&=-f(t,t)dt+\int_t^T d_tf(t,s)ds\\&=-f(t,t)dt+\int_t^T\alpha(t,s)dsdt+\int_t^T\sigma(t,s)dsdB_t\\&=-r_tdt+\left(\int_t^T\alpha(t,s)ds\right)dt+\left(\int_t^T\sigma(t,s)ds\right)dB_t,\end{aligned}$$

hence

$$|d_tX_t|^2=\left(\int_t^T\sigma(t,s)ds\right)^2dt.$$

By Itô's calculus, we find

$$\begin{aligned}d_tP(t,T)&=d_t\,\mathrm{e}^{-X_t}\\&=-\mathrm{e}^{-X_t}d_tX_t+\frac{1}{2}\mathrm{e}^{-X_t}(d_tX_t)^2\\&=-\mathrm{e}^{-X_t}d_tX_t+\frac{1}{2}\mathrm{e}^{-X_t}\left(\int_t^T\sigma(t,s)ds\right)^2dt\\&=-\mathrm{e}^{-X_t}\left(-r_tdt+\int_t^T\alpha(t,s)dsdt+\int_t^T\sigma(t,s)dsdB_t\right)\\&\quad+\frac{1}{2}\mathrm{e}^{-X_t}\left(\int_t^T\sigma(t,s)ds\right)^2dt,\end{aligned}$$

and the discounted bond price satisfies

$$\begin{aligned}
& d_t\left(\exp\left(-\int_0^t r_s ds\right) P(t,T)\right) \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t\right) dt + \exp\left(-\int_0^t r_s ds\right) d_t P(t,T) \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t\right) dt - \exp\left(-\int_0^t r_s ds - X_t\right) d_t X_t \\
&\quad + \frac{1}{2}\exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \sigma(t,s) ds\right)^2 dt \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t\right) dt \\
&\quad - \exp\left(-\int_0^t r_s ds - X_t\right)\left(-r_t dt + \int_t^T \alpha(t,s) ds dt + \int_t^T \sigma(t,s) ds dB_t\right) \\
&\quad + \frac{1}{2}\exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \sigma(t,s) ds\right)^2 dt \\
&= -\exp\left(-\int_0^t r_s ds - X_t\right)\int_t^T \sigma(t,s) ds dB_t \\
&\quad - \exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \alpha(t,s) ds - \frac{1}{2}\left(\int_t^T \sigma(t,s) ds\right)^2\right) dt.
\end{aligned}$$

Thus, the discounted bond price process

$$t \mapsto \exp\left(-\int_0^t r_s ds\right) P(t,T)$$

will be a martingale provided that

$$\int_t^T \alpha(t,s) ds - \frac{1}{2}\left(\int_t^T \sigma(t,s) ds\right)^2 = 0, \qquad 0 \leqslant t \leqslant T. \tag{18.31}$$

Differentiating the above relation with respect to T yields

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s) ds,$$

which is in fact equivalent to (18.31). □

Forward Vasicek rates in the HJM model

The HJM coefficients in the Vasicek model are in fact deterministic, for example, taking $a = 0$, by (18.9) we have

$$d_t f(t,T) = \sigma^2 \mathrm{e}^{-(T-t)b} \int_t^T \mathrm{e}^{(t-s)b} ds dt + \sigma \mathrm{e}^{-(T-t)b} dB_t,$$

i.e.

$$\alpha(t,T) = \sigma^2 \mathrm{e}^{-(T-t)b} \int_t^T \mathrm{e}^{(t-s)b} ds = \sigma^2 \mathrm{e}^{-(T-t)b} \frac{1 - \mathrm{e}^{-(T-t)b}}{b},$$

and $\sigma(t,T) = \sigma \mathrm{e}^{-(T-t)b}$, and the HJM condition reads

$$\alpha(t,T) = \sigma^2 \mathrm{e}^{-(T-t)b} \int_t^T \mathrm{e}^{(t-s)b} ds = \sigma(t,T) \int_t^T \sigma(t,s) ds. \tag{18.32}$$

Random simulations of the Vasicek instantaneous forward rates are provided in Figures 18.10 and 18.11 using the Musiela convention (18.26).

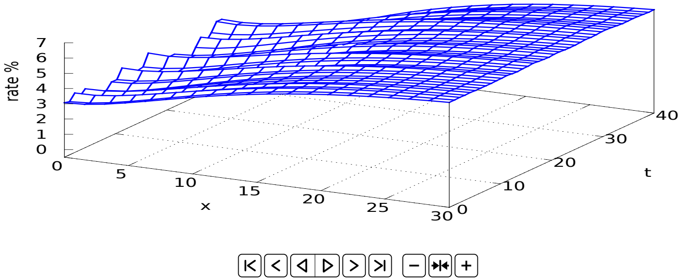

Fig. 18.10: Forward instantaneous curve $(t,x) \mapsto f(t,t+x)$ in the Vasicek model.*

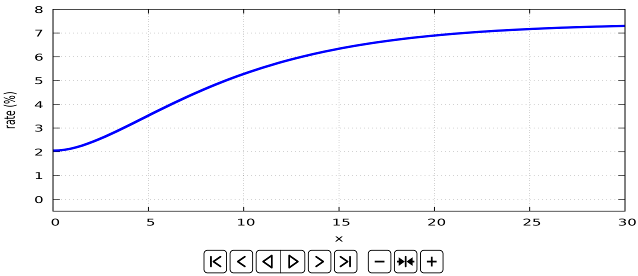

Fig. 18.11: Forward instantaneous curve $x \mapsto f(0,x)$ in the Vasicek model.†

This version: May 3, 2024
https://personal.ntu.edu.sg/nprivault/indext.html

For $x = 0$ the first "slice" of this surface is actually the short rate Vasicek process $r_t = f(t,t) = f(t,t+0)$ which is represented in Figure 18.12 using another discretization.

