Chapter 5
Continuous-Time Market Model

The continuous-time market model allows for the incorporation of portfolio re-allocation algorithms in a stochastic dynamic programming setting. This chapter starts with a review of the concepts of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also state and solve the equation satisfied by geometric Brownian motion, which will be used for the modeling of continuous asset price processes.

5.1 Asset Price Modeling

The prices at time $t \geqslant 0$ of $d+1$ assets numbered n^o $0, 1, \ldots, d$ is denoted by the *random* vector

$$\overline{S}_t = \big(S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)}\big)$$

which forms a stochastic process $(\overline{S}_t)_{t\in\mathbb{R}_+}$. As in discrete time, the asset n^o 0 is a riskless asset (of savings account type) yielding an interest rate r, *i.e.* we have

$$S_t^{(0)} = S_0^{(0)} \,\mathrm{e}^{rt}, \qquad t \geqslant 0.$$

Definition 5.1. Discounting. *Let*

$$\overline{X}_t := \big(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\big), \qquad t \in \mathbb{R},$$

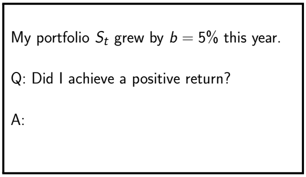
denote the vector of discounted asset prices, defined as:

$$\widetilde{S}_t^{(k)} = \mathrm{e}^{-rt} S_t^{(k)}, \qquad t \geqslant 0, \quad k = 0, 1, \ldots, d.$$

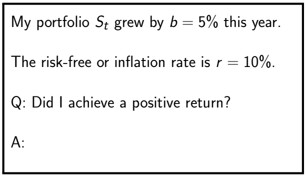
We can also write

$$\overline{X}_t := \mathrm{e}^{-rt} \overline{S}_t, \qquad t \geqslant 0.$$

The concept of discounting is illustrated in the following figures.



(a) Scenario A.



(b) Scenario B.

(a) *Without* inflation adjustment.

(b) *With* inflation adjustment.

Fig. 5.1: Why apply discounting?

Definition 5.2. *A portfolio strategy is a stochastic process* $(\bar{\xi}_t)_{t\in\mathbb{R}_+} \subset \mathbb{R}^{d+1}$*, where* $\xi_t^{(k)}$ *denotes the (possibly fractional) quantity of asset n^o k held at time* $t \geqslant 0$.

The *value* at time $t \geqslant 0$ of the portfolio strategy $(\bar{\xi}_t)_{t\in\mathbb{R}_+} \subset \mathbb{R}^{d+1}$ is defined by

$$V_t := \bar{\xi}_t \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \geqslant 0.$$

The *discounted* value of the portfolio is defined by

$$\begin{aligned} \widetilde{V}_t &:= \mathrm{e}^{-rt} V_t \\ &= \mathrm{e}^{-rt} \bar{\xi}_t \cdot \overline{S}_t \end{aligned}$$

$$\begin{aligned}
&= \mathrm{e}^{-rt} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \\
&= \sum_{k=0}^{d} \xi_t^{(k)} \widetilde{S}_t^{(k)} \\
&= \bar{\xi}_t \cdot \overline{X}_t, \qquad t \geqslant 0.
\end{aligned}$$

The effect of discounting from time t to time 0 is to divide prices by e^{rt}, making all prices comparable at time 0.

5.2 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the one-step and discrete-time models. In what follows we will only consider *admissible* portfolio strategies whose total value V_t remains nonnegative for all times $t \in [0, T]$.

Definition 5.3. *A portfolio strategy* $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ *with value*

$$V_t = \bar{\xi}_t \cdot \overline{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in [0, T],$$

constitutes an arbitrage opportunity if all three *following conditions are satisfied:*

i) $V_0 \leqslant 0$ *at time* $t = 0$, [Start from a zero-cost portfolio, or with a debt.]
ii) $V_T \geqslant 0$ *at time* $t = T$, [Finish with a nonnegative amount.]
iii) $\mathbb{P}(V_T > 0) > 0$ *at time* $t = T$. [Profit is made with nonzero probability.]

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with zero capital or even with a debt.

Next, we turn to the definition of risk-neutral probability measures (or martingale measures) in continuous time, which states that under a risk-neutral probability measure $\mathbb{P}^*$, the return of the risky asset over the time interval $[u, t]$ equals the return of the riskless asset given by

$$S_t^{(0)} = \mathrm{e}^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t.$$

Recall that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, *i.e.*

$$\mathcal{F}_t = \sigma(B_u \ : \ 0 \leqslant u \leqslant t), \qquad t \geqslant 0.$$

Definition 5.4. *A probability measure* $\mathbb{P}^*$ *on* Ω *is called a risk-neutral measure if it satisfies*

$$\mathbb{E}^*\big[S_t^{(k)} \,\big|\, \mathcal{F}_u\big] = \mathrm{e}^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d. \tag{5.1}$$

where $\mathbb{E}^*$ *denotes the expectation under* $\mathbb{P}^*$*.*

As in the discrete-time case, $\mathbb{P}^\sharp$ would be called a risk premium measure if it satisfied

$$\mathbb{E}^\sharp\big[S_t^{(k)} \,\big|\, \mathcal{F}_u\big] > \mathrm{e}^{(t-u)r} S_u, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d,$$

meaning that by taking risks in buying $S_t^{(i)}$, one could make an expected return higher than that of the riskless asset

$$S_t^{(0)} = \mathrm{e}^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t.$$

Similarly, a negative risk premium measure $\mathbb{P}^\flat$ satisfies

$$\mathbb{E}^\flat\big[S_t^{(k)} \,\big|\, \mathcal{F}_u\big] < \mathrm{e}^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d.$$

From the relation

$$S_t^{(0)} = \mathrm{e}^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t,$$

we interpret (5.1) by saying that the expected return of the risky asset $S_t^{(k)}$ under $\mathbb{P}^*$ equals the return of the riskless asset $S_t^{(0)}$, $k = 1, 2, \ldots, d$. Recall that the discounted (in \$ at time 0) price $\widetilde{S}_t^{(k)}$ of the risky asset n^o k is defined by

$$\widetilde{S}_t^{(k)} := \mathrm{e}^{-rt} S_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)} / S_0^{(0)}}, \qquad t \geqslant 0, \quad k = 0, 1, \ldots, d,$$

i.e. $S_t^{(0)} / S_0^{(0)}$ plays the role of a *numéraire* process in the sense of Chapter 16.

As in the discrete-time case, see Proposition 2.13, the martingale property in continuous time, see Definition 4.2, can be used to characterize risk-neutral probability measures, for the derivation of pricing partial differential equations (PDEs), and for the computation of conditional expectations.

Proposition 5.5. *The probability measure* $\mathbb{P}^*$ *is risk-neutral if and only if the discounted risky asset price process* $(\widetilde{S}_t^{(k)})_{t\in\mathbb{R}_+}$ *is a martingale* under $\mathbb{P}^*$, $k = 1, 2, \ldots, d$.

