Chapter 16
Change of Numéraire and Forward Measures

Change of numéraire is a powerful technique for the pricing of options under random discount factors by the use of forward measures. It has applications to the pricing of interest rate derivatives and other types of options, including exchange options (Margrabe formula) and foreign exchange options (Garman-Kohlagen formula). The computation of self-financing hedging strategies by change of numéraire is treated in Section 16.5, and the change of numéraire technique will be applied to the pricing of interest rate derivatives such as bond options and swaptions in Chapter 19.

16.1 Notion of Numéraire

A *numéraire* is any strictly positive $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted stochastic process $(N_t)_{t\in\mathbb{R}_+}$ that can be taken as a unit of reference when pricing an asset or a claim.

In general, the price S_t of an asset, when quoted in terms of the numéraire N_t, is given by

$$\widehat{S}_t := \frac{S_t}{N_t}, \qquad t \geqslant 0.$$

Deterministic numéraire transformations are easy to handle, as change of numéraire by a constant factor is a formal algebraic transformation that does not involve any risk. This can be the case for example when a currency

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is pegged to another currency,* for example the exchange rate of the French franc to the Euro was locked at €1 = FRF 6.55957 on 31 December 1998.

On the other hand, a random numéraire may involve new risks, and can allow for arbitrage opportunities.

Examples of numéraire processes $(N_t)_{t\in\mathbb{R}_+}$ include:

- *Money market account.*

Given $(r_t)_{t\in\mathbb{R}_+}$ a possibly random, time-dependent and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted risk-free interest rate process, let†

$$N_t := \exp\left(\int_0^t r_s ds\right).$$

In this case,

$$\widehat{S}_t = \frac{S_t}{N_t} = \mathrm{e}^{-\int_0^t r_s ds} S_t, \qquad t \geqslant 0,$$

represents the discounted price of the asset at time 0.

- *Currency exchange rates*

In this case, $N_t := R_t$ denotes *e.g.* the EUR/SGD (EURSGD=X) exchange rate from a foreign currency (*e.g.* EUR) to a domestic currency (*e.g.* SGD), *i.e.* one unit of foreign currency (EUR) corresponds to R_t units of local currency (SGD). Let

$$\widehat{S}_t := \frac{S_t}{R_t}, \qquad t \geqslant 0,$$

denote the price of a local (SG) asset quoted in units of the foreign currency (EUR). For example, if $R_t = 1.63$ and $S_t = S\$1$, then

$$\widehat{S}_t = \frac{S_t}{R_t} = \frac{1}{1.63} \times \$1 \simeq €0.61,$$

and $1/R_t$ is the domestic SGD/EUR exchange rate. A question of interest is whether a local asset $\$S_t$, discounted according to a foreign risk-free rate r^f and priced in foreign currency as

$$\mathrm{e}^{-r^f t}\frac{S_t}{R_t} = \mathrm{e}^{-r^f t}\widehat{S}_t,$$

can be a martingale on the foreign market.

* Major currencies have started floating against each other since 1973, following the end of the system of fixed exchanged rates agreed upon at the Bretton Woods Conference, July 1-22, 1944.

† "Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", K.E. Boulding (1973), page 248.

My foreign currency account S_t grew by 5% this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My foreign currency account S_t grew by 5% this year.

The foreign exchange rate dropped by 10%.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

Fig. 16.1: Why change of numéraire?

- *Forward numéraire.*

The price $P(t,T)$ of a bond paying $P(T,T) = \$1$ at maturity T can be taken as numéraire. In this case, we have

$$N_t := P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \;\Big|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

Recall that

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} \;\Big|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T,$$

is an $\mathcal{F}_t$- martingale.

- *Annuity numéraire.*

Processes of the form

$$N_t = P(t, T_0, T_n) := \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k), \qquad 0 \leqslant t \leqslant T_0,$$

where $P(t,T_1), P(t,T_2), \ldots, P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$ arranged according to a *tenor structure*.

- *Combinations of the above*: for example a foreign money market account $\mathrm{e}^{\int_0^t r_s^{\mathrm{f}} ds} R_t$, expressed in local (or domestic) units of currency, where $(r_t^{\mathrm{f}})_{t \in \mathbb{R}_+}$ represents a short-term interest rate on the foreign market.

When the numéraire is a random process, the pricing of a claim whose value has been transformed under change of numéraire, *e.g.* under a change of currency, has to take into account the risks existing on the foreign market.

In particular, in order to perform a fair pricing, one has to determine a probability measure under which the transformed (or forward, or deflated) process $\widehat{S}_t = S_t / N_t$ will be a martingale, see Section 16.3 for details.

For example, in case $N_t := e^{\int_0^t r_s ds}$ is the money market account, the risk-neutral probability measure $\mathbb{P}^*$ is a measure under which the discounted price process

$$\widehat{S}_t = \frac{S_t}{N_t} = e^{-\int_0^t r_s ds} S_t, \qquad t \geqslant 0,$$

is a martingale. In the next section, we will see that this property can be extended to any type of numéraire.

See Exercises 16.5 and 16.6 for other examples of numéraires.

16.2 Change of Numéraire

In this section we review the pricing of options by a change of measure associated to a numéraire N_t, cf. *e.g.* Geman et al. (1995) and references therein.

Most of the results of this chapter rely on the following assumption, which expresses absence of arbitrage. In the foreign exchange setting where $N_t = R_t$, this condition states that the price of one unit of foreign currency is a martingale when quoted and discounted in the domestic currency.

Assumption (A). *The discounted numéraire*

$$t \mapsto M_t := e^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale under the risk-neutral probability measure $\mathbb{P}^*$.

Definition 16.1. *Given $(N_t)_{t\in[0,T]}$ a numéraire process, the associated* forward measure $\widehat{\mathbb{P}}$ *is defined by its Radon-Nikodym density*

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} := \frac{M_T}{M_0} = e^{-\int_0^T r_s ds} \frac{N_T}{N_0}. \tag{16.1}$$

In particular, we note that $\widehat{\mathbb{P}} = \mathbb{P}^*$ when $N_t := \exp\left(\int_0^t r_s ds\right)$ is the money market account. Recall also that from Section 7.3, Relation (16.1) rewrites as

$$d\widehat{\mathbb{P}} = \frac{M_T}{M_0} d\mathbb{P}^* = e^{-\int_0^T r_s ds} \frac{N_T}{N_0} d\mathbb{P}^*,$$

which is equivalent to stating that

$$\begin{aligned}
\widehat{\mathbb{E}}[F] &= \int_\Omega F(\omega) d\widehat{\mathbb{P}}(\omega) \\
&= \int_\Omega F(\omega)\, e^{-\int_0^T r_s ds} \frac{N_T}{N_0} d\mathbb{P}^*(\omega) \\
&= \mathbb{E}^*\left[F\, e^{-\int_0^T r_s ds} \frac{N_T}{N_0}\right],
\end{aligned}$$

for all integrable $\mathcal{F}_T$-measurable random variables F. More generally, by (16.1) and the fact that the process

$$t \mapsto M_t := \mathrm{e}^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ by Assumption (A), we find that

$$\begin{aligned}\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}^*\left[\frac{N_T}{N_0}\mathrm{e}^{-\int_0^T r_s ds} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\frac{M_T}{M_0} \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{M_t}{M_0} \\ &= \frac{N_t}{N_0}\mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leqslant t \leqslant T. \end{aligned} \tag{16.2}$$

In Proposition 16.5 we will show, as a consequence of next Lemma 16.2 below, that for any integrable random claim payoff C we have

$$\mathbb{E}^*\left[C\,\mathrm{e}^{-\int_t^T r_s ds} N_T \,\middle|\, \mathcal{F}_t\right] = N_t \widehat{\mathbb{E}}[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T.$$

Similarly to the above, the Radon-Nikodym density $d\widehat{\mathbb{P}}_{|\mathcal{F}_t} / d\mathbb{P}^*_{|\mathcal{F}_t}$ of $\widehat{\mathbb{P}}_{|\mathcal{F}_t}$ with respect to $\mathbb{P}^*_{|\mathcal{F}_t}$ satisfies the relation

$$\begin{aligned}\widehat{\mathbb{E}}[F \mid \mathcal{F}_t] &= \int_\Omega F(\omega) d\widehat{\mathbb{P}}_{|\mathcal{F}_t}(\omega) \\ &= \int_\Omega F(\omega) \frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} d\mathbb{P}^*_{|\mathcal{F}_t}(\omega) \\ &= \mathbb{E}^*\left[F \frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \,\middle|\, \mathcal{F}_t\right],\end{aligned}$$

for all integrable $\mathcal{F}_T$-measurable random variables F, $0 \leqslant t \leqslant T$. Note that (16.2), which is $\mathcal{F}_t$-measurable, should not be confused with (16.3), which is $\mathcal{F}_T$-measurable.