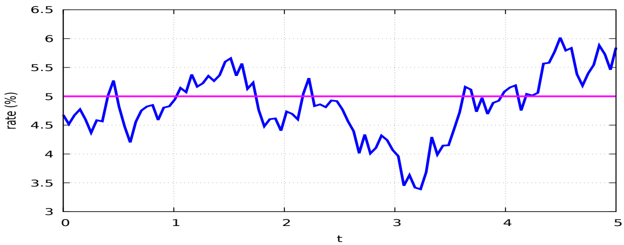

Fig. 18.12: Short-term interest rate curve $t \mapsto r_t$ in the Vasicek model.

HJM-SOFR Model

In the HJM-SOFR model, the instantaneous forward rate $f(t,T)$ is extended to $t > T$ by taking

$$d_t f(t,T) = \mathbb{1}_{[0,T]}(t)\alpha(t,T)dt + \mathbb{1}_{[0,T]}(t)\sigma(t,T)dB_t, \qquad t \geqslant T,$$

i.e.

$$f(t,T) = f(T,T) = r_T, \qquad t \geqslant T,$$

see Lyashenko and Mercurio (2020).

18.4 Yield Curve Modeling

Nelson-Siegel parametrization of instantaneous forward rates

In the Nelson and Siegel (1987) parametrization the instantaneous forward rate curves are parametrized by 4 coefficients z_1, z_2, z_3, z_4, as

$$g(x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4}, \qquad x \geqslant 0.$$

An example of graph of forward rate $f(t,T,T+x) = g(x)$ obtained by the Nelson-Siegel parametrization is given in Figure 18.13, for $z_1 = 1$, $z_2 = -10$, $z_3 = 100$, $z_4 = 10$.

† The animation works in Acrobat Reader on the entire pdf file.

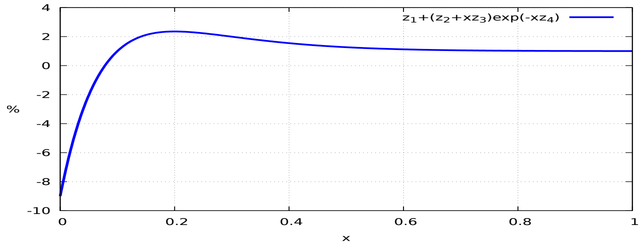

Fig. 18.13: Graph of $x \mapsto g(x)$ in the Nelson-Siegel model.

Svensson parametrization of instantaneous forward rates

The Svensson (1994) parametrization has the advantage to reproduce two humps instead of one, the location and height of which can be chosen *via* 6 parameters z_1, z_2, z_3, z_4, z_5, z_6 as

$$g(x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4} + z_5 x\,\mathrm{e}^{-xz_6}, \qquad x \geqslant 0.$$

An typical example of graph of forward rate $f(t,T,T+x) = g(x)$ obtained by the Svensson parametrization is given in Figure 18.14, for $z_1 = 6.6$, $z_2 = -5$, $z_3 = -100$, $z_4 = 10$, $z_5 = -1/2$, $z_6 = 1$.

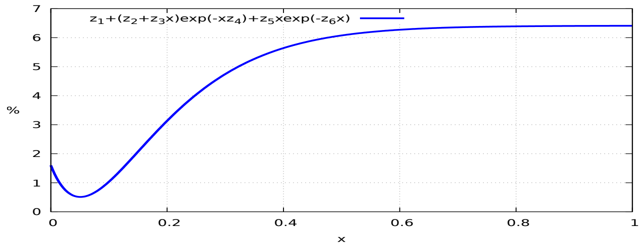

Fig. 18.14: Graph of $x \mapsto g(x)$ in the Svensson model.

Figure 18.15 presents a fit of the market data of Figure 18.1 using a Svensson curve.

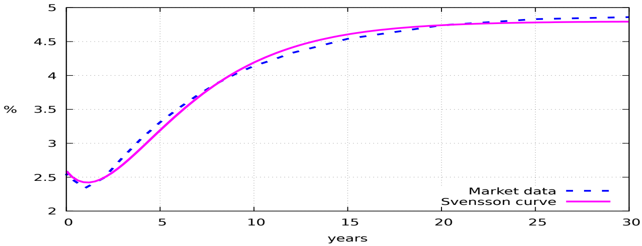

Fig. 18.15: Fitting of a Svensson curve to market data.

The attached **IPython notebook** can be run **here** or **here** to fit a Svensson curve to market data.