Proof. If $\mathbb{P}^*$ is a risk-neutral probability measure, we have

$$\begin{aligned}
\mathbb{E}^*\big[\widetilde{S}_t^{(k)} \,\big|\, \mathcal{F}_u\big] &= \mathbb{E}^*\big[\mathrm{e}^{-rt} S_t^{(k)} \,\big|\, \mathcal{F}_u\big] \\
&= \mathrm{e}^{-rt}\mathbb{E}^*\big[S_t^{(k)} \,\big|\, \mathcal{F}_u\big] \\
&= \mathrm{e}^{-rt}\mathrm{e}^{(t-u)r} S_u^{(k)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-ru} S_u^{(k)} \\
&= \widetilde{S}_u^{(k)}, \qquad 0 \leqslant u \leqslant t,
\end{aligned}
$$

hence $\big(\widetilde{S}_t^{(k)}\big)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$, $k = 1, 2, \ldots, d$. Conversely, if $\big(\widetilde{S}_t^{(k)}\big)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$, then

$$
\begin{aligned}
\mathbb{E}^*\big[S_t^{(k)} \,\big|\, \mathcal{F}_u\big] &= \mathbb{E}^*\big[\mathrm{e}^{rt}\widetilde{S}_t^{(k)} \,\big|\, \mathcal{F}_u\big] \\
&= \mathrm{e}^{rt}\mathbb{E}^*\big[\widetilde{S}_t^{(k)} \,\big|\, \mathcal{F}_u\big] \\
&= \mathrm{e}^{rt}\widetilde{S}_u^{(k)} \\
&= \mathrm{e}^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d,
\end{aligned}
$$

hence the probability measure $\mathbb{P}^*$ is risk-neutral according to Definition 5.4.

□

In what follows we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

Definition 5.6. *A probability measure* $\mathbb{P}^*$ *on* $(\Omega, \mathcal{F})$ *is said to be* equivalent *to another probability measure* $\mathbb{P}$ *when*

$$
\mathbb{P}^*(A) = 0 \quad \textit{if and only if} \quad \mathbb{P}(A) = 0, \quad \textit{for all} \quad A \in \mathcal{F}. \tag{5.2}
$$

Next, we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

Theorem 5.7. *A market is* without *arbitrage opportunity if and only if it admits at least one* equivalent *risk-neutral probability measure* $\mathbb{P}^*$*.*

Proof. See Harrison and Pliska (1981) and Chapter VII-4a of Shiryaev (1999).

□

5.3 Self-Financing Portfolio Strategies

Let $\xi_t^{(i)}$ denote the (possibly fractional) quantity invested at time t over the time interval $[t, t+dt)$, in the asset $S_t^{(k)}$, $k = 0, 1, \ldots, d$, and let

$$
\bar{\xi}_t = \big(\xi_t^{(k)}\big)_{k=0,1,\ldots,d}, \qquad \overline{S}_t = \big(S_t^{(k)}\big)_{k=0,1,\ldots,d}, \quad t \geqslant 0,
$$

denote the associated portfolio value and asset price processes. The portfolio value V_t at time $t \geqslant 0$ is given by

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \geqslant 0. \tag{5.3}$$

Our description of portfolio strategies proceeds in four equivalent formulations (5.4), (5.6) (5.8) and (5.9), which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy $(\bar{\xi}_t)_{t\in\mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from $\bar{\xi}_t$ to $\bar{\xi}_{t+dt}$, *i.e.*

$$\bar{\xi}_t \cdot \bar{S}_{t+dt} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t+dt}^{(k)} = \sum_{k=0}^{d} \xi_{t+dt}^{(k)} S_{t+dt}^{(k)} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}, \tag{5.4}$$

which is the continuous-time analog of the self-financing condition already encountered in the discrete setting of Chapter 2, see Definition 2.4. A major difference with the discrete-time case of Definition 2.4, however, is that the continuous-time differentials dS_t and $d\xi_t$ do not make pathwise sense as continuous-time stochastic integrals are defined by L^2 limits, cf. Proposition 4.21, or by convergence in probability.

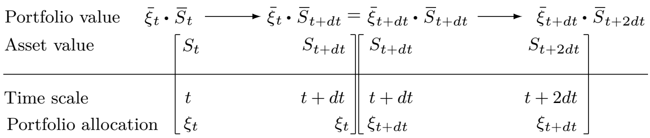


Fig. 5.2: Illustration of the self-financing condition (5.4).

Equivalently, Condition (5.4) can be rewritten as

$$\sum_{k=0}^{d} S_{t+dt}^{(k)} (\xi_{t+dt}^{(k)} - \xi_t^{(k)}) = 0, \tag{5.5}$$

or, letting

$$d\xi_t^{(k)} := \xi_{t+dt}^{(k)} - \xi_t^{(k)}, \qquad k = 0, 1, \ldots, d,$$

denote the respective variations in portfolio allocations, as

$$\sum_{k=0}^{d} S_{t+dt}^{(k)} d\xi_t^{(k)} = 0. \tag{5.6}$$

Condition (5.5) can be rewritten as

$$\sum_{k=0}^{d} S_t^{(k)}\big(\xi_{t+dt}^{(k)}-\xi_t^{(k)}\big)+\sum_{k=0}^{d}\big(S_{t+dt}^{(k)}-S_t^{(k)}\big)\big(\xi_{t+dt}^{(k)}-\xi_t^{(k)}\big)=0, \tag{5.7}$$

which shows that (5.4) and (5.6) are equivalent to

$$\sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)}+\sum_{k=0}^{d} dS_t^{(k)}\bullet d\xi_t^{(k)}=0 \tag{5.8}$$

in differential notation, which may also be rewritten as

$$\overline{S}_t\bullet d\bar{\xi}_t+d\overline{S}_t\bullet d\bar{\xi}_t=0.$$

Self-financing portfolio differential

In practice, the self-financing portfolio property will be characterized by the following proposition, which states that the value of a self-financing portfolio can be written as the sum of its trading Profits and Losses (P/L).

Proposition 5.8. *A portfolio strategy* $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ *with value*

$$V_t=\bar{\xi}_t\bullet\overline{S}_t=\sum_{k=0}^{d}\xi_t^{(k)}S_t^{(k)}, \qquad t\geqslant 0,$$

is self-financing according to (5.4) *if and only if the relation*

$$dV_t=\sum_{k=0}^{d}\underbrace{\xi_t^{(k)}dS_t^{(k)}}_{\text{P/L for asset } n^o\ k} \tag{5.9}$$

holds.

Proof. By Itô's calculus, using (4.27) we have

$$dV_t=\sum_{k=0}^{d}\xi_t^{(k)}dS_t^{(k)}+\sum_{k=0}^{d}S_t^{(k)}d\xi_t^{(k)}+\sum_{k=0}^{d}dS_t^{(k)}\bullet d\xi_t^{(k)},$$

which shows that (5.8) is equivalent to (5.9). □

Market Completeness

We refer to Definition 1.9 for the definition of contingent claim.