Lemma 16.2. *We have*

$$\frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} = \frac{M_T}{M_t} = \mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t}, \qquad 0 \leqslant t \leqslant T. \tag{16.3}$$

Proof. The proof of (16.3) relies on an abstract version of the Bayes formula. For all bounded $\mathcal{F}_t$-measurable random variable G, by (16.2), the tower property (A.33) of conditional expectations and the characterization (A.45) in Proposition A.21, we have

$$\begin{aligned}
\widehat{\mathbb{E}}\big[G\widehat{X}\big] &= \mathbb{E}^*\left[G\widehat{X}\,\mathrm{e}^{-\int_0^T r_s ds}\frac{N_T}{N_0}\right]\\
&= \mathbb{E}^*\left[G\frac{N_t}{N_0}\,\mathrm{e}^{-\int_0^t r_s ds}\mathbb{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\,\Big|\,\mathcal{F}_t\right]\right]\\
&= \mathbb{E}^*\left[G\mathbb{E}^*\left[\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}^*}\,\Big|\,\mathcal{F}_t\right]\mathbb{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\,\Big|\,\mathcal{F}_t\right]\right]\\
&= \mathbb{E}^*\left[G\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}^*}\mathbb{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\,\Big|\,\mathcal{F}_t\right]\right]\\
&= \widehat{\mathbb{E}}\left[G\mathbb{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\,\Big|\,\mathcal{F}_t\right]\right],
\end{aligned}$$

for all bounded random variables $\widehat{X}$, which shows that

$$\widehat{\mathbb{E}}\big[\widehat{X}\mid\mathcal{F}_t\big] = \mathbb{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\,\Big|\,\mathcal{F}_t\right],$$

i.e. (16.3) holds. □

We note that in case the numéraire $N_t = \mathrm{e}^{\int_0^t r_s ds}$ is equal to the money market account we simply have $\widehat{\mathbb{P}} = \mathbb{P}^*$.

Definition 16.3. *Given $(X_t)_{t\in\mathbb{R}_+}$ an asset price process, we define the process of forward (or deflated) prices*

$$\widehat{X}_t := \frac{X_t}{N_t}, \qquad 0 \leqslant t \leqslant T. \tag{16.4}$$

The process $(\widehat{X}_t)_{t\in\mathbb{R}_+}$ epresents the values at times t of X_t, expressed in units of the numéraire N_t. In the sequel, it will be useful to determine the dynamics of $(\widehat{X}_t)_{t\in\mathbb{R}_+}$ under the forward measure $\widehat{\mathbb{P}}$. The next proposition shows in particular that the process $\left(\mathrm{e}^{\int_0^t r_s ds}/N_t\right)_{t\in\mathbb{R}_+}$ is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$.

Proposition 16.4. *Let $(X_t)_{t\in\mathbb{R}_+}$ denote a continuous $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted asset price process such that*

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds}X_t, \qquad t \geqslant 0,$$

is a martingale under $\mathbb{P}^*$. Then, under change of numéraire,

the deflated process $(\widehat{X}_t)_{t\in[0,T]} = (X_t/N_t)_{t\in[0,T]}$ of forward prices is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$,

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provided that $(\widehat{X}_t)_{t\in[0,T]}$ *is integrable under* $\widehat{\mathbb{P}}$.

Proof. We show that

$$\widehat{\mathbb{E}}\left[\frac{X_t}{N_t}\,\Big|\,\mathcal{F}_s\right] = \frac{X_s}{N_s}, \qquad 0 \leqslant s \leqslant t, \tag{16.5}$$

using the characterization (A.45) of conditional expectation in Proposition A.21. Namely, for all bounded $\mathcal{F}_s$-measurable random variables G we note that using (16.2) under Assumption (A) we have

$$\begin{aligned}
\widehat{\mathbb{E}}\left[G\frac{X_t}{N_t}\right] &= \mathbb{E}^*\left[G\frac{X_t}{N_t}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[G\frac{X_t}{N_t}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^*\left[G\frac{X_t}{N_t}\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^*\left[G\,\mathrm{e}^{-\int_0^t r_u du}\frac{X_t}{N_0}\right] \qquad (16.6)\\
&= \mathbb{E}^*\left[G\,\mathrm{e}^{-\int_0^s r_u du}\frac{X_s}{N_0}\right] \qquad (16.7)\\
&= \mathbb{E}^*\left[G\frac{X_s}{N_s}\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathcal{F}_s\right]\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[G\frac{X_s}{N_s}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathcal{F}_s\right]\right] \\
&= \mathbb{E}^*\left[G\frac{X_s}{N_s}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\right] \\
&= \widehat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right], \qquad 0 \leqslant s \leqslant t,
\end{aligned}$$

where from (16.6) to (16.7)we used the fact that

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} X_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$. Finally, the identity

$$\widehat{\mathbb{E}}\big[G\widehat{X}_t\big] = \widehat{\mathbb{E}}\left[G\frac{X_t}{N_t}\right] = \widehat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right] = \widehat{\mathbb{E}}\big[G\widehat{X}_s\big], \qquad 0 \leqslant s \leqslant t,$$

for all bounded $\mathcal{F}_s$-measurable G, implies (16.5). □

Pricing using change of numéraire

The change of numéraire technique is especially useful for pricing under random interest rates, in which case an expectation of the form

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds} C \,\Big|\, \mathcal{F}_t \right]$$

becomes a *path integral*, see *e.g.* Dash (2004) for an account of path integral methods in quantitative finance. The next proposition is the basic result of this section, it provides a way to price an option with arbitrary payoff C under a random discount factor $e^{-\int_t^T r_s ds}$ by use of the forward measure. It will be applied in Chapter 19 to the pricing of bond options and caplets, cf. Propositions 19.1, 19.3 and 19.5 below.

Proposition 16.5. *An option with integrable claim payoff* $C \in L^1(\Omega, \mathbb{P}^*, \mathcal{F}_T)$ *is priced at time* t *as*

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds} C \,\Big|\, \mathcal{F}_t \right] = N_t \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\Big|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T, \tag{16.8}$$

provided that $C/N_T \in L^1(\Omega, \widehat{\mathbb{P}}, \mathcal{F}_T)$.

Proof. By Relation (16.3) in Lemma 16.2 we have

$$\begin{aligned}
\mathbb{E}^*\left[e^{-\int_t^T r_s ds} C \,\Big|\, \mathcal{F}_t \right] &= \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \frac{N_t}{N_T} C \,\Big|\, \mathcal{F}_t \right] \\
&= N_t \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \frac{C}{N_T} \,\Big|\, \mathcal{F}_t \right] \\
&= N_t \int_\Omega \frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \frac{C}{N_T} d\mathbb{P}^*_{|\mathcal{F}_t} \\
&= N_t \int_\Omega \frac{C}{N_T} d\widehat{\mathbb{P}}_{|\mathcal{F}_t} \\
&= N_t \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\Big|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Equivalently, we can write

$$\begin{aligned}
N_t \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\Big|\, \mathcal{F}_t \right] &= N_t \mathbb{E}^*\left[\frac{C}{N_T} \frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \,\Big|\, \mathcal{F}_t \right] \\
&= \mathbb{E}^*\left[e^{-\int_t^T r_s ds} C \,\Big|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

□

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Each application of the change of numéraire formula (16.8) will require to:

a) pick a suitable numéraire $(N_t)_{t\in\mathbb{R}_+}$ satisfying Assumption (A),
b) make sure that the ratio C/N_T takes a sufficiently simple form,
c) use the Girsanov theorem in order to determine the dynamics of asset prices under the new probability measure $\widehat{\mathbb{P}}$,

so as to compute the expectation under $\widehat{\mathbb{P}}$ on the right-hand side of (16.8).