Vasicek parametrization

In the Vasicek model, the instantaneous forward rate process is given from (18.9) and (18.26) as

$$f(t,T) = \frac{a}{b} - \frac{\sigma^2}{2b^2} + \left(r_t - \frac{a}{b} + \frac{\sigma^2}{b^2}\right) \mathrm{e}^{-bx} - \frac{\sigma^2}{2b^2} \mathrm{e}^{-2bx}, \qquad (18.33)$$

in the Musiela notation ($x = T - t$), and we have

$$\frac{\partial f}{\partial T}(t,T) = \left(a - br_t - \frac{\sigma^2}{b}(1 - \mathrm{e}^{-(T-t)b})\right) \mathrm{e}^{-(T-t)b}.$$

We check that the derivative $\partial f / \partial T$ vanishes when $a - br_t + a - \sigma^2(1 - \mathrm{e}^{-bx})/b = 0$, *i.e.*

$$\mathrm{e}^{-bx} = 1 + \frac{b}{\sigma^2}(br_t - a),$$

which admits at most one solution, provided that $a > br_t$. As a consequence, the possible forward curves in the Vasicek model are limited to one change of "regime" per curve, as illustrated in Figure 18.16 for various values of r_t, and in Figure 18.17.

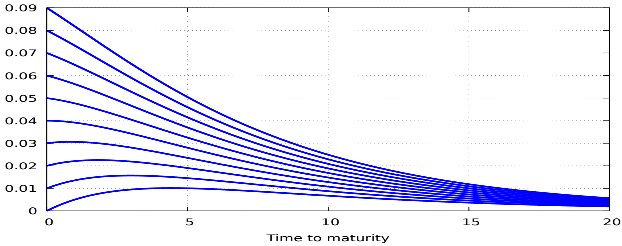

Fig. 18.16: Graphs of forward rates with $b = 0.16$, $a/b = 0.04$, $r_0 = 2\%$, $\sigma = 4.5\%$.

The next Figure 18.17 is also using the parameters $b = 0.16$, $a/b = 0.04$, $r_0 = 2\%$, and $\sigma = 4.5\%$.

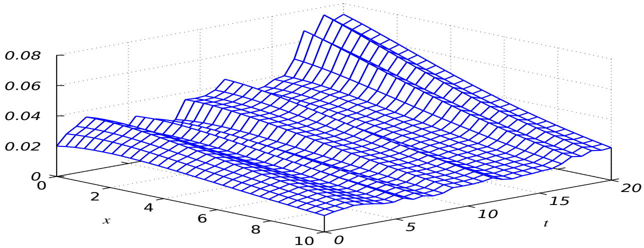

Fig. 18.17: Forward instantaneous curve $(t, x) \mapsto f(t, t + x)$ in the Vasicek model.

One may think of constructing an instantaneous forward rate process taking values in the Svensson space, however this type of modeling is not consistent with absence of arbitrage, and it can be proved that the HJM curves cannot live in the Nelson-Siegel or Svensson spaces, see §3.5 of Björk (2004b). In other words, it can be shown that the forward yield curves produced by the Vasicek model are included neither in the Nelson-Siegel space, nor in the Svensson space. In addition, the Vasicek yield curves do not appear to correctly model the market forward curves cf. also Figure 18.1 above.

Another way to deal with the curve fitting problem is to use deterministic shifts for the fitting of one forward curve, such as the initial curve at $t = 0$, cf. *e.g.* § 6.3 in Privault (2021b).

Fitting the Nelson-Siegel and Svensson models to yield curve data

Recall that in the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by four coefficients z_1, z_2, z_3, z_4, as

$$f(t,t+x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4}, \qquad x \geqslant 0. \tag{18.34}$$

Taking $x = T - t$, the yield $f(t,t,T)$ is given from (18.7) as

$$\begin{aligned} f(t,t,T) &= \frac{1}{T-t}\int_t^T f(t,s)ds \\ &= \frac{1}{x}\int_0^x f(t,t+y)dy \\ &= z_1 + \frac{z_2}{x}\int_0^x \mathrm{e}^{-yz_4}dy + \frac{z_3}{x}\int_0^x y\,\mathrm{e}^{-yz_4}dy \\ &= z_1 + z_2\frac{1-\mathrm{e}^{-xz_4}}{xz_4} + z_3\frac{1-\mathrm{e}^{-xz_4}+x\,\mathrm{e}^{-xz_4}}{xz_4}. \end{aligned}$$

The yield $f(t,t,T)$ can then be reparametrized as

$$f(t,t+x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4} = \beta_0 + \beta_1\,\mathrm{e}^{-x/\lambda} + \frac{\beta_2}{\lambda}x\,\mathrm{e}^{-x/\lambda}, \qquad x \geqslant 0,$$

see Charpentier (2014), with $\beta_0 = z_1$, $\beta_1 = z_2$, $\beta_2 = z_3/z_4$, $\lambda = 1/z_4$, and similarly in the Svensson model.

```
require(YieldCurve);data(ECBYieldCurve)
mat.ECB<-c(3/12,0.5,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,
      24,25,26,27,28,29,30)
first(ECBYieldCurve, '1 month');Nelson.Siegel(first(ECBYieldCurve, '1 month'), mat.ECB)
```

```
for (n in seq(from=70, to=290, by=10)) {
ECB.NS <- Nelson.Siegel(ECBYieldCurve[n,], mat.ECB)
ECB.S <- Svensson(ECBYieldCurve[n,], mat.ECB)
ECB.NS.yield.curve <- NSrates(ECB.NS, mat.ECB)
ECB.S.yield.curve <- Srates(ECB.S, mat.ECB,"Spot")
plot(mat.ECB, as.numeric(ECBYieldCurve[n,]), type="o", lty=1, col=1,ylab="Interest rates",
      xlab="Maturity in years", ylim=c(3.2,4.8),cex.lab=1.6,cex.axis=1.6)
lines(mat.ECB, as.numeric(ECB.NS.yield.curve), type="l", lty=3,col=2,lwd=2)
lines(mat.ECB, as.numeric(ECB.S.yield.curve), type="l", lty=2,col=6,lwd=2)
title(main=paste("ECB yield curve observed at",time(ECBYieldCurve[n], sep=" "),"vs fitted
      yield curve"))
legend('bottomright', legend=c("ECB data","Nelson-Siegel","Svensson"),col=c(1,2,6), lty=1,
      bg='gray90')
grid();}
```