Definition 5.9. *A contingent claim with payoff C is said to be attainable if there exists a (self-financing) portfolio strategy $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ such that at the maturity time T the equality*

$$V_T = \bar{\xi}_T \cdot \overline{S}_T = \sum_{k=0}^{d} \xi_T^{(k)} S_T^{(k)} = C$$

holds (almost surely) between random variables.

When a claim with payoff C is attainable, its price at time t will be given by the value V_t of a self-financing portfolio hedging C.

Definition 5.10. *A market model is said to be* complete *if every contingent claim is attainable.*

The next result is the continuous-time statement of the second fundamental theorem of asset pricing.

Theorem 5.11. *A market model without arbitrage opportunities is complete if and only if it admits only one* equivalent *risk-neutral probability measure $\mathbb{P}^*$.*

Proof. See Harrison and Pliska (1981) and Chapter VII-4a of Shiryaev (1999). □

5.4 Two-Asset Portfolio Model

In the Black and Scholes (1973) model, one can show the existence of a unique risk-neutral probability measure, hence the model is without arbitrage and complete. From now on we work with $d = 1$, *i.e.* with a market based on a riskless asset with price $(A_t)_{t\in\mathbb{R}_+}$ and a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$.

Self-financing portfolio strategies

Let ξ_t and η_t denote the (possibly fractional) quantities respectively invested at time t over the time interval $[t, t+dt)$, into the risky asset S_t and riskless asset A_t, and let

$$\bar{\xi}_t = (\eta_t, \xi_t), \qquad \overline{S}_t = (A_t, S_t), \qquad t \geqslant 0,$$

denote the associated portfolio value and asset price processes. The portfolio value V_t at time t is given by

$$V_t = \bar{\xi}_t \cdot \overline{S}_t = \eta_t A_t + \xi_t S_t, \qquad t \geqslant 0.$$

Our description of portfolio strategies proceeds in four equivalent formulations presented below in Equations (5.10), (5.11), (5.14) and (5.15), which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from (η_t, ξ_t) to $(\eta_{t+dt}, \xi_{t+dt})$, *i.e.*

$$\bar{\xi}_t \cdot \overline{S}_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} = \eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} = \bar{\xi}_{t+dt} \cdot \overline{S}_{t+dt}. \tag{5.10}$$

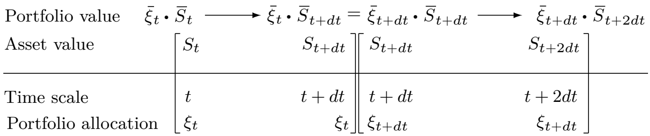


Fig. 5.3: Illustration of the self-financing condition (5.10).

Equivalently, Condition (5.10) can be rewritten as

$$A_{t+dt} d\eta_t + S_{t+dt} d\xi_t = 0, \tag{5.11}$$

where

$$d\eta_t := \eta_{t+dt} - \eta_t \quad \text{and} \quad d\xi_t := \xi_{t+dt} - \xi_t$$

denote the respective changes in portfolio allocations, hence we have

$$A_{t+dt}(\eta_t - \eta_{t+dt}) = S_{t+dt}(\xi_{t+dt} - \xi_t). \tag{5.12}$$

In other words, when one sells a (possibly fractional) quantity $\eta_t - \eta_{t+dt} > 0$ of the riskless asset valued A_{t+dt} at the end of the time interval $[t, t+dt]$ for the total amount $A_{t+dt}(\eta_t - \eta_{t+dt})$, one should entirely spend this income to buy the corresponding quantity $\xi_{t+dt} - \xi_t > 0$ of the risky asset for the same amount $S_{t+dt}(\xi_{t+dt} - \xi_t) > 0$.

Similarly, if one sells a quantity $-d\xi_t > 0$ of the risky asset S_{t+dt} between the time intervals $[t, t+dt]$ and $[t+dt, t+2dt]$ for a total amount $-S_{t+dt} d\xi_t$, one should entirely use this income to buy a quantity $d\eta_t > 0$ of the riskless asset for an amount $A_{t+dt} d\eta_t > 0$, *i.e.*

$$A_{t+dt} d\eta_t = -S_{t+dt} d\xi_t.$$

Condition (5.12) can also be rewritten as

$$S_t(\xi_{t+dt} - \xi_t) + A_t(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) \\ + (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = 0, \tag{5.13}$$

which shows that (5.10) and (5.11) are equivalent to

$$S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t = 0 \tag{5.14}$$

in differential notation, with

$$dA_t \cdot d\eta_t \simeq (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = rA_t(dt \cdot d\eta_t) = 0$$

in the sense of Itô's calculus, by the Itô multiplication Table 4.1. This yields the following proposition, which is also consequence of Proposition 5.8.

Proposition 5.12. *A portfolio allocation* $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ *with value*

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \geqslant 0,$$

is self-financing according to (5.10) *if and only if the relation*

$$dV_t = \underbrace{\eta_t dA_t}_{\text{Risk}-\text{free P/L}} + \underbrace{\xi_t dS_t}_{\text{Risky P/L}} \tag{5.15}$$

holds.

Proof. By the Itô formula (4.27), we have

$$dV_t = [\eta_t dA_t + \xi_t dS_t] + [S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t],$$

which shows that (5.15) is equivalent to (5.14). Equivalently, we can also check the equality

$$\begin{aligned} dV_t &= V_{t+dt} - V_t \\ &= \eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} - (\eta_t A_t + \xi_t S_t) \\ &= \eta_t (A_{t+dt} - A_t) + \xi_t (S_{t+dt} - S_t) + S_t(\xi_{t+dt} - \xi_t) + A_t(\eta_{t+dt} - \eta_t) \\ &\quad + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) + (A_{t+dt} - A_t)(\eta_{t+dt} - \eta_t). \end{aligned}$$

□

Let

$$\widetilde{V}_t := \mathrm{e}^{-rt} V_t \qquad \text{and} \qquad \widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \geqslant 0,$$

respectively denote the discounted portfolio value and discounted risky asset price at time $t \geqslant 0$.