Next, we consider further examples of numéraires and associated examples of option prices.

Examples:

a) *Money market account.*

Take $N_t := \mathrm{e}^{\int_0^t r_s ds}$, where $(r_t)_{t\in\mathbb{R}_+}$ is a possibly random and time-dependent risk-free interest rate. In this case, Assumption (A) is clearly satisfied, we have $\widehat{\mathbb{P}} = \mathbb{P}^*$ and $d\mathbb{P}^*/d\widehat{\mathbb{P}}$, and (16.8) simply reads

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathcal{F}_t\right] = \mathrm{e}^{\int_0^t r_s ds}\mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} C \,\middle|\, \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

which yields no particular information.

b) *Forward numéraire.*

Here, $N_t := P(t,T)$ is the price $P(t,T)$ of a bond maturing at time T, $0 \leqslant t \leqslant T$, and the discounted bond price process $\left(\mathrm{e}^{-\int_0^t r_s ds}P(t,T)\right)_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, *i.e.* Assumption (A) is satisfied and $N_t = P(t,T)$ can be taken as numéraire. In this case, (16.8) shows that a random claim payoff C can be priced as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathcal{F}_t\right] = P(t,T)\widehat{\mathbb{E}}[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T, \tag{16.9}$$

since $N_T = P(T,T) = 1$, where the forward measure $\widehat{\mathbb{P}}$ satisfies

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^T r_s ds}\frac{P(T,T)}{P(0,T)} = \frac{\mathrm{e}^{-\int_0^T r_s ds}}{P(0,T)} \tag{16.10}$$

by (16.1).

c) *Annuity numéraires.*

We take

$$N_t := \sum_{k=1}^n (T_k - T_{k-1})P(t,T_k)$$

where $P(t,T_1),\ldots,P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$. Here, (16.8) shows that a swaption on the cash flow $P(T,T_n) - P(T,T_1) - \kappa N_T$ can be priced as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,T_n)-P(T,T_1)-\kappa N_T)^+ \,\Big|\, \mathcal{F}_t\right]$$
$$= N_t \widehat{\mathbb{E}}\left[\left(\frac{P(T,T_n)-P(T,T_1)}{N_T}-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right],$$

$0 \leqslant t \leqslant T < T_1$, where $(P(T,T_n)-P(T,T_1))/N_T$ becomes a *swap rate*, cf. (18.21) in Proposition 18.12 and Section 19.5.

Girsanov theorem

We refer to *e.g.* Theorem III-35 page 132 of Protter (2004) for the following version of the Girsanov Theorem.

Theorem 16.6. *Assume that $\widehat{\mathbb{P}}$ is equivalent* to $\mathbb{P}^*$ with Radon-Nikodym density $\mathrm{d}\widehat{\mathbb{P}}/\mathrm{d}\mathbb{P}^*$, and let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion under $\mathbb{P}^*$. Then, letting*

$$\Phi_t := \mathbb{E}^*\left[\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}^*}\,\Big|\,\mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T, \tag{16.11}$$

the process $\big(\widehat{W}_t\big)_{t\in[0,T]}$ defined by

$$d\widehat{W}_t := dW_t - \frac{1}{\Phi_t}\mathrm{d}\Phi_t \bullet dW_t, \qquad 0 \leqslant t \leqslant T, \tag{16.12}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$.

In case the martingale $(\Phi_t)_{t\in[0,T]}$ takes the form

$$\Phi_t = \exp\left(-\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t |\psi_s|^2 ds\right), \qquad 0 \leqslant t \leqslant T,$$

i.e.

$$\mathrm{d}\Phi_t = -\psi_t \Phi_t dW_t, \qquad 0 \leqslant t \leqslant T,$$

the Itô multiplication Table 4.1 shows that Relation (16.12) reads

$$\begin{aligned} d\widehat{W}_t &= dW_t - \frac{1}{\Phi_t}\mathrm{d}\Phi_t \bullet dW_t \\ &= dW_t - \frac{1}{\Phi_t}(-\psi_t\Phi_t dW_t) \bullet dW_t \\ &= dW_t + \psi_t dt, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

* This means that the Radon-Nikodym densities $\mathrm{d}\widehat{\mathbb{P}}/\mathrm{d}\mathbb{P}^*$ and $\mathrm{d}\mathbb{P}^*/\mathrm{d}\widehat{\mathbb{P}}$ exist and are strictly positive with $\mathbb{P}^*$ and $\widehat{\mathbb{P}}$-probability one, respectively.

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and shows that the shifted process $\left(\widehat{W}_t\right)_{t\in[0,T]} = \left(W_t + \int_0^t \psi_s ds\right)_{t\in[0,T]}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$, which is consistent with the Girsanov Theorem 7.3. The next result is another application of the Girsanov Theorem.

Proposition 16.7. *The process* $\left(\widehat{W}_t\right)_{t\in[0,T]}$ *defined by*

$$d\widehat{W}_t := dW_t - \frac{1}{N_t} dN_t \cdot dW_t, \qquad 0 \leqslant t \leqslant T, \tag{16.13}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$.

Proof. Relation (16.2) shows that Φ_t defined in (16.11) satisfies

$$\begin{aligned}\Phi_t &= \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathcal{F}_t\right]\\ &= \mathbb{E}^*\left[\frac{N_T}{N_0}\mathrm{e}^{-\int_0^T r_s ds}\,\Big|\,\mathcal{F}_t\right]\\ &= \frac{N_t}{N_0}\mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leqslant t \leqslant T,\end{aligned}$$

hence

$$\begin{aligned}d\Phi_t &= d\left(\frac{N_t}{N_0}\mathrm{e}^{-\int_0^t r_s ds}\right)\\ &= -\Phi_t r_t dt + \mathrm{e}^{-\int_0^t r_s ds} d\left(\frac{N_t}{N_0}\right)\\ &= -\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t,\end{aligned}$$

which, by (16.12), yields

$$\begin{aligned}d\widehat{W}_t &= dW_t - \frac{1}{\Phi_t} d\Phi_t \cdot dW_t\\ &= dW_t - \frac{1}{\Phi_t}\left(-\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t\right) \cdot dW_t\\ &= dW_t - \frac{1}{N_t} dN_t \cdot dW_t,\end{aligned}$$

which is (16.13), from Relation (16.12) and the Itô multiplication Table 4.1. □

The next Proposition 16.8 is consistent with the statement of Proposition 16.4, and in addition it specifies the dynamics of $\left(\widehat{X}_t\right)_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}$ using the Girsanov Theorem 16.7. As a consequence, we have the next proposition, see Exercise 16.1 for another calculation based on geometric Brownian motion, and Exercise 16.10 for an extension to correlated Brownian motions.

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Proposition 16.8. *Assume that* $(X_t)_{t\in\mathbb{R}_+}$ *and* $(N_t)_{t\in\mathbb{R}_+}$ *satisfy the stochastic differential equations*

$$dX_t = r_t X_t dt + \sigma_t^X X_t dW_t, \quad \text{and} \quad dN_t = r_t N_t dt + \sigma_t^N N_t dW_t, \qquad (16.14)$$

where $(\sigma_t^X)_{t\in\mathbb{R}_+}$ *and* $(\sigma_t^N)_{t\in\mathbb{R}_+}$ *are* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$*-adapted volatility processes and* $(W_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion under* $\mathbb{P}^*$*. Then the forward (or deflated) process* $\big(\widehat{X}_t\big)_{t\in[0,T]} = (X_t/N_t)_{t\in[0,T]}$ *satisfies*

$$d\widehat{X}_t = \big(\sigma_t^X - \sigma_t^N\big)\widehat{X}_t d\widehat{W}_t, \qquad (16.15)$$

hence $(\widehat{X}_t)_{t\in[0,T]}$ *is given by the driftless geometric Brownian motion*

$$\widehat{X}_t = \widehat{X}_0 \exp\left(\int_0^t (\sigma_s^X - \sigma_s^N) d\widehat{W}_s - \frac{1}{2}\int_0^t (\sigma_s^X - \sigma_s^N)^2 ds\right), \quad 0 \leqslant t \leqslant T.$$