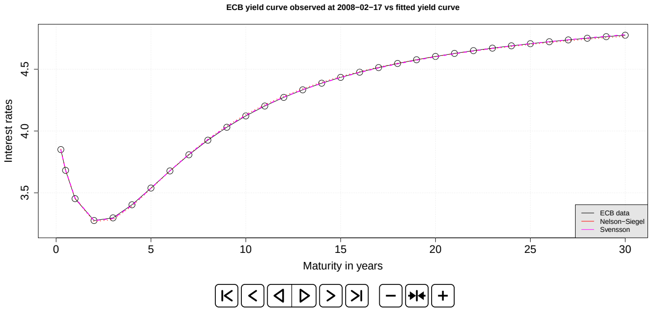

Fig. 18.18: ECB data *vs.* fitted yield curve.*

18.5 Two-Factor Model

The correlation problem is another issue of concern when using the affine models considered so far, see (17.9) and (17.28). Let us compare three bond price simulations with maturity $T_1 = 10$, $T_2 = 20$, and $T_3 = 30$ based on the same Brownian path, as given in Figure 18.19. Clearly, the bond prices

$$F(r_t, T_i) = P(t, T_i) = \mathrm{e}^{A(t,T_i)+r_t C(t,T_i)}, \qquad 0 \leqslant t \leqslant T_i, \quad i = 1, 2,$$

with maturities T_1 and T_2 are linked by the relation

$$P(t, T_2) = P(t, T_1) \exp\big(A(t, T_2) - A(t, T_1) + r_t(C(t, T_2) - C(t, T_1))\big), \tag{18.35}$$

meaning that bond prices with different maturities could be deduced from each other, which is unrealistic.

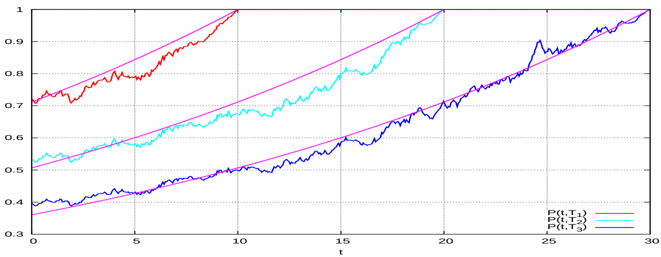

Fig. 18.19: Graph of $t \mapsto P(t, T_1), P(t, T_2), P(t, T_3)$.

* The animation works in Acrobat Reader on the entire pdf file.

In affine short rate models, by (18.35), $\log P(t,T_1)$ and $\log P(t,T_2)$ are linked by the affine relationship

$$\log P(t,T_2) = \log P(t,T_1) + A(t,T_2) - A(t,T_1) + r_t(C(t,T_2) - C(t,T_1))$$
$$= \log P(t,T_1) + A(t,T_2) - A(t,T_1) + (C(t,T_2) - C(t,T_1))\frac{\log P(t,T_1) - A(t,T_1)}{C(t,T_1)}$$
$$= \left(1 + \frac{C(t,T_2) - C(t,T_1)}{A(t,T_1)}\right)\log P(t,T_1) + A(t,T_2) - A(t,T_1)\frac{C(t,T_2)}{C(t,T_1)}$$

with constant coefficients, which yields the perfect correlation or anticorrelation

$$\mathrm{Cor}(\log P(t,T_1), \log P(t,T_2)) = \pm 1,$$

depending on the sign of the coefficient $1 + (C(t,T_2) - C(t,T_1))/A(t,T_1)$, cf. § 6.4 in Privault (2021b),

A solution to the correlation problem is to consider a two-factor model based on two state processes $(X_t)_{t\in\mathbb{R}_+}$, $(Y_t)_{t\in\mathbb{R}_+}$ which are solution of

$$\begin{cases} dX_t = \mu_1(t,X_t)dt + \sigma_1(t,X_t)dB_t^{(1)}, \\ \\ dY_t = \mu_2(t,Y_t)dt + \sigma_2(t,Y_t)dB_t^{(2)}, \end{cases} \tag{18.36}$$

where $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$, $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are correlated Brownian motion, with

$$\mathrm{Cov}\left(B_s^{(1)}, B_t^{(2)}\right) = \rho \min(s,t), \qquad s,t \geqslant 0, \tag{18.37}$$

and

$$dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt, \tag{18.38}$$

for some correlation parameter $\rho \in [-1,1]$. In practice, $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ can be constructed from two independent Brownian motions $\big(W_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(W_t^{(2)}\big)_{t\in\mathbb{R}_+}$, by letting

$$\begin{cases} B_t^{(1)} = W_t^{(1)}, \\ \\ B_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2}W_t^{(2)}, \qquad t \geqslant 0, \end{cases}$$

and Relations (18.37) and (18.38) are easily satisfied from this construction.