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https://personal.ntu.edu.sg/nprivault/indext.html

Geometric Brownian motion

Recall that the riskless asset price process $(A_t)_{t\in\mathbb{R}_+}$ admits the following equivalent constructions:

$$\frac{A_{t+dt}-A_t}{A_t}=rdt, \qquad \frac{dA_t}{A_t}=rdt, \qquad \frac{dA_t}{dt}=rA_t, \quad t\geqslant 0,$$

with the solution

$$A_t = A_0\,\mathrm{e}^{rt}, \qquad t\geqslant 0, \tag{5.16}$$

where $r>0$ is the risk-free interest rate.* The risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ will be modeled using a geometric Brownian motion defined from the equation

$$\frac{S_{t+dt}-S_t}{S_t}=\frac{dS_t}{S_t}=\mu dt+\sigma dB_t, \qquad t\geqslant 0, \tag{5.17}$$

see Section 5.5, which can be solved numerically, according to the following R code, see also Sections 22.1-22.2.

```
nsim <- 10; N=2000; t <- 0:N; dt <- 1.0/N; mu=0.75; sigma=0.2; S <- matrix(0, nsim, N+1)
dB <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N+1)
for (i in 1:nsim){S[i,1]=1.0;
for (j in 1:N+1){S[i,j]=S[i,j-1]+mu*S[i,j-1]*dt+sigma*S[i,j-1]*dB[i,j]}}
plot(t*dt, rep(0, N+1), xlab = "Time", ylab = "Geometric Brownian motion", lwd=2, ylim =
      c(min(S),max(S)), type = "l", col = 0,las=1, cex.axis=1.5,cex.lab=1.5, xaxs='i', yaxs='i')
for (i in 1:nsim){lines(t*dt, S[i, ], lwd=2, type = "l", col = i)}
lines(t*dt, exp(mu*t*dt),xlab = "",type = "l", col = 1, lwd=5)
```

Note that by Proposition 5.15 below, we also have

$$S_t = S_0\exp\left(\sigma B_t+\left(\mu-\frac{1}{2}\sigma^2\right)t\right), \qquad t\geqslant 0,$$

which can be simulated by the following R code.

```
N=2000; t <- 0:N; dt <- 1.0/N; mu=1.5;sigma=0.3; nsim <- 20; par(oma=c(0,1,0,0))
dB <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
S <- cbind(rep(0, nsim), t(apply(dB, 1, cumsum)))
for (i in 1:nsim){S[i,] <- exp(mu*t*dt+sigma*S[i,]-sigma*sigma*t*dt/2)}
plot(t*dt, rep(0, N+1), xlab = "Time", ylab = "Geometric Brownian motion", lwd=2, ylim =
      c(0.8,6), type = "l", col = 0,las=1, cex.axis=1.5, cex.lab=1.5, xaxs='i', yaxs='i')
for (i in 1:nsim){lines(t*dt, S[i, ], lwd=2, type = "l", col = i)}
lines(t*dt, exp(mu*t*dt),xlab = "",type = "l", col = 1, lwd=5)
```

Figure 5.4 presents a collection of of geometric Brownian motion sample paths.

* "Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", K. E. Boulding (1973), page 248.

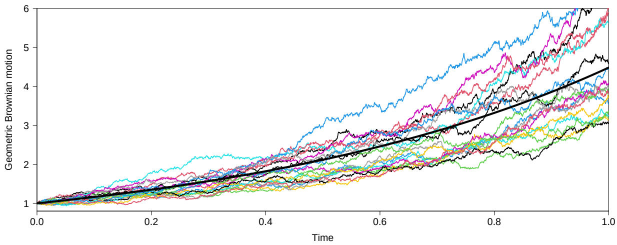

Fig. 5.4: Ten sample paths of geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$.

Lemma 5.13. Discounting lemma. *Consider an asset price process* $(S_t)_{t\in\mathbb{R}_+}$ *be as in* (5.17), *i.e.*

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \geqslant 0.$$

Then, the discounted asset price process $\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ *satisfies the equation*

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.$$

Proof. By (4.27) and the Itô multiplication Table 4.1, we have

$$\begin{aligned}
d\widetilde{S}_t &= d(\mathrm{e}^{-rt}S_t) \\
&= S_t d(\mathrm{e}^{-rt}) + \mathrm{e}^{-rt}dS_t + (d\,\mathrm{e}^{-rt}) \bullet dS_t \\
&= -r\,\mathrm{e}^{-rt}S_t dt + \mathrm{e}^{-rt}dS_t + (-r\,\mathrm{e}^{-rt}dt) \bullet dS_t \\
&= -r\,\mathrm{e}^{-rt}S_t dt + \mu\,\mathrm{e}^{-rt}S_t dt + \sigma\,\mathrm{e}^{-rt}S_t dB_t \\
&= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.
\end{aligned}$$

□

In Lemma 5.14, which is the continuous-time analog of Lemma 3.2, we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted trading profits and losses (number of risky assets ξ_t times discounted price variation $d\widetilde{S}_t$).

Note that in Equation (5.18) below, no profit or loss arises from trading the discounted riskless asset $\widetilde{A}_t := \mathrm{e}^{-rt}A_t = A_0$, because its price remains constant over time.

Lemma 5.14. *Let* $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ *be a portfolio strategy with value*

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \geqslant 0.$$

The following statements are equivalent:

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(i) the portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ *is self-financing,*

(ii) the discounted portfolio value $\widetilde{V}_t = \mathrm{e}^{-rt}V_t$ *can be written as the stochastic integral sum*

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \underbrace{\xi_u d\widetilde{S}_u,}_{\text{Discounted P/L}} \qquad t \geqslant 0, \tag{5.18}$$

of discounted profits and losses.

Proof. Assuming that (i) holds, the self-financing condition and (5.16)-(5.17) show that

$$\begin{aligned}
dV_t &= \eta_t dA_t + \xi_t dS_t \\
&= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \qquad t \geqslant 0,
\end{aligned}$$

where we used $V_t = \eta_t A_t + \xi_t S_t$, hence

$$\mathrm{e}^{-rt} dV_t = r\,\mathrm{e}^{-rt} V_t dt + (\mu - r)\,\mathrm{e}^{-rt}\xi_t S_t dt + \sigma\,\mathrm{e}^{-rt}\xi_t S_t dB_t, \quad t \geqslant 0,$$

and

$$\begin{aligned}
d\widetilde{V}_t &= d\left(\mathrm{e}^{-rt} V_t\right) \\
&= -r\,\mathrm{e}^{-rt} V_t dt + \mathrm{e}^{-rt} dV_t \\
&= (\mu - r)\xi_t\,\mathrm{e}^{-rt} S_t dt + \sigma\xi_t\,\mathrm{e}^{-rt} S_t dB_t \\
&= (\mu - r)\xi_t \widetilde{S}_t dt + \sigma\xi_t \widetilde{S}_t dB_t \\
&= \xi_t d\widetilde{S}_t, \qquad t \geqslant 0,
\end{aligned}$$

i.e. (5.18) holds by integrating on both sides as

$$\widetilde{V}_t - \widetilde{V}_0 = \int_0^t d\widetilde{V}_u = \int_0^t \xi_u d\widetilde{S}_u, \qquad t \geqslant 0.$$

(ii) Conversely, if (5.18) is satisfied we have

$$\begin{aligned}
dV_t &= d(\mathrm{e}^{rt}\widetilde{V}_t) \\
&= r\,\mathrm{e}^{rt}\widetilde{V}_t dt + \mathrm{e}^{rt} d\widetilde{V}_t \\
&= r\,\mathrm{e}^{rt}\widetilde{V}_t dt + \mathrm{e}^{rt}\xi_t d\widetilde{S}_t \\
&= rV_t dt + \mathrm{e}^{rt}\xi_t d\widetilde{S}_t \\
&= rV_t dt + \mathrm{e}^{rt}\xi_t \widetilde{S}_t((\mu - r)dt + \sigma dB_t) \\
&= rV_t dt + \xi_t S_t((\mu - r)dt + \sigma dB_t) \\
&= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t
\end{aligned}$$

$$= \eta_t dA_t + \xi_t dS_t,$$

hence the portfolio is self-financing according to Definition 5.8. □

As a consequence of Relation (5.18), the problem of hedging a claim payoff C with maturity T also reduces to that of finding the process $(\xi_t)_{t\in[0,T]}$ appearing in the decomposition of the discounted claim payoff $\widetilde{C} = \mathrm{e}^{-rT}C$ as a stochastic integral:

$$\widetilde{C} = \widetilde{V}_T = \widetilde{V}_0 + \int_0^T \xi_t d\widetilde{S}_t,$$

see Section 7.5 on hedging by the martingale method.