Proof. First, we note that by (16.13) and (16.14),

$$d\widehat{W}_t = dW_t - \frac{1}{N_t} dN_t \bullet dW_t = dW_t - \sigma_t^N dt, \qquad t \geqslant 0,$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$. Next, by Itô's calculus and the Itô multiplication Table 4.1 and (16.14) we have

$$\begin{aligned}
d\left(\frac{1}{N_t}\right) &= -\frac{1}{N_t^2} dN_t + \frac{1}{N_t^3} dN_t \bullet dN_t \\
&= -\frac{1}{N_t^2}\big(r_t N_t dt + \sigma_t^N N_t dW_t\big) + \frac{|\sigma_t^N|^2}{N_t} dt \\
&= -\frac{1}{N_t^2}\big(r_t N_t dt + \sigma_t^N N_t \big(d\widehat{W}_t + \sigma_t^N dt\big)\big) + \frac{|\sigma_t^N|^2}{N_t} dt \\
&= -\frac{1}{N_t}\big(r_t dt + \sigma_t^N d\widehat{W}_t\big), \qquad (16.16)
\end{aligned}$$

hence

$$\begin{aligned}
d\widehat{X}_t &= d\left(\frac{X_t}{N_t}\right) \\
&= \frac{dX_t}{N_t} + X_t d\left(\frac{1}{N_t}\right) + dX_t \bullet d\left(\frac{1}{N_t}\right) \\
&= \frac{1}{N_t}\big(r_t X_t dt + \sigma_t^X X_t dW_t\big) - \frac{X_t}{N_t}\big(r_t dt + \sigma_t^N dW_t - |\sigma_t^N|^2 dt\big) \\
&\quad -\frac{1}{N_t}\big(r_t X_t dt + \sigma_t^X X_t dW_t\big) \cdot \big(r_t dt + \sigma_t^N dW_t - |\sigma_t^N|^2 dt\big) \\
&= \frac{1}{N_t}\big(r_t X_t dt + \sigma_t^X X_t dW_t\big) - \frac{X_t}{N_t}\big(r_t dt + \sigma_t^N dW_t\big)
\end{aligned}$$

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$$\begin{aligned}
&+\frac{X_t}{N_t}|\sigma_t^N|^2 dt - \frac{X_t}{N_t}\sigma_t^X\sigma_t^N dt \\
&= \frac{X_t}{N_t}\sigma_t^X dW_t - \frac{X_t}{N_t}\sigma_t^N dW_t - \frac{X_t}{N_t}\sigma_t^X\sigma_t^N dt + X_t\frac{|\sigma_t^N|^2}{N_t}dt \\
&= \frac{X_t}{N_t}\big(\sigma_t^X dW_t - \sigma_t^N dW_t - \sigma_t^X\sigma_t^N dt + |\sigma_t^N|^2 dt\big) \\
&= \widehat{X}_t\big(\sigma_t^X - \sigma_t^N\big)dW_t - \widehat{X}_t\big(\sigma_t^X - \sigma_t^N\big)\sigma_t^N dt \\
&= \widehat{X}_t\big(\sigma_t^X - \sigma_t^N\big)d\widehat{W}_t,
\end{aligned}$$

since $d\widehat{W}_t = dW_t - \sigma_t^N dt$, $0 \leqslant t \leqslant T$. □

We end this section with some comments on inverse changes of measure.

Inverse changes of measure

In the next proposition we compute the conditional inverse Radon-Nikodym density $\mathrm{d}\mathbb{P}^*/\mathrm{d}\widehat{\mathbb{P}}$, see also (16.2).

Proposition 16.9. *We have*

$$\widehat{\mathbb{E}}\left[\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\widehat{\mathbb{P}}}\,\Big|\,\mathcal{F}_t\right] = \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right), \qquad 0 \leqslant t \leqslant T, \tag{16.17}$$

and the process

$$t \mapsto \frac{M_0}{M_t} = \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right), \qquad 0 \leqslant t \leqslant T,$$

is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$.

Proof. For all bounded and $\mathcal{F}_t$-measurable random variables F we have,

$$\begin{aligned}
\widehat{\mathbb{E}}\left[F\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\widehat{\mathbb{P}}}\right] &= \mathbb{E}^*[F] \\
&= \mathbb{E}^*\left[F\frac{N_t}{N_t}\right] \\
&= \mathbb{E}^*\left[F\frac{N_T}{N_t}\exp\left(-\int_t^T r_s ds\right)\right] \\
&= \widehat{\mathbb{E}}\left[F\frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right)\right].
\end{aligned}$$

□

By (16.16) we also have

$$d\left(\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\right) = -\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\sigma_t^N d\widehat{W}_t,$$

which recovers the second part of Proposition 16.9, *i.e.* the martingale property of

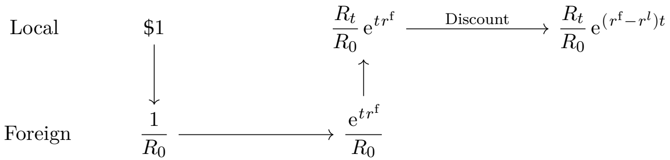
$$t \mapsto \frac{M_0}{M_t} = \frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)$$

under $\widehat{\mathbb{P}}$.

16.3 Foreign Exchange

Currency exchange is a typical application of change of numéraire, that illustrates the absence of arbitrage principle.

Let R_t denote the foreign exchange rate, *i.e.* R_t is the (possibly fractional) quantity of local currency that correspond to one unit of foreign currency, while $1/R_t$ represents the quantity of foreign currency that correspond to a unit of local currency.



Consider an investor that intends to exploit an "overseas investment opportunity" by

a) at time 0, changing one unit of local currency into $1/R_0$ units of foreign currency,
b) investing $1/R_0$ on the foreign market at the rate r^f, which will yield the amount e^{tr^f}/R_0 at time $t > 0$,
c) changing back e^{tr^f}/R_0 into a quantity $e^{tr^f}R_t/R_0 = N_t/R_0$ of local currency.

Fig. 16.2: Overseas investment opportunity.*

In other words, the foreign money market account e^{tr^f} is valued $e^{tr^f} R_t$ on the local (or domestic) market, and its discounted value on the local market is

$$e^{-tr^l + tr^f} R_t, \qquad t > 0.$$

The outcome of this investment will be obtained by a martingale comparison of $e^{tr^f} R_t / R_0$ to the amount e^{tr^l} that could have been obtained by investing on the local market.

Taking

$$N_t := e^{tr^f} R_t, \qquad t \geqslant 0, \tag{16.18}$$

as *numéraire*, absence of arbitrage is expressed by Assumption (A), which states that the discounted numéraire process

$$t \mapsto e^{-r^l t} N_t = e^{-t(r^l - r^f)} R_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$.

Next, we find a characterization of this arbitrage condition using the model parameters r, r^f, μ, by modeling the foreign exchange rates R_t according to a geometric Brownian motion (16.19).

Proposition 16.10. *Assume that the foreign exchange rate R_t satisfies a stochastic differential equation of the form*

$$dR_t = \mu R_t dt + \sigma R_t dW_t, \tag{16.19}$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$. Under the absence of arbitrage Assumption (A) for the numéraire (16.18), we have

$$\mu = r^l - r^f, \tag{16.20}$$

hence the exchange rate process satisfies

$$dR_t = \left(r^l - r^f\right) R_t dt + \sigma R_t dW_t. \tag{16.21}$$

under $\mathbb{P}^*$.