In two-factor models one chooses to build the short-term interest rate r_t *via*

$$r_t := X_t + Y_t, \qquad t \geqslant 0.$$

By the previous standard arbitrage arguments we define the price of a bond with maturity T as

$$\begin{aligned}
P(t,T) &:= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| X_t,\ Y_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T (X_s+Y_s) ds\right) \Big| X_t,\ Y_t\right] \\
&= F(t, X_t, Y_t), \qquad (18.39)
\end{aligned}$$

since the couple $(X_t, Y_t)_{t\in\mathbb{R}_+}$ is Markovian. Applying the Itô formula with two variables to

$$t \mapsto F(t, X_t, Y_t) = P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right],$$

and using the fact that the discounted process

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_0^T r_s ds\right) \Big| \mathcal{F}_t\right]$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, we can derive the PDE

$$\begin{aligned}
&-(x+y)F(t,x,y) + \mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y) + \mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y) \\
&+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y) \\
&+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x \partial y}(t,x,y) + \frac{\partial F}{\partial t}(t,x,y) = 0, \qquad (18.40)
\end{aligned}$$

on $\mathbb{R}^2$ for the bond price $P(t,T)$. In the Vasicek model

$$\begin{cases} dX_t = -aX_t dt + \sigma dB_t^{(1)}, \\ dY_t = -bY_t dt + \eta dB_t^{(2)}, \end{cases}$$

this yields the solution $F(t,x,y)$ of (18.40) as

$$P(t,T) = F(t, X_t, Y_t) = F_1(t, X_t) F_2(t, Y_t) \exp\left(\rho U(t,T)\right), \qquad (18.41)$$

where $F_1(t, X_t)$ and $F_2(t, Y_t)$ are the bond prices associated to X_t and Y_t in the Vasicek model, and

$$U(t,T) := \frac{\sigma\eta}{ab}\left(T-t+\frac{\mathrm{e}^{-(T-t)a}-1}{a}+\frac{\mathrm{e}^{-(T-t)b}-1}{b}-\frac{\mathrm{e}^{-(a+b)(T-t)}-1}{a+b}\right)$$

is a correlation term which vanishes when $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are independent, *i.e.* when $\rho=0$, cf. Ch. 4, Appendix A in Brigo and Mercurio (2006), § 6.5 of Privault (2021b).

Partial differentiation of $\log P(t,T)$ with respect to T leads to the instantaneous forward rate

$$f(t,T) = f_1(t,T)+f_2(t,T)-\rho\frac{\sigma\eta}{ab}\big(1-\mathrm{e}^{-(T-t)a}\big)\big(1-\mathrm{e}^{-(T-t)b}\big), \quad (18.42)$$

where $f_1(t,T)$, $f_2(t,T)$ are the instantaneous forward rates corresponding to X_t and Y_t respectively, cf. § 6.5 of Privault (2021b).

An example of a forward rate curve obtained in this way is given in Figure 18.20.

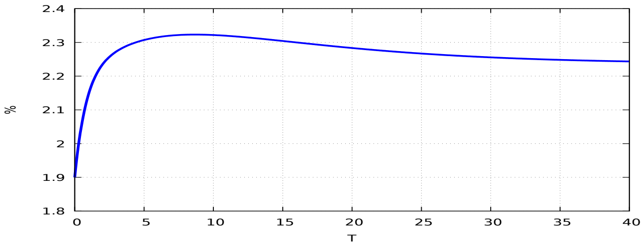

Fig. 18.20: Graph of forward rates in a two-factor model.

Next, in Figure 18.21 we present a graph of the evolution of forward curves in a two-factor model.

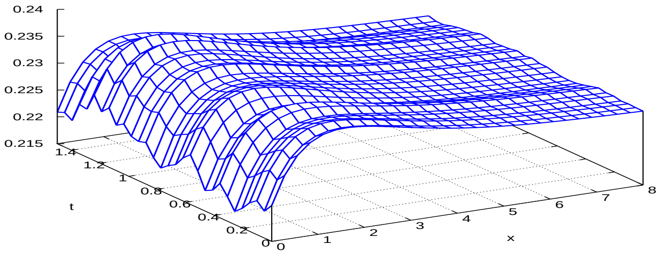

Fig. 18.21: Random evolution of instantaneous forward rates in a two-factor model.

18.6 The BGM Model

The models (HJM, affine, etc.) considered in the previous chapter suffer from various drawbacks such as nonpositivity of interest rates in Vasicek model, and lack of closed-form solutions in more complex models. The Brace et al. (1997) (BGM) model has the advantage of yielding positive interest rates, and to permit to derive explicit formulas for the computation of prices for interest rate derivatives such as interest rate caps and swaptions on the LIBOR market.

In the BGM model we consider two bond prices $P(t, T_1)$, $P(t, T_2)$ with maturities T_1, T_2, and the forward probability measure $\widehat{\mathbb{P}}_2$ defined as

$$\frac{d\widehat{\mathbb{P}}_2}{d\mathbb{P}^*} = \frac{\mathrm{e}^{-\int_0^{T_2} r_s ds}}{P(0, T_2)},$$

with numéraire $P(t, T_2)$, cf. (16.10). The forward LIBOR rate $L(t, T_1, T_2)$ is modeled as a driftless geometric Brownian motion under $\widehat{\mathbb{P}}_2$, *i.e.*

$$\frac{dL(t, T_1, T_2)}{L(t, T_1, T_2)} = \gamma_1(t) dB_t, \tag{18.43}$$

$0 \leqslant t \leqslant T_1$, for some deterministic volatility function of time $\gamma_1(t)$, with solution

$$L(u, T_1, T_2) = L(t, T_1, T_2) \exp\left(\int_t^u \gamma_1(s) dB_s - \frac{1}{2} \int_t^u |\gamma_1|^2(s) ds\right),$$

i.e. for $u = T_1$,

$$L(T_1, T_1, T_2) = L(t, T_1, T_2) \exp\left(\int_t^{T_1} \gamma_1(s) dB_s - \frac{1}{2} \int_t^{T_1} |\gamma_1|^2(s) ds\right).$$

Since $L(t, T_1, T_2)$ is a geometric Brownian motion under $\widehat{\mathbb{P}}_2$, standard caplets can be priced at time $t \in [0, T_1]$ from the Black-Scholes formula.