Example. *Power options in the Bachelier model.*

In the Bachelier (1900) model with interest rate $r = 0$, the underlying asset price can be modeled by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, and may therefore become negative.* The claim payoff $C = (B_T)^2$ of a power option with at maturity $T > 0$ admits the stochastic integral decomposition

$$(B_T)^2 = T + 2\int_0^T B_t dB_t$$

which shows that the claim can be hedged using $\xi_t = 2B_t$ units of the underlying asset at time $t \in [0, T]$, see Exercise 6.1.

Similarly, in the case of power claim payoff $C = (B_T)^3$ we have

$$(B_T)^3 = 3\int_0^T \big(T - t + (B_t)^2\big)dB_t,$$

cf. Exercise 4.12.

Note that according to (5.18), the (non-discounted) self-financing portfolio value V_t can be written as

$$V_t = \mathrm{e}^{rt}V_0 + (\mu - r)\int_0^t \mathrm{e}^{(t-u)r}\xi_u S_u du + \sigma\int_0^t \mathrm{e}^{(t-u)r}\xi_u S_u dB_u, \quad t \geqslant 0. \tag{5.19}$$

5.5 Geometric Brownian Motion

In this section we solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

which is used to model the S_t the risky asset price at time t, where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation is rewritten in *integral form* as

* Negative oil prices have been observed in May 2020 when the prices of oil futures contracts fell below zero.

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$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s, \qquad t \geqslant 0. \tag{5.20}$$

```
N=1000; t <- 0:N; dt <- 1.0/N; sigma=0.2; mu=0.5
dB <- rnorm(N,mean=0,sd=sqrt(dt));
plot(t*dt, exp(mu*t*dt), xlab = "time", ylab = "Geometric Brownian motion", type = "l", ylim
      = c(0.75, 2), col = 1,lwd=3)
lines(t*dt, exp(sigma*c(0,cumsum(dB))+mu*t*dt-sigma*sigma*t*dt/2),xlab = "time",type =
      "l",ylim = c(0, 4), col = 4)
```

Figure 5.5 presents an illustration of the geometric Brownian process using the expression provided in Proposition 5.15.

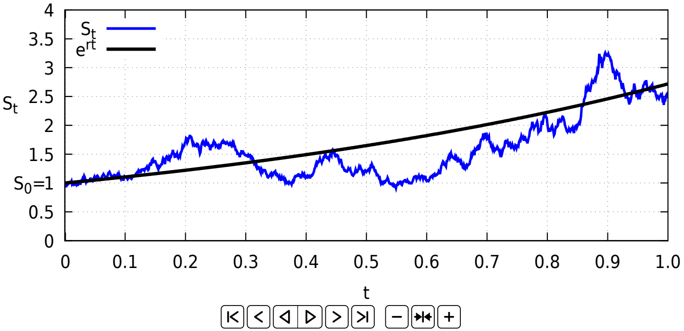

Fig. 5.5: Geometric Brownian motion started at $S_0 = 1$, with $\mu = r = 1$ and $\sigma^2 = 0.5$.*

Proposition 5.15. *The solution of the stochastic differential equation*

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{5.21}$$

is given by

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \geqslant 0. \tag{5.22}$$

Proof. *a*) Using log-returns. We apply Itô's formula to the Itô process $(S_t)_{t\in\mathbb{R}_+}$ with $v_t = \mu S_t$ and $u_t = \sigma S_t$, by taking

$$f(S_t) = \log S_t, \qquad \text{with} \qquad f(x) := \log x.$$

Using (4.29), this yields the log-return dynamics

$$d \log S_t = f'(S_t) dS_t + \frac{1}{2} f''(S_t) dS_t \cdot dS_t$$

* The animation works in Acrobat Reader on the entire pdf file.

$$= \mu S_t f'(S_t)dt + \sigma S_t f'(S_t)dB_t + \frac{\sigma^2}{2} S_t^2 f''(S_t)dt$$
$$= \mu dt + \sigma dB_t - \frac{\sigma^2}{2} dt,$$

hence

$$\begin{aligned}\log S_t - \log S_0 &= \int_0^t d\log S_s \\ &= \left(\mu - \frac{\sigma^2}{2}\right)\int_0^t ds + \sigma \int_0^t dB_s \\ &= \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t,\end{aligned}$$

and

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right), \qquad t \geqslant 0.$$

b) Let us provide an alternative proof by searching for a solution of the form

$$S_t = f(t, B_t)$$

where $f(t,x)$ is a function to be determined. By Itô's formula (4.25) we have

$$dS_t = df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt.$$

Comparing this expression to (5.21) and identifying the terms in dB_t we get

$$\begin{cases} \dfrac{\partial f}{\partial x}(t, B_t) = \sigma S_t, \\ \\ \dfrac{\partial f}{\partial t}(t, B_t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t, B_t) = \mu S_t. \end{cases}$$

Using the relation $S_t = f(t, B_t)$, these two equations rewrite as

$$\begin{cases} \dfrac{\partial f}{\partial x}(t, B_t) = \sigma f(t, B_t), \\ \\ \dfrac{\partial f}{\partial t}(t, B_t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t, B_t) = \mu f(t, B_t). \end{cases}$$

Since B_t is a Gaussian random variable taking all possible values in $\mathbb{R}$, the equations should hold for all $x \in \mathbb{R}$, as follows:

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,x) = \sigma f(t,x), & (5.23a) \\ \\ \dfrac{\partial f}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,x) = \mu f(t,x). & (5.23b) \end{cases}$$

To find the solution $f(t,x) = f(t,0)\,\mathrm{e}^{\sigma x}$ of (5.23a) we let $g(t,x) = \log f(t,x)$ and rewrite (5.23a) as

$$\frac{\partial g}{\partial x}(t,x) = \frac{\partial}{\partial x}\log f(t,x) = \frac{1}{f(t,x)}\frac{\partial f}{\partial x}(t,x) = \sigma,$$

i.e.

$$\frac{\partial g}{\partial x}(t,x) = \sigma,$$

which is solved as

$$g(t,x) = g(t,0) + \sigma x,$$

hence

$$f(t,x) = \mathrm{e}^{g(t,0)}\,\mathrm{e}^{\sigma x} = f(t,0)\,\mathrm{e}^{\sigma x}.$$