Proof. The equation (16.19) has solution

$$R_t = R_0 e^{\mu t + \sigma W_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

hence the discounted value of the foreign money market account e^{tr^f} on the local market is

* For illustration purposes only. *Not* an advertisement.

$$\mathrm{e}^{-tr^l} N_t = \mathrm{e}^{-tr^l+tr^\mathrm{f}} R_t = R_0 \, \mathrm{e}^{(r^\mathrm{f}-r^l+\mu)t+\sigma W_t-\sigma^2 t/2}, \qquad t \geqslant 0.$$

Under the absence of arbitrage Assumption (A), the process $\mathrm{e}^{-(r^l-r^\mathrm{f})t} R_t = \mathrm{e}^{-tr^l} N_t$ should be an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, and this holds provided that $r^\mathrm{f} - r + \mu = 0$, which yields (16.20) and (16.21). □

As a consequence of Proposition 16.10, under absence of arbitrage a local investor who buys a unit of foreign currency in the hope of a higher return $r^\mathrm{f} >> r$ will have to face a lower (or even more negative) drift

$$\mu = r^l - r^\mathrm{f} << 0$$

in his exchange rate R_t. The drift $\mu = r^l - r^\mathrm{f}$ is also called the *cost of carrying* the foreign currency.

The local money market account $X_t := \mathrm{e}^{tr^l}$ is valued e^{tr^l}/R_t on the foreign market, and its discounted value at time $t \geqslant 0$ on the foreign market is

$$\begin{aligned} \frac{\mathrm{e}^{(r^l-r^\mathrm{f})t}}{R_t} &= \frac{X_t}{N_t} = \widehat{X}_t \qquad (16.22) \\ &= \frac{1}{R_0} \mathrm{e}^{(r^l-r^\mathrm{f})t-\mu t-\sigma W_t+\sigma^2 t/2} \\ &= \frac{1}{R_0} \mathrm{e}^{(r^l-r^\mathrm{f})t-\mu t-\sigma \widehat{W}_t-\sigma^2 t/2}, \end{aligned}$$

where

$$\begin{aligned} d\widehat{W}_t &= dW_t - \frac{1}{N_t} dN_t \cdot dW_t \\ &= dW_t - \frac{1}{R_t} dR_t \cdot dW_t \\ &= dW_t - \sigma dt, \qquad t \geqslant 0, \end{aligned}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$ by (16.13). Under absence of arbitrage, the process $\mathrm{e}^{-(r^l-r^\mathrm{f})t} R_t$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ and (16.22) is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$ by Proposition 16.4, which recovers (16.20).

```
library(quantmod)
getSymbols("EURTRY=X",src = "yahoo",from = "2018-01-01",to = "2021-12-31")
getSymbols("INTDSRTRM193N",src = "FRED")
Interestrate<-`INTDSRTRM193N`["2018-01-01::2021-31-12"]
EURTRY<-Ad(`EURTRY=X`); myPars <- chart_pars();myPars$cex<-1.2
Cumulative<-cumprod(1+Interestrate/100/12)
normalizedfxrate<-1+(as.numeric(last(Cumulative))-1)*(EURTRY-as.numeric(EURTRY[1]))/
     (as.numeric(last(EURTRY))-as.numeric(EURTRY[1]))
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
dev.new(width=16,height=8)
chart_Series(Cumulative,pars=myPars, theme = myTheme)
add_TA(normalizedfxrate, col='black', lw =2, on = 1)
add_TA(Interestrate, col='purple', lw =2)
```

The above code plots an evolution of currency exchange rates compared with the evolution of interest rates, as shown in Figure 16.3.

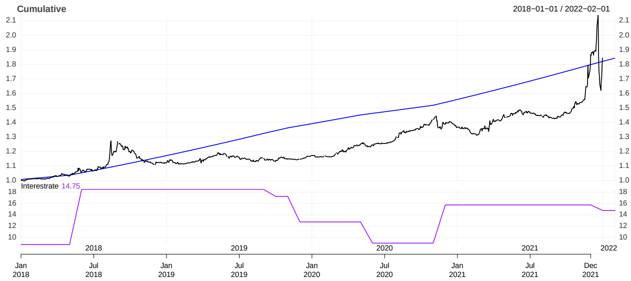


Fig. 16.3: Evolution of exchange rate *vs.* interest rate.*

Proposition 16.11. *Under the absence of arbitrage condition* (16.20)*, the inverse exchange rate* $1/R_t$ *satisfies*

$$d\left(\frac{1}{R_t}\right) = \frac{r^{\mathrm{f}} - r^l}{R_t}dt - \frac{\sigma}{R_t}d\widehat{W}_t, \tag{16.23}$$

under $\widehat{\mathbb{P}}$ *, where* $(R_t)_{t\in\mathbb{R}_+}$ *is given by* (16.21)*.*

Proof. By (16.20), the exchange rate $1/R_t$ is written using Itô's calculus as

$$\begin{aligned} d\left(\frac{1}{R_t}\right) &= -\frac{1}{R_t^2}(\mu R_t dt + \sigma R_t dW_t) + \frac{1}{R_t^3}\sigma^2 R_t^2 dt \\ &= -\frac{\mu - \sigma^2}{R_t}dt - \frac{\sigma}{R_t}dW_t \\ &= -\frac{\mu}{R_t}dt - \frac{\sigma}{R_t}d\widehat{W}_t \end{aligned}$$

$$= \frac{r^{\rm f} - r^l}{R_t} dt - \frac{\sigma}{R_t} d\widehat{W}_t,$$

where $(\widehat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$. □

Consequently, under absence of arbitrage, a foreign investor who buys a unit of the local currency in the hope of a higher return $r >> r^{\rm f}$ will have to face a lower (or even more negative) drift $-\mu = r^{\rm f} - r$ in his exchange rate $1/R_t$ as written in (16.23) under $\widehat{\mathbb{P}}$.

Foreign exchange options

We now price a foreign exchange call option with payoff $(R_T - \kappa)^+$ under $\mathbb{P}^*$ on the exchange rate R_T by the Black-Scholes formula as in the next proposition, also known as the Garman and Kohlhagen (1983) formula. The foreign exchange call option is designed for a local buyer of foreign currency.

Proposition 16.12. *(Garman and Kohlhagen (1983) formula for call options). Consider the exchange rate process $(R_t)_{t\in\mathbb{R}_+}$ given by (16.21). The price of the foreign exchange call option on R_T with maturity T and strike price $\kappa > 0$ is given in local currency units as*

$$\mathrm{e}^{-(T-t)r^l}\mathbb{E}^*\big[(R_T-\kappa)^+ \big| \mathcal{F}_t\big] = \mathrm{e}^{-(T-t)r^{\rm f}} R_t \Phi_+(t, R_t) - \kappa\, \mathrm{e}^{-(T-t)r^l} \Phi_-(t, R_t) \tag{16.24}$$

$0 \leqslant t \leqslant T$, where

$$\Phi_+(t,x) = \Phi\left(\frac{\log(x/\kappa) + (T-t)\big(r^l - r^{\rm f} + \sigma^2/2\big)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) = \Phi\left(\frac{\log(x/\kappa) + (T-t)\big(r^l - r^{\rm f} - \sigma^2/2\big)}{\sigma\sqrt{T-t}}\right).$$

Proof. As a consequence of (16.21), we find the numéraire dynamics

$$\begin{aligned} dN_t &= d\big(\mathrm{e}^{tr^{\rm f}} R_t\big) \\ &= r^{\rm f}\, \mathrm{e}^{tr^{\rm f}} R_t dt + \mathrm{e}^{tr^{\rm f}} dR_t \\ &= r^{\rm f}\, \mathrm{e}^{tr^{\rm f}} R_t dt + \sigma\, \mathrm{e}^{tr^{\rm f}} R_t dW_t \\ &= r^{\rm f} N_t dt + \sigma N_t dW_t. \end{aligned}$$