In Table 18.1 we summarize some stochastic models used for interest rates.

	Model
Short rate r_t	Mean reverting SDEs
Instantaneous forward rate $f(t, s)$	HJM model
Forward rate $f(t, T, S)$	BGM model

Table 18.1: Stochastic interest rate models.

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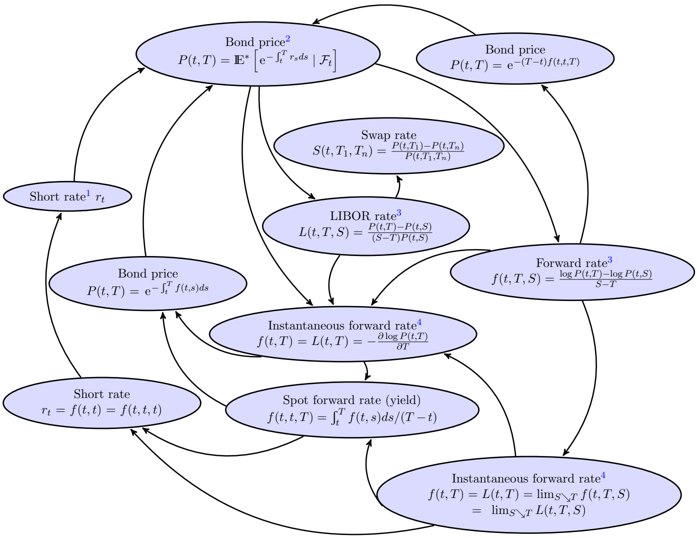
https://personal.ntu.edu.sg/nprivault/indext.html

The following Graph 18.22 summarizes the notions introduced in this chapter.



[1]Can be modeled by Vasiçek and other short rate models
[2]Can be modeled from $dP(t,T)/P(t,T)$.
[3]Can be modeled in the BGM model
[4]Can be modeled in the HJM model

Fig. 18.22: Roadmap of stochastic interest rate modeling.

Exercises

Exercise 18.1 We consider a bond with maturity T, priced $P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \;\middle|\; \mathcal{F}_t\right]$ at time $t \in [0,T]$.

a) Using the forward measure $\widehat{\mathbb{P}}$ with numéraire $N_t = P(t,T)$, apply the change of numéraire formula (16.9) to compute the derivative $\frac{\partial P}{\partial T}(t,T)$.
b) Using Relation (18.5), find an expression of the instantaneous forward rate $f(t,T)$ using the short rate r_T and the forward expectation $\widehat{\mathbb{E}}$.
c) Show that the instantaneous forward rate $(f(t,T))_{t\in[0,T]}$ is a martingale under the forward measure $\widehat{\mathbb{P}}$.

Exercise 18.2 Consider a tenor structure $\{T_1, T_2\}$ and a bond with maturity T_2 and price given at time $t \in [0, T_2]$ by

$$P(t, T_2) = \exp\left(-\int_t^{T_2} f(t,s)ds\right), \qquad t \in [0, T_2],$$

where the instantaneous yield curve $f(t,s)$ is parametrized as

$$f(t,s) = r_1 \mathbb{1}_{[0,T_1]}(s) + r_2 \mathbb{1}_{[T_1,T_2]}(s), \qquad t \leqslant s \leqslant T_2.$$

Find a formula to estimate the values of r_1 and r_2 from the data of $P(0, T_2)$ and $P(T_1, T_2)$.

Same question when $f(t,s)$ is parametrized as

$$f(t,s) = r_1 s \mathbb{1}_{[0,T_1]}(s) + (r_1 T_1 + (s - T_1) r_2) \mathbb{1}_{[T_1,T_2]}(s), \quad t \leqslant s \leqslant T_2.$$

Exercise 18.3 (El Karoui et al. (1997)) Consider a short term interest rate process $(r_t)_{t\in[0,T]}$ and a bond priced $P(t,T)$ at time $t \in [0,T]$.

a) Using Jensen's inequality, find an inequality between

   a) the yield $y(t,T) = f(t,t,T)$, and
   b) the average short rate $\displaystyle\frac{1}{T-t}\int_t^T r_s ds$.

b) Show that in the Vasicek model in which the short-term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ solves the equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \tag{18.44}$$

where $a, \sigma \in \mathbb{R}$, $b > 0$, and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, we have

$$\lim_{T\to\infty} \frac{1}{T-t}\int_t^T \mathbb{E}[r_s \mid \mathcal{F}_t] ds = \frac{a}{b} \quad t \geqslant 0,$$

and

$$\lim_{T\to\infty} y(t,T) = \frac{a}{b} - \frac{\sigma^2}{2b^2}.$$

Exercise 18.4 (Exercise 4.17 continued). Bridge model. Assume that the price $P(t,T)$ of a zero-coupon bond with maturity $T > 0$ is modeled as

$$P(t,T) = \mathrm{e}^{-\mu(T-t)+X_t^T}, \qquad t \in [0,T],$$

where $(X_t^T)_{t\in[0,T)}$ is the solution of the stochastic differential equation

$$dX_t^T = \sigma dB_t - \frac{X_t^T}{T-t}dt, \qquad t \in [0,T),$$

under the initial condition $X_0^T = 0$, *i.e.*

$$X_t^T = (T-t)\int_0^t \frac{\sigma}{T-s} dB_s, \qquad 0 \leqslant t < T,$$

with $\mu, \sigma > 0$.