Plugging back this expression into the second equation (5.23b) yields

$$\mathrm{e}^{\sigma x}\frac{\partial f}{\partial t}(t,0) + \frac{1}{2}\sigma^2\,\mathrm{e}^{\sigma x} f(t,0) = \mu f(t,0)\,\mathrm{e}^{\sigma x},$$

i.e.

$$\frac{\partial f}{\partial t}(t,0) = \left(\mu - \frac{\sigma^2}{2}\right) f(t,0),$$

or

$$\frac{\partial g}{\partial t}(t,0) = \mu - \frac{\sigma^2}{2}.$$

This yields

$$g(t,0) = g(0,0) + \left(\mu - \frac{\sigma^2}{2}\right) t,$$

hence

$$\begin{aligned} f(t,x) &= \mathrm{e}^{g(t,x)} = \mathrm{e}^{g(t,0)+\sigma x} \\ &= \mathrm{e}^{g(0,0)+\sigma x+(\mu-\sigma^2/2)t} \\ &= f(0,0)\,\mathrm{e}^{\sigma x+(\mu-\sigma^2/2)t}, \qquad t \geqslant 0. \end{aligned}$$

We conclude that

$$S_t = f(t,B_t) = f(0,0)\,\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t},$$

and since $f(0,0) = f(0,B_0) = S_0$, the solution to (5.21) is given by

$$S_t = S_0 \,\mathrm{e}^{\sigma B_t + (\mu - \sigma^2/2)t}, \qquad t \geqslant 0.$$

□

Conversely, taking $S_t = f(t, B_t)$ with $f(t,x) = S_0 \,\mathrm{e}^{\mu t + \sigma x - \sigma^2 t/2}$, we may apply Itô's formula to check that

$$\begin{aligned}
dS_t &= df(t, B_t) \qquad (5.24)\\
&= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt \\
&= \left(\mu - \frac{\sigma^2}{2}\right) S_0 \,\mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} dt + \sigma S_0 \,\mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} dB_t \\
&\quad + \frac{1}{2}\sigma^2 S_0 \,\mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} dt \\
&= \mu S_0 \,\mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} dt + \sigma S_0 \,\mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} dB_t \\
&= \mu S_t dt + \sigma S_t dB_t.
\end{aligned}$$

Exercises

Exercise 5.1 Show that at any time $T > 0$, the random variable $S_T := S_0 \,\mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T}$ has the *lognormal distribution* with probability density function

$$x \longmapsto f(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \,\mathrm{e}^{-(-(\mu - \sigma^2/2)T + \log(x/S_0))^2/(2\sigma^2 T)}, \qquad x > 0,$$

with log-variance σ^2 and log-mean $(\mu - \sigma^2/2)T + \log S_0$, see Figures 3.10 and 5.6.

```
N=1000; t <- 0:N; dt <- 1.0/N; nsim <- 100
sigma=0.2;r=0.5;a=(1+r*dt)*(1-sigma*sqrt(dt))-1;b=(1+r*dt)*(1+sigma*sqrt(dt))-1
X <- matrix(a+(b-a)*rbinom( nsim * N, 1, 0.5), nsim, N) # using Bernoulli samples
X<-cbind(rep(0,nsim),t(apply((1+X),1,cumprod))); X[,1]=1;H<-hist(X[,N],plot=FALSE);
      dev.new(width=16,height=7);
layout(matrix(c(1,2), nrow =1, byrow = TRUE)); par(mar=c(2,2,2,0), oma = c(2, 2, 2, 2))
plot(t*dt,X[1,],xlab="time",ylab="",type="l",ylim=c(0.8,3), col = 0,xaxs='i',las=1,
      cex.axis=1.6)
for (i in 1:nsim){lines(t*dt, X[i, ], xlab = "time", type = "l", col = i)}
lines((1+r*dt)^t, type="l", lty=1, col="black",lwd=3,xlab="",ylab="", main="")
for (i in 1:nsim){points(0.999, X[i,N], pch=1, lwd = 5, col = i)}; x <- seq(0.01,3, length=100);
px <- exp(-(-(r-sigma^2/2)+log(x))^2/2/sigma^2)/x/sigma/sqrt(2*pi); par(mar = c(2,2,2,2))
plot(NULL , xlab="", ylab="", xlim = c(0, max(px,H$density)),ylim=c(0.8,3),axes=F, las=1)
rect(0, H$breaks[1:(length(H$breaks) - 1)], col=rainbow(20,start=0.08,end=0.6), H$density,
      H$breaks[2:length(H$breaks)])
lines(px,x, type="l", lty=1, col="black",lwd=3,xlab="",ylab="", main="")
```

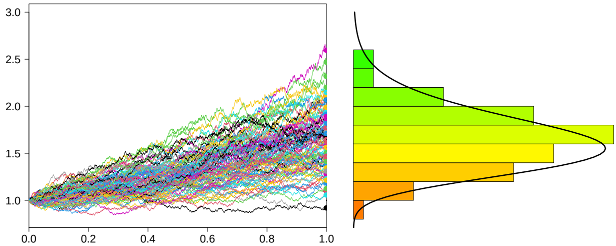

Fig. 5.6: Statistics of geometric Brownian paths *vs.* lognormal distribution.

Exercise 5.2

a) Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \geqslant 0, \tag{5.25}$$

where $r, \sigma \in \mathbb{R}$ are constants and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Compute $d \log S_t$ using the Itô formula.

b) Solve the ordinary differential equation $df(t) = cf(t)dt$ for $f(t)$, and the stochastic differential equation (5.25) for S_t.

c) Using the Gaussian moment generating function (MGF) formula (A.41), compute the *n-th* order moment $\mathbb{E}[S_t^n]$ for all $n > 0$.

d) Compute the lognormal mean and variance

$$\mathbb{E}[S_t] = S_0 \, e^{rt} \quad \text{and} \quad \mathrm{Var}[S_t] = S_0^2 \, e^{2rt} \big(e^{\sigma^2 t} - 1 \big), \quad t \geqslant 0.$$

e) Recover the lognormal mean and variance of Question (d) by deriving differential equations for the functions $u(t) := \mathbb{E}[S_t]$ and $v(t) := \mathbb{E}\big[S_t^2\big]$, $t \geqslant 0$, using stochastic calculus.

Exercise 5.3 Assume that $(B_t)_{t\in\mathbb{R}_+}$ and $(W_t)_{t\in\mathbb{R}_+}$ are standard Brownian motions, correlated according to the Itô rule $dW_t \bullet dB_t = \rho dt$ for $\rho \in [-1, 1]$, and consider the solution $(Y_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation $dY_t = \mu Y_t dt + \eta Y_t dW_t$, $t \geqslant 0$, where $\mu, \eta \in \mathbb{R}$ are constants. Compute $df(S_t, Y_t)$, for f a $\mathcal{C}^2$ function on $\mathbb{R}^2$ using the bivariate Itô formula (4.26).