Hence, a standard application of the Black-Scholes formula yields

$$\begin{aligned}
\mathrm{e}^{-(T-t)r^l}\mathbb{E}^*\big[(R_T-\kappa)^+ \,\big|\, \mathcal{F}_t\big] &= \mathrm{e}^{-(T-t)r^l}\mathbb{E}^*\big[\big(\mathrm{e}^{-Tr^{\mathrm{f}}}N_T-\kappa\big)^+ \,\big|\, \mathcal{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r^l}\,\mathrm{e}^{-Tr^{\mathrm{f}}}\mathbb{E}^*\big[\big(N_T-\kappa\,\mathrm{e}^{Tr^{\mathrm{f}}}\big)^+ \mid \mathcal{F}_t\big] \\
&= \mathrm{e}^{-Tr^{\mathrm{f}}}\left(N_t\Phi\left(\frac{\log\big(N_t\,\mathrm{e}^{-Tr^{\mathrm{f}}}/\kappa\big)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\
&\qquad \left. -\kappa\,\mathrm{e}^{Tr^{\mathrm{f}}-(T-t)r^l}\Phi\left(\frac{\log\big(N_t\,\mathrm{e}^{-Tr^{\mathrm{f}}}/\kappa\big)+\big(r^l-\sigma^2/2\big)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= \mathrm{e}^{-Tr^{\mathrm{f}}}\left(N_t\Phi\left(\frac{\log(R_t/\kappa)+(T-t)\big(r^l-r^{\mathrm{f}}+\sigma^2/2\big)}{\sigma\sqrt{T-t}}\right)\right. \\
&\qquad \left. -\kappa\,\mathrm{e}^{Tr^{\mathrm{f}}-(T-t)r^l}\Phi\left(\frac{\log(R_t/\kappa)+(T-t)\big(r^l-r^{\mathrm{f}}-\sigma^2/2\big)}{\sigma\sqrt{T-t}}\right)\right) \\
&= \mathrm{e}^{-(T-t)r^{\mathrm{f}}}R_t\Phi_+(t,R_t)-\kappa\,\mathrm{e}^{-(T-t)r^l}\Phi_-(t,R_t).
\end{aligned}$$

A similar conclusion can be reached by directly applying (16.21). □

Similarly, from (16.23) rewritten as

$$d\left(\frac{\mathrm{e}^{tr^l}}{R_t}\right) = r^{\mathrm{f}}\frac{\mathrm{e}^{tr^l}}{R_t}dt - \sigma\frac{\mathrm{e}^{tr^l}}{R_t}d\widehat{W}_t,$$

a foreign exchange put option with payoff $(1/\kappa - 1/R_T)^+$ can be priced under $\widehat{\mathbb{P}}$ in a Black-Scholes model by taking e^{tr^l}/R_t as underlying asset price, r^{f} as risk-free interest rate, and $-\sigma$ as volatility parameter. The foreign exchange put option is designed for the foreign seller of local currency, see for example the **buy back guarantee*** which is a typical example of a foreign exchange put option.

Proposition 16.13. *(Garman and Kohlhagen (1983) formula for put options). Consider the exchange rate process* $(R_t)_{t\in\mathbb{R}_+}$ *given by* (16.21)*. The price of the foreign exchange put option on* R_T *with maturity* T *and strike price* $1/\kappa > 0$ *is given in foreign currency units as*

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\widehat{\mathbb{E}}\left[\left(\frac{1}{\kappa}-\frac{1}{R_T}\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&\qquad = \frac{\mathrm{e}^{-(T-t)r^{\mathrm{f}}}}{\kappa}\Phi_-\left(t,\frac{1}{R_t}\right) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t}\Phi_+\left(t,\frac{1}{R_t}\right),
\end{aligned} \tag{16.25}$$

* Right-click to open or save the attachment.

$0 \leqslant t \leqslant T$, *where*

$$\Phi_+(t,x) := \Phi\left(-\frac{\log(\kappa x)+(T-t)\big(r^{\rm f}-r^l+\sigma^2/2\big)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) := \Phi\left(-\frac{\log(\kappa x)+(T-t)\big(r^{\rm f}-r^l-\sigma^2/2\big)}{\sigma\sqrt{T-t}}\right).$$

Proof. The Black-Scholes formula (6.12) yields

$$\begin{aligned}
\mathrm{e}^{-(T-t)r^{\rm f}}\widehat{\mathbb{E}}\left[\left(\frac{1}{\kappa}-\frac{1}{R_T}\right)^+ \,\Big|\, \mathcal{F}_t\right] &= \mathrm{e}^{-(T-t)r^{\rm f}}\,\mathrm{e}^{-Tr^l}\widehat{\mathbb{E}}\left[\left(\frac{\mathrm{e}^{Tr^l}}{\kappa}-\frac{\mathrm{e}^{Tr^l}}{R_T}\right)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= \frac{1}{\kappa}\,\mathrm{e}^{-(T-t)r^{\rm f}}\Phi_-\left(t,\frac{1}{R_t}\right) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t}\Phi_+\left(t,\frac{1}{R_t}\right),
\end{aligned}$$

which is the symmetric of (16.24) by exchanging R_t with $1/R_t$, and r with r^f. □

Call/put duality for foreign exchange options

Let $N_t = \mathrm{e}^{tr^{\rm f}}R_t$, where R_t is an exchange rate with respect to a foreign currency and r_f is the foreign market interest rate.

Proposition 16.14. *The foreign exchange call and put options on the local and foreign markets are linked by the call/put duality relation*

$$\mathrm{e}^{-(T-t)r^l}\mathbb{E}^*\big[(R_T-\kappa)^+ \,\big|\, \mathcal{F}_t\big] = \kappa R_t\,\mathrm{e}^{-(T-t)r^{\rm f}}\widehat{\mathbb{E}}\left[\left(\frac{1}{\kappa}-\frac{1}{R_T}\right)^+ \,\Big|\, \mathcal{F}_t\right], \tag{16.26}$$

between a put option with strike price $1/\kappa$ *and a (possibly fractional) quantity* $1/(\kappa R_t)$ *of call option(s) with strike price* κ.

Proof. By application of change of numéraire from Proposition 16.5 and (16.8) we have

$$\widehat{\mathbb{E}}\left[\frac{1}{\mathrm{e}^{Tr^{\rm f}}R_T}(R_T-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] = \frac{1}{N_t}\,\mathrm{e}^{-(T-t)r^l}\mathbb{E}^*\big[(R_T-\kappa)^+ \,\big|\, \mathcal{F}_t\big],$$

hence

$$\begin{aligned}
\mathrm{e}^{-(T-t)r^{\rm f}}\widehat{\mathbb{E}}\left[\left(\frac{1}{\kappa}-\frac{1}{R_T}\right)^+ \,\Big|\, \mathcal{F}_t\right] &= \mathrm{e}^{-(T-t)r^{\rm f}}\widehat{\mathbb{E}}\left[\frac{1}{\kappa R_T}(R_T-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= \frac{1}{\kappa}\,\mathrm{e}^{tr^{\rm f}}\widehat{\mathbb{E}}\left[\frac{1}{\mathrm{e}^{Tr^{\rm f}}R_T}(R_T-\kappa)^+ \,\Big|\, \mathcal{F}_t\right]
\end{aligned}$$

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$$= \frac{1}{\kappa N_t} \mathrm{e}^{t r^{\mathrm{f}} - (T-t) r^l} \mathbb{E}^* \big[(R_T - \kappa)^+ \,\big|\, \mathcal{F}_t \big]$$
$$= \frac{1}{\kappa R_t} \mathrm{e}^{-(T-t) r^l} \mathbb{E}^* \big[(R_T - \kappa)^+ \,\big|\, \mathcal{F}_t \big].$$

□

In the Black-Scholes case, the duality (16.26) can be directly checked by verifying that (16.25) coincides with

$$\begin{aligned}
&\frac{1}{\kappa R_t} \mathrm{e}^{-(T-t) r^l} \mathbb{E}^* \big[(R_T - \kappa)^+ \,\big|\, \mathcal{F}_t \big] \\
&\quad = \frac{1}{\kappa R_t} \mathrm{e}^{-(T-t) r^l} \mathrm{e}^{-T r^{\mathrm{f}}} \mathbb{E}^* \big[\big(\mathrm{e}^{T r^{\mathrm{f}}} R_T - \kappa \, \mathrm{e}^{T r^{\mathrm{f}}} \big)^+ \,\big|\, \mathcal{F}_t \big] \\
&\quad = \frac{1}{\kappa R_t} \mathrm{e}^{-(T-t) r^l} \mathrm{e}^{-T r^{\mathrm{f}}} \mathbb{E}^* \big[\big(N_T - \kappa \, \mathrm{e}^{T r^{\mathrm{f}}} \big)^+ \,\big|\, \mathcal{F}_t \big] \\
&\quad = \frac{1}{\kappa R_t} \big(\mathrm{e}^{-(T-t) r^{\mathrm{f}}} R_t \Phi^{\mathrm{c}}_+ (t, R_t) - \kappa \, \mathrm{e}^{-(T-t) r^l} \Phi^{\mathrm{c}}_- (t, R_t) \big) \\
&\quad = \frac{1}{\kappa} \mathrm{e}^{-(T-t) r^{\mathrm{f}}} \Phi^{\mathrm{c}}_+ (t, R_t) - \frac{\mathrm{e}^{-(T-t) r^l}}{R_t} \Phi^{\mathrm{c}}_- (t, R_t) \\
&\quad = \frac{1}{\kappa} \mathrm{e}^{-(T-t) r^{\mathrm{f}}} \Phi_- \left(t, \frac{1}{R_t} \right) - \frac{\mathrm{e}^{-(T-t) r^l}}{R_t} \Phi_+ \left(t, \frac{1}{R_t} \right),
\end{aligned}$$