a) Show that the terminal condition $P(T,T) = 1$ is satisfied.
b) Compute the forward rate

$$f(t,T,S) = -\frac{1}{S-T}(\log P(t,S) - \log P(t,T)).$$

c) Compute the instantaneous forward rate

$$f(t,T) = -\lim_{S\searrow T} \frac{1}{S-T}(\log P(t,S) - \log P(t,T)).$$

d) Show that the limit $\lim_{T\searrow t} f(t,T)$ does not exist in $L^2(\Omega)$.
e) Show that $P(t,T)$ satisfies the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + \frac{\sigma^2}{2}dt - \frac{\log P(t,T)}{T-t}dt, \qquad t \in [0,T].$$

f) Rewrite the equation of Question (e) as

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + r_t^T dt, \qquad t \in [0,T],$$

where $(r_t^{\mathrm{T}})_{t\in[0,T]}$ is a process to be determined.
g) Show that we have the expression

$$P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s^{\mathrm{T}} ds} \,\middle|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

h) Compute the conditional Radon-Nikodym density

$$\mathbb{E}^*\left[\frac{\mathrm{d}\widehat{\mathbb{P}}_T}{\mathrm{d}\mathbb{P}^*} \,\middle|\, \mathcal{F}_t \right] = \frac{P(t,T)}{P(0,T)} \mathrm{e}^{-\int_0^t r_s^{\mathrm{T}} ds}$$

of the forward measure $\widehat{\mathbb{P}}_T$ with respect to $\mathbb{P}^*$.
i) Show that the process

$$\widehat{B}_t := B_t - \sigma t, \qquad 0 \leqslant t \leqslant T,$$

is a standard Brownian motion under $\widehat{\mathbb{P}}_T$.
j) Compute the dynamics of X_t^S and $P(t,S)$ under $\widehat{\mathbb{P}}_T$.
Hint: Show that

$$-\mu(S-T)+\sigma(S-T)\int_0^t \frac{1}{S-s}dB_s = \frac{S-T}{S-t}\log P(t,S).$$

k) Compute the bond option price

$$\mathbb{E}^*\left[e^{-\int_t^T r_s^{\mathrm{T}} ds}(P(T,S)-K)^+ \;\Big|\; \mathcal{F}_t\right] = P(t,T)\widehat{\mathbb{E}}_T\big[(P(T,S)-K)^+ \mid \mathcal{F}_t\big],$$

$0 \leqslant t < T < S$.

Hint: Given X a Gaussian random variable with mean m and variance v^2 given $\mathcal{F}_t$, we have:

$$\begin{aligned}\mathbb{E}\big[\big(e^X-\kappa\big)^+ \mid \mathcal{F}_t\big] = {} & e^{m+v^2/2}\Phi\left(\frac{1}{v}(m+v^2-\log\kappa)\right) \\ & -\kappa\Phi\left(\frac{1}{v}(m-\log\kappa)\right).\end{aligned}$$

Exercise 18.5 Consider a short rate process $(r_t)_{t\in\mathbb{R}_+}$ of the form $r_t = h(t)+X_t$, where $h(t)$ is a deterministic function of time and $(X_t)_{\mathbb{R}_+}$ is a Vasicek process started at $X_0=0$.

a) Compute the price $P(0,T)$ at time $t=0$ of a bond with maturity T, using $h(t)$ and the function $A(T)$ defined in (17.35) for the pricing of Vasicek bonds.
b) Show how the function $h(t)$ can be estimated from the market data of the initial instantaneous forward rate curve $f(0,t)$.

Exercise 18.6 (Exercise 4.14 continued). Consider two assets whose prices $S_t^{(1)}$, $S_t^{(2)}$ at time $t\in[0,T]$ follow the Bachelier dynamics

$$dS_t^{(1)} = rS_t^{(1)}dt+\sigma_1 dW_t^{(1)} \quad dS_t^{(2)} = rS_t^{(2)}dt+\sigma_2 dW_t^{(2)} \quad t\in[0,T],$$

where $\big(W_t^{(1)}\big)_{t\in[0,T]}$, $\big(W_t^{(2)}\big)_{t\in[0,T]}$ are two standard Brownian motions with correlation $\rho\in[-1,1]$ under a risk-neutral probability measure $\mathbb{P}^*$.

Compute the price $e^{-rT}\mathbb{E}^*[(S_T-K)^+]$ of the spread option on $S_T := S_T^{(2)}-S_T^{(1)}$ with maturity $T>0$ and strike price $K>0$.

Exercise 18.7

a) Given two LIBOR spot rates $R(t,t,T)$ and $R(t,t,S)$, express the LIBOR forward rate $R(t,T,S)$ in terms of $R(t,t,T)$ and $R(t,t,S)$.
b) Assuming $t=0$, $T=1$ year, $S=2$ years, $R(0,0,T)=2\%$, $R(0,0,S)=2.5\%$ would you sign a LIBOR forward rate agreement at $t=0$ with rate $R(0,T,S)$ over $[T,S]$ if you believe that $R(T,T,S)$ will remain at the

level $R(T,T,S) = R(0,0,T) = 2\%$?