Exercise 5.4 Consider the asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Find the stochastic integral decomposition of the random variable S_T, *i.e.*, find the *constant* $C(S_0, r, T)$ and the process $(\zeta_{t,T})_{t\in[0,T]}$ such that

$$S_T = C(S_0, r, T) + \int_0^T \zeta_{t,T} dB_t. \qquad (5.26)$$

Hint: Use the fact that the discounted price process $(X_t)_{t\in[0,T]} := (\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ satisfies the relation $dX_t = \sigma X_t dB_t$.

Exercise 5.5 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and the process $(S_t)_{t\in\mathbb{R}_+}$ defined by

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dB_s + \int_0^t u_s ds\right), \qquad t \geqslant 0,$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ and $(u_t)_{t\in\mathbb{R}_+}$ are $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes.

a) Compute dS_t using Itô calculus.
b) Show that S_t satisfies a stochastic differential equation to be determined.

Exercise 5.6 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, and let $\sigma > 0$.

a) Compute the mean and variance of the random variable S_t defined as

$$S_t := 1 + \sigma \int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2} dB_s, \qquad t \geqslant 0. \qquad (5.27)$$

b) Express $d\log(S_t)$ using (5.27) and the Itô formula.
c) Show that $S_t = e^{\sigma B_t - \sigma^2 t/2}$ for $t \geqslant 0$.

Exercise 5.7 We consider a leveraged fund with factor $\beta : 1$ on an index $(S_t)_{t\in\mathbb{R}_+}$ modeled as the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \geqslant 0,$$

under the risk-neutral probability measure $\mathbb{P}^*$. Examples of leveraged funds include ProShares Ultra S&P500 and ProShares UltraShort S&P500.

a) Find the portfolio allocation (ξ_t, η_t) of the leveraged fund value

$$F_t = \xi_t S_t + \eta_t A_t, \qquad t \geqslant 0,$$

where $A_t := A_0 \mathrm{e}^{rt}$ represents the risk-free money market account price.

Hint: Leveraging with a factor $\beta : 1$ means that the risky component $\xi_t S_t$of the portfolio should represent β times the invested amount F_t at any time $t \geqslant 0$.

b) Find the stochastic differential equation satisfied by $(F_t)_{t\in\mathbb{R}_+}$ under the self-financing condition $dF_t = \xi_t dS_t + \eta_t dA_t$.
c) Find the relation between the fund value F_t and the index S_t by solving the stochastic differential equation obtained for F_t in Question (b). For simplicity we take $F_0 := S_0^\beta$.

Exercise 5.8 Consider two assets whose prices $S_t^{(1)}$, $S_t^{(2)}$ at time $t \in [0,T]$ follow the geometric Brownian dynamics

$$dS_t^{(1)} = \mu S_t^{(1)} dt + \sigma_1 S_t^{(1)} dW_t^{(1)} \quad \text{and} \quad dS_t^{(2)} = \mu S_t^{(2)} dt + \sigma_2 S_t^{(2)} dW_t^{(2)},$$

$t \in [0,T]$, where $\big(W_t^{(1)}\big)_{t\in[0,T]}$, $\big(W_t^{(2)}\big)_{t\in[0,T]}$ are two Brownian motions with correlation $\rho \in [-1,1]$, *i.e.* we have $\mathbb{E}\big[W_t^{(1)} W_t^{(2)}\big] = \rho t$.

a) Compute $\mathbb{E}\big[S_t^{(i)}\big]$, $t \in [0,T]$, $i = 1,2$.
b) Compute $\mathrm{Var}\,\big[S_t^{(i)}\big]$, $t \in [0,T]$, $i = 1,2$.
c) Compute $\mathrm{Var}\,\big[S_t^{(2)} - S_t^{(1)}\big]$, $t \in [0,T]$.

Exercise 5.9 Solve the stochastic differential equation

$$dX_t = h(t) X_t dt + \sigma X_t dB_t,$$

where $\sigma > 0$ and $h(t)$ is a deterministic, integrable function of $t \geqslant 0$.

Hint: Look for a solution of the form $X_t = f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$, where $f(t)$ is a function to be determined, $t \geqslant 0$.

Exercise 5.10 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

a) Letting $X_t := \sigma B_t + \nu t$, $\sigma > 0$, $\nu \in \mathbb{R}$, compute $S_t := \mathrm{e}^{X_t}$ by the Itô formula

$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s) dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s) ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s) ds, \tag{5.28}$$

applied to $f(x) = \mathrm{e}^x$, by writing X_t as $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$.
b) Let $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = r S_t dt + \sigma S_t dB_t \quad ?$$

c) Given $\sigma > 0$, let $X_t := (B_T - B_t)\sigma$, and compute $\mathrm{Var}[X_t]$, $0 \leqslant t \leqslant T$.

d) Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0\, \mathrm{e}^{\sigma B_t + \nu t}$, $t \geqslant 0$. Using the result of Exercise A.2, show that the conditional probability that $S_T > K$ given $S_t = x$ can be computed as

$$\mathbb{P}(S_T > K \mid S_t = x) = \Phi\left(\frac{\log(x/K) + (T-t)\nu}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t < T,$$

where $\Phi(x)$ denotes the standard Gaussian Cumulative Distribution Function.
Hint: Use the time splitting decomposition

$$S_T = S_t \frac{S_T}{S_t} = S_t\, \mathrm{e}^{(B_T - B_t)\sigma + (T-t)\nu}, \qquad 0 \leqslant t \leqslant T.$$

Problem 5.11 Stop-loss/start-gain strategy (Lipton (2001) § 8.3.3, Exercise 4.21 continued). Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at $B_0 \in \mathbb{R}$.

a) We consider a simplified foreign exchange model in which the AUD is a risky asset and the AUD/SGD exchange rate at time t is modeled by B_t, *i.e.* AU\$1 equals SG\$$B_t$ at time t. A foreign exchange (FX) European call option gives to its holder the right (but not the obligation) to receive AU\$1 in exchange for $K =$ SG\$1 at maturity T. Give the option payoff at maturity, quoted in SGD.

   In what follows, for simplicity we assume no time value of money ($r = 0$), *i.e.* the (riskless) SGD account is priced $A_t = A_0 = 1$, $0 \leqslant t \leqslant T$.

b) Consider the following hedging strategy for the European call option of Question (a):

   i) If $B_0 > 1$, charge the premium $B_0 - 1$ at time 0, and borrow SG\$1 to purchase AU\$1.
   ii) If $B_0 < 1$, issue the option for free.
   iii) From time 0 to time T, purchase* AU\$1 every time B_t crosses $K = 1$ from below, and sell† AU\$1 each time B_t crosses $K = 1$ from above.

   Show that this strategy effectively hedges the foreign exchange European call option at maturity T.

   Hint: Note that it suffices to consider four scenarios based on $B_0 < 1$ *vs.* $B_0 < 1$ and $B_T > 1$ *vs.* $B_T < 1$.

c) Determine the quantities η_t of SGD cash and ξ_t of (risky) AUDs to be held in the portfolio and express the portfolio value

$$V_t = \eta_t + \xi_t B_t$$

* We need to borrow SG\$1 if this is the first AUD purchase.