where

$$\Phi^{\mathrm{c}}_+ (t, x) := \Phi \left(\frac{\log (x / \kappa) + (T-t) \big(r^l - r^{\mathrm{f}} + \sigma^2 / 2 \big)}{\sigma \sqrt{T-t}} \right),$$

and

$$\Phi^{\mathrm{c}}_- (t, x) := \Phi \left(\frac{\log (x / \kappa) + (T-t) \big(r^l - r^{\mathrm{f}} - \sigma^2 / 2 \big)}{\sigma \sqrt{T-t}} \right).$$

	"Local" market	"Foreign" market
Measure	$\mathbb{P}^*$	$\widehat{\mathbb{P}}$
Discount factor	$t \mapsto \mathrm{e}^{-r^l t}$	$t \mapsto \mathrm{e}^{-r^{\mathrm{f}} t}$
Martingale	$t \mapsto \mathrm{e}^{-tr^l} N_t = \mathrm{e}^{-t(r^l - r^{\mathrm{f}})} R_t$	$t \mapsto \dfrac{X_t}{N_t} = \widehat{X}_t = \dfrac{\mathrm{e}^{(r^l - r^{\mathrm{f}})t}}{R_t}$
Option price	$\mathrm{e}^{-(T-t)r^l} \mathbb{E}^* \big[(R_T - \kappa)^+ \, \big\vert \, \mathcal{F}_t \big]$	$\mathrm{e}^{-(T-t)r^{\mathrm{f}}} \widehat{\mathbb{E}} \left[\left(\dfrac{1}{\kappa} - \dfrac{1}{R_T} \right)^+ \, \Big\vert \, \mathcal{F}_t \right]$
Application	Local purchase of foreign currency	Foreign selling of local currency

Table 16.1: Local *vs.* foreign exchange options.

Example - Buy back guarantee

The put option priced

$$\begin{aligned} & \mathrm{e}^{-(T-t)r^{\mathrm{f}}} \widehat{\mathbb{E}} \left[\left(\frac{1}{\kappa} - \frac{1}{R_T} \right)^+ \, \Big\vert \, \mathcal{F}_t \right] \\ & = \frac{1}{\kappa} \mathrm{e}^{-(T-t)r^{\mathrm{f}}} \Phi_- \left(t, \frac{1}{R_t} \right) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t} \Phi_+ \left(t, \frac{1}{R_t} \right) \end{aligned}$$

on the foreign market corresponds to a **buy back guarantee*** in currency exchange. In the case of an option "at the money" with $\kappa = R_t$ and $r^{\mathrm{l}} = r^{\mathrm{f}} \simeq 0$, we find

$$\begin{aligned} \widehat{\mathbb{E}} \left[\left(\frac{1}{R_t} - \frac{1}{R_T} \right)^+ \, \Big\vert \, \mathcal{F}_t \right] & = \frac{1}{R_t} \times \left(\Phi \left(\frac{\sigma \sqrt{T-t}}{2} \right) - \Phi \left(- \frac{\sigma \sqrt{T-t}}{2} \right) \right) \\ & = \frac{1}{R_t} \times \left(2 \Phi \left(\frac{\sigma \sqrt{T-t}}{2} \right) - 1 \right). \end{aligned}$$

For example, let R_t denote the USD/EUR (USDEUR=X) exchange rate from a foreign currency (USD) to a local currency (EUR), *i.e.* one unit of the foreign currency (USD) corresponds to $R_t = 1/1.23$ units of local currency (EUR). Taking $T - t = 30$ days and $\sigma = 10\%$, we find that the foreign currency put option allowing for the foreign sale of one EURO back into USDs is priced at the money in USD as

* Right-click to open or save the attachment.

$$\widehat{\mathbb{E}}\left[\left(\frac{1}{R_t}-\frac{1}{R_T}\right)^+ \Bigm| \mathcal{F}_t\right] = 1.23\big(2\Phi\big(0.05\times\sqrt{31/365}\big)-1\big)$$
$$= 1.23(2\times 0.505813-1)$$
$$= \$0.01429998$$

per USD, or €0.011626 per exchanged unit of EURO. Based on a displayed option price of €4.5 per unit of foreign currency (USD), in order to make the contract fair this would translate into an average amount of

$$\frac{4.5}{0.011626} \simeq \ N387$$

exchanged per contract at the counter by the foreign customers subscribing to the buy back guarantee.

16.4 Pricing Exchange Options

Based on Proposition 16.4, we model the process $\widehat{X}_t$ of forward prices as a continuous martingale under $\widehat{\mathbb{P}}$, written as

$$d\widehat{X}_t = \widehat{\sigma}_t d\widehat{W}_t, \qquad t \geqslant 0, \tag{16.27}$$

where $\big(\widehat{W}_t\big)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$ and $(\widehat{\sigma}_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted stochastic volatility process. More precisely, we assume that $\big(\widehat{X}_t\big)_{t\in\mathbb{R}_+}$ has the dynamics

$$d\widehat{X}_t = \widehat{\sigma}_t\big(\widehat{X}_t\big)d\widehat{W}_t, \tag{16.28}$$

where $x \mapsto \widehat{\sigma}_t(x)$ is a local volatility function which is Lipschitz in x, uniformly in $t \geqslant 0$. The Markov property of the diffusion process $\big(\widehat{X}_t\big)_{t\in\mathbb{R}_+}$, cf. Theorem V-6-32 of Protter (2004), shows that when $\widehat{g}$ is a deterministic payoff function, the conditional expectation $\widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \bigm| \mathcal{F}_t\big]$ can be written as

$$\widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \bigm| \mathcal{F}_t\big] = \widehat{C}\big(t,\widehat{X}_t\big), \qquad 0 \leqslant t \leqslant T,$$

where $\widehat{C}(t,x)$ is a (measurable) function of t and $\widehat{X}_t$. Consequently, a vanilla option with claim payoff $C := N_T\widehat{g}\big(\widehat{X}_T\big)$ can be priced from Proposition 16.5 as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} N_T\widehat{g}\big(\widehat{X}_T\big) \Bigm| \mathcal{F}_t\right] = N_t\widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \bigm| \mathcal{F}_t\big]$$
$$= N_t\widehat{C}\big(t,\widehat{X}_t\big), \quad 0 \leqslant t \leqslant T. \tag{16.29}$$

In the next Proposition 16.15 we state the Margrabe (1978) formula for the pricing of exchange options by the zero interest rate Black-Scholes formula. It will be applied in particular in Proposition 19.3 below for the pricing of bond options. Here, $(N_t)_{t\in\mathbb{R}_+}$ denotes any numéraire process satisfying Assumption (A).

Proposition 16.15. *(Margrabe (1978) formula). Assume that* $\widehat{\sigma}_t(\widehat{X}_t) = \widehat{\sigma}(t)\widehat{X}_t$, *i.e. the martingale* $(\widehat{X}_t)_{t\in[0,T]}$ *is a (driftless) geometric Brownian motion under* $\widehat{\mathbb{P}}$ *with deterministic volatility* $(\widehat{\sigma}(t))_{t\in[0,T]}$. *Then we have*

$$\mathbb{E}^*\left[e^{-\int_t^T r_s ds} (X_T - \kappa N_T)^+ \,\Big|\, \mathcal{F}_t \right] = X_t \Phi^0_+(t, \widehat{X}_t) - \kappa N_t \Phi^0_-(t, \widehat{X}_t), \tag{16.30}$$

$t \in [0,T]$, *where*

$$\Phi^0_+(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2} \right), \quad \Phi^0_-(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} - \frac{v(t,T)}{2} \right), \tag{16.31}$$

and $v^2(t,T) = \int_t^T \widehat{\sigma}^2(s)ds$.