Exercise 18.8 Consider a bond market with two bonds with maturities T, S, whose prices $P(t,T), P(t,S)$ at time t are given by

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \zeta_T(t) dW_t, \qquad \frac{dP(t,S)}{P(t,S)} = r_t dt + \zeta_S(t) dW_t,$$

where $(r_t)_{t\in\mathbb{R}_+}$ is a short-term interest rate process, $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and $\zeta_T(t), \zeta_S(t)$ are volatility processes. Compute the coefficients μ_t and σ_t in the stochastic differential equation

$$\frac{dL(t,T,S)}{L(t,T,S)} = \mu_t dt + \sigma_t dW_t$$

satisfied by the LIBOR rate

$$L(t,T,S) := \frac{P(t,T) - P(t,S)}{P(t,S)}.$$

Exercise 18.9 (Exercise 17.5 continued).

a) Compute the forward rate $f(t,T,S)$ in the Ho-Lee model (17.50) with constant deterministic volatility.

   In the next questions we take $a = 0$.
b) Compute the instantaneous forward rate $f(t,T)$ in this model.
c) Derive the stochastic equation satisfied by the instantaneous forward rate $f(t,T)$.
d) Check that the HJM absence of arbitrage condition is satisfied in this equation.

Exercise 18.10 Consider the two-factor Vasicek model

$$\begin{cases} dX_t = -bX_t dt + \sigma dB_t^{(1)}, \\ \\ dY_t = -bY_t dt + \sigma dB_t^{(2)}, \end{cases}$$

where $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$, $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are correlated Brownian motion such that $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$, for $\rho \in [-1,1]$.

a) Write down the expressions of the short rates X_t and Y_t.

   Hint: They can be found in Section 17.1.
b) Compute the variances $\mathrm{Var}[X_t]$, $\mathrm{Var}[Y_t]$, and the covariance $\mathrm{Cov}(X_t, Y_t)$.

   Hint: The expressions of $\mathrm{Var}[X_t]$ and $\mathrm{Var}[Y_t]$ can be found in Section 17.1.

c) Compute the covariance $\text{Cov}(\log P(t,T_1), \log P(t,T_2))$ for the two-factor bond prices

$$P(t,T_1) = F_1(t,X_t,T_1)F_2(t,Y_t,T_1)\,\mathrm{e}^{\rho U(t,T_1)}$$

and

$$P(t,T_2) = F_1(t,X_t,T_2)F_2(t,Y_t,T_2)\,\mathrm{e}^{\rho U(t,T_2)},$$

where

$$\log F_1(t,x,T) = C_1^T + xA_1^T \quad \text{and} \quad \log F_2(t,x) = C_2^T + xA_2^T.$$

Hint: We have $\text{Cov}(X+Y,Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z)$ and $\text{Cov}(c,X) = 0$ when c is a constant.

Exercise 18.11 Stochastic string model (Santa-Clara and Sornette (2001)). Consider an instantaneous forward rate $f(t,x)$ solution of

$$d_t f(t,x) = \alpha x^2 dt + \sigma d_t B(t,x), \tag{18.45}$$

with a flat initial curve $f(0,x) = r$, where x represents the time to maturity, and $(B(t,x))_{(t,x)\in\mathbb{R}_+^2}$ is a standard *Brownian sheet* with covariance

$$\mathbb{E}[B(s,x)B(t,y)] = (\min(s,t))(\min(x,y)), \qquad s,t,x,y \geqslant 0, \tag{18.46}$$

and initial conditions $B(t,0) = B(0,x) = 0$ for all $t,x \geqslant 0$.

a) Solve the equation (18.45) for $f(t,x)$.
b) Compute the short-term interest rate $r_t = f(t,0)$.
c) Compute the value at time $t \in [0,T]$ of the bond price

$$P(t,T) = \exp\left(-\int_0^{T-t} f(t,x)dx\right)$$

with maturity T.

d) Compute the variance $\mathbb{E}\left[\left(\int_0^{T-t} B(t,x)dx\right)^2\right]$ of the centered Gaussian random variable $\int_0^{T-t} B(t,x)dx$.
e) Compute the expected value $\mathbb{E}^*[P(t,T)]$.
f) Find the value of α such that the discounted bond price

$$\mathrm{e}^{-rt}P(t,T) = \exp\left(-rT - \frac{\alpha}{3}t(T-t)^3 - \sigma\int_0^{T-t} B(t,x)dx\right), \quad t \in [0,T].$$

satisfies $\mathbb{E}^*[P(t,T)] = \mathrm{e}^{-(T-t)r}$.

g) Compute the bond option price $\mathbb{E}^*\left[\exp\left(-\int_0^T r_s ds\right)(P(T,S)-K)^+\right]$ by the Black-Scholes formula, knowing that for any centered Gaussian random variable $X \simeq \mathcal{N}(0,v^2)$ with variance v^2 we have

$$\begin{aligned}&\mathbb{E}[(x\,\mathrm{e}^{m+X}-K)^+]\\&=x\,\mathrm{e}^{m+v^2/2}\Phi(v+(m+\log(x/K))/v)-K\Phi((m+\log(x/K))/v).\end{aligned}$$