† We use the SG\$1 product of the sale to refund the loan.

at all times $t \in [0, T]$.

d) Compute the integral summation

$$\int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s$$

of portfolio profits and losses at any time $t \in [0, T]$.

Hint: Apply the Itô-Tanaka formula (4.48), see Question (e) in Exercise 4.21.

e) Is the portfolio strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ self-financing? How to interpret the answer in practice?

Problem 5.12 Liquidity pools are smart contracts found on decentralized exchanges (DEX) such as Uniswap, in which liquidity providers (LPs) may earn passive income by depositing two digital assets. Liquidity pools enable the automatic trading of digital assets using Automated Market Makers (AMMs), such as the Constant Product Automated Market Maker (CPAMM) introduced by Vitalik Buterin in a legendary 2016 Reddit post.

In the Geometric Mean Market Maker (G3M) model used in Uniswap v2,* Sushiswap or Balancer, the quantities $X_t > 0$, $Y_t > 0$ of the assets X, Y present in the pool are linked at all times $t \geqslant 0$ by the relation

$$C = X_t^{\alpha} Y_t^{1-\alpha}, \qquad t \geqslant 0, \tag{5.29}$$

where $C > 0$ and $\alpha \in (0, 1)$ are constants. External traders are accessing the liquidity pool to perform the exchange of a quantity $\Delta Y_t := Y_t - Y_{t^-} \in \mathbb{R}$ of the asset Y into a quantity $\Delta X_t := X_t - X_{t^-} \in \mathbb{R}$ of the asset X, where X_{t^-}

* See the 2020 White Paper.

and Y_{t^-} are the left limits*

$$X_{t^-} := \lim_{s \nearrow t} X_s \quad \text{and} \quad Y_{t^-} := \lim_{s \nearrow t} Y_s, \qquad t > 0.$$

The exchange rate S_t at time $t \geqslant 0$ is defined as the ratio of the number of units of Y *vs.* number of units of X exchanged during an "infinitesimal" transaction, *i.e.*

$$S_t := \lim_{\Delta X_t \to 0} \frac{\mp \Delta Y_t}{\pm \Delta X_t} = - \lim_{\Delta X_t \to 0} \frac{\Delta Y_t}{\Delta X_t},$$

where the limit is assumed to exist at all times $t \geqslant 0$. In addition, in what follows we will regard X_t and Y_t as *continuous* processes.

The following 10 questions are interdependent and should be treated in sequence.

a) Express the exchange rate S_t in terms of X_t, Y_t and $\gamma := \alpha/(1-\alpha)$ under the G3M condition (5.29).
b) At time $t = 0$, the liquidity provider deposits Y_0 units of Y in the pool, together with X_0 units of X. Write down the value LP_0 of the liquidity pool at time $t = 0$ in terms of C, α, γ and S_0 only, quoted in units of Y.
c) Write down the liquidity pool value LP_t at any time $t \geqslant 0$ in terms of C α, γ and S_t only, quoted in units of Y.
d) We consider an investor who builds a long portfolio with $Y_0 > 0$ units of Y and $X_0 > 0$ of X at time $t = 0$, and holds those assets at all times. Write down the value V_t of this long portfolio at any time $t \geqslant 0$ in terms of C, α, γ, S_0 and S_t only, quoted in units of Y.
e) Show that the difference in value $V_t - \mathrm{LP}_t$ between the long portfolio and the liquidity pool can be written as

$$V_t - \mathrm{LP}_t = \phi\left(\frac{S_t}{S_0}\right),$$

where $\phi(x)$ is a function to be determined explicitly, and parameterized by C, S_0, α and γ.
f) Show that the value V_t of the investor's long portfolio always overperforms the liquidity pool value LP_t.
g) Find a lower bound for the expected loss $\mathbb{E}[V_t - \mathrm{LP}_t]$ of the liquidity provider in terms of C, S_0, $\mathbb{E}[S_t]$, α and γ.

Hint: Use Jensen's inequality.
h) In this question only, we consider the CPAMM setting with $\alpha = 1/2$ and we model the exchange rate $(S_t)_{t=0,1}$ in a one-step model on the probability space $\Omega = \{\omega^-, \omega^+\}$, with

* ΔX_t and ΔY_t may be positive or negative, depending on the direction of the transaction.

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$$S_1(\omega^-) = a, \qquad \text{and} \qquad S_1(\omega^+) = b,$$

and $ab = S_0^2$, $a < S_0 < b$. Show that the potential loss $V_1 - \text{LP}_1$ of the liquidity provider at time $t = 1$ can be exactly hedged by holding a quantity ξ of an option with payoff $|S_1 - S_0|$, and compute ξ in terms of C, S_0 and/or a, b.

Hints: Here the payoff is $C = V_1 - \text{LP}_1$ and the underlying asset value at time $t = 1$ is $|S_1 - S_0|$. The portfolio allocation is $(\xi, 0)$, as no riskless asset is used here.

i) In this question only, we model the exchange rate $(S_t)_{t\in\mathbb{R}_+}$ as the geometric Brownian motion* solution of

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Compute the expected loss $\mathbb{E}[V_t - \text{LP}_t]$ incurred by the liquidity provider in comparison to the investor's long portfolio.

j) In this question, the liquidity provider is rewarded† by a proportional fee payed on *deposits* at the rate κ, and stored outside of the liquidity pool. This results into an ask rate from the pool

$$A_t := \lim_{\Delta X_t \nearrow 0} \frac{\mp \Delta Y_t}{\pm \Delta X_t} = - \lim_{\Delta X_t \nearrow 0} \frac{\Delta Y_t}{\Delta X_t}$$

when the trader wishes to exchange Y into X, and a bid rate from the pool

$$B_t := \lim_{\Delta X_t \searrow 0} \frac{\mp \Delta Y_t}{\pm \Delta X_t} = - \lim_{\Delta X_t \searrow 0} \frac{\Delta Y_t}{\Delta X_t}$$

when the trader wishes to exchange X into Y, where the limits are assumed to exist at all times $t \geqslant 0$.
Express the ask and bid exchange rates A_t, B_t in terms of X_t, Y_t, κ and γ under the G3M condition (5.29).

Hint: Treat the cases $\Delta X_t > 0$ and $\Delta Y_t > 0$ separately.

* Question (i) requires knowledge from Chapters 5-6.

† Yield farming: a high-risk, volatile investment strategy where an investor stakes, or lends, crypto assets on a decentralized (DeFi) platform to earn a higher return.

Further reading

Briola et al. (2023) “Specifically, two main events are identified as the fuse for the Terra collapse. First, private market actors short sold Bitcoin (BTC) with the final aim of spreading panic into the market. Second, on 07 May 2022, the liquidity pool Curve-3pool suffered a "liquidity pool attack", which caused the first UST de-pegging, below $0.99. It is worth noting that, on 01 April 2022, Kwon announced the launch of a new liquidity pool (4pool) together with DeFi majors Frax Finance and Redacted Cartel.