Proof. Taking $g(x) := (x-\kappa)^+$ in (16.29), the call option with payoff

$$\begin{aligned}(X_T - \kappa N_T)^+ &= N_T(\widehat{X}_T - \kappa)^+ \\ &= N_T\left(\widehat{X}_t \exp\left(\int_t^T \widehat{\sigma}(t) d\widehat{W}_t - \frac{1}{2}\int_t^T |\widehat{\sigma}(t)|^2 dt \right) - \kappa \right)^+,\end{aligned}$$

and floating strike price κN_T is priced by (16.29) as

$$\begin{aligned}\mathbb{E}^*\left[e^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \,\Big|\, \mathcal{F}_t \right] &= \mathbb{E}^*\left[e^{-\int_t^T r_s ds} N_T (\widehat{X}_T - \kappa)^+ \,\Big|\, \mathcal{F}_t \right] \\ &= N_t \widehat{\mathbb{E}}\big[(\widehat{X}_T - \kappa)^+ \,\big|\, \mathcal{F}_t \big] \\ &= N_t \widehat{C}(t, \widehat{X}_t),\end{aligned}$$

where the function $\widehat{C}(t, \widehat{X}_t)$ is given by the Black-Scholes formula

$$\widehat{C}(t,x) = x\Phi^0_+(t,x) - \kappa\Phi^0_-(t,x),$$

with zero interest rate, since $(\widehat{X}_t)_{t\in[0,T]}$ is a driftless geometric Brownian motion which is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$, and $\widehat{X}_T$ is a lognormal random variable with variance coefficient $v^2(t,T) = \int_t^T \widehat{\sigma}^2(s)ds$. Hence we have

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T-\kappa N_T)^+ \;\middle|\; \mathcal{F}_t\right] = N_t\widehat{C}(t,\widehat{X}_t)$$
$$= N_t\widehat{X}_t\Phi_+^0(t,\widehat{X}_t) - \kappa N_t\Phi_-^0(t,\widehat{X}_t),$$

$t \geqslant 0$. □

In particular, from Proposition 16.8 and (16.15), we can take $\widehat{\sigma}(t) = \sigma_t^X - \sigma_t^N$ when $(\sigma_t^X)_{t\in\mathbb{R}_+}$ and $(\sigma_t^N)_{t\in\mathbb{R}_+}$ are deterministic.

Examples:

a) When the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function of time and $N_t = \mathrm{e}^{\int_t^T r(s)ds}$, $0 \leqslant t \leqslant T$, we have $\widehat{\mathbb{P}} = \mathbb{P}^*$ and Proposition 16.15 yields the Merton (1973) "zero interest rate" version of the Black-Scholes formula

$$\mathrm{e}^{-\int_t^T r(s)ds}\mathbb{E}^*\left[(X_T-\kappa)^+ \mid \mathcal{F}_t\right]$$
$$= X_t\Phi_+^0\left(t, \mathrm{e}^{\int_t^T r(s)ds}X_t\right) - \kappa\,\mathrm{e}^{-\int_t^T r(s)ds}\Phi_-^0\left(t, \mathrm{e}^{\int_t^T r(s)ds}X_t\right),$$

where Φ_+^0 and Φ_-^0 are defined in (16.31) and $(X_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$\frac{dX_t}{X_t} = r(t)dt + \widehat{\sigma}(t)dW_t, \quad i.e. \quad \frac{d\widehat{X}_t}{\widehat{X}_t} = \widehat{\sigma}(t)dW_t, \quad 0 \leqslant t \leqslant T.$$

b) In the case of pricing under a *forward numéraire*, *i.e.* when $N_t = P(t,T)$, $t \in [0,T]$, we get

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T-\kappa)^+ \;\middle|\; \mathcal{F}_t\right] = X_t\Phi_+^0(t,\widehat{X}_t) - \kappa P(t,T)\Phi_-^0(t,\widehat{X}_t),$$

$0 \leqslant t \leqslant T$, since $N_T = P(T,T) = 1$. In particular, when $X_t = P(t,S)$ the above formula allows us to price a bond call option on $P(T,S)$ as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \;\middle|\; \mathcal{F}_t\right] = P(t,S)\Phi_+^0(t,\widehat{X}_t) - \kappa P(t,T)\Phi_-^0(t,\widehat{X}_t),$$

$0 \leqslant t \leqslant T$, provided that the martingale $\widehat{X}_t = P(t,S)/P(t,T)$ under $\widehat{\mathbb{P}}$ is given by a geometric Brownian motion, cf. Section 19.2.

16.5 Hedging by Change of Numéraire

In this section we reconsider and extend the Black-Scholes self-financing hedging strategies found in (7.42)-(7.43) and Proposition 7.13 of Chapter 7. For this, we use the stochastic integral representation of the forward claim payoffs and change of numéraire in order to compute self-financing portfolio strategies. Our hedging portfolios will be built on the assets (X_t, N_t), not on X_t

and the money market account $B_t = \mathrm{e}^{\int_0^t r_s ds}$, extending the classical hedging portfolios that are available in from the Black-Scholes formula, using a technique from Jamshidian (1996), cf. also Privault and Teng (2012).

Consider a claim with random payoff C, typically an interest rate derivative, cf. Chapter 19. Assume that the forward claim payoff $C/N_T \in L^2(\Omega)$ has the stochastic integral representation

$$\widehat{C} := \frac{C}{N_T} = \widehat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^T \widehat{\phi}_t d\widehat{X}_t, \tag{16.32}$$

where $(\widehat{X}_t)_{t\in[0,T]}$ is given by (16.27) and $(\widehat{\phi}_t)_{t\in[0,T]}$ is a square-integrable adapted process under $\widehat{\mathbb{P}}$, from which it follows that the forward claim price

$$\widehat{V}_t := \frac{V_t}{N_t} = \frac{1}{N_t}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathcal{F}_t\right] = \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\middle|\, \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T,$$

is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$, that can be decomposed as

$$\widehat{V}_t = \widehat{\mathbb{E}}\big[\widehat{C} \mid \mathcal{F}_t\big] = \widehat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^t \widehat{\phi}_s d\widehat{X}_s, \qquad 0 \leqslant t \leqslant T. \tag{16.33}$$

The next Proposition 16.16 extends the argument of Jamshidian (1996) to the general framework of pricing using change of numéraire. Note that this result differs from the standard formula that uses the money market account $B_t = \mathrm{e}^{\int_0^t r_s ds}$ for hedging instead of N_t, cf. *e.g.* Geman et al. (1995) pages 453-454. The notion of self-financing portfolio is similar to that of Definition 5.8.

Proposition 16.16. *Letting* $\widehat{\eta}_t := \widehat{V}_t - \widehat{X}_t\widehat{\phi}_t$, *with* $\widehat{\phi}_t$ *defined in* (16.33), $0 \leqslant t \leqslant T$, *the portfolio allocation*

$$\big(\widehat{\phi}_t, \widehat{\eta}_t\big)_{t\in[0,T]}$$

with value

$$V_t = \widehat{\phi}_t X_t + \widehat{\eta}_t N_t, \qquad 0 \leqslant t \leqslant T,$$

is self-financing in the sense that

$$dV_t = \widehat{\phi}_t dX_t + \widehat{\eta}_t dN_t,$$

and it hedges the claim payoff C, *i.e.*

$$V_t = \widehat{\phi}_t X_t + \widehat{\eta}_t N_t = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T. \tag{16.34}$$

Proof. In order to check that the portfolio allocation $\big(\widehat{\phi}_t, \widehat{\eta}_t\big)_{t\in[0,T]}$ hedges the claim payoff C it suffices to check that (16.34) holds since by (16.8) the price V_t at time $t \in [0,T]$ of the hedging portfolio satisfies

$$V_t = N_t \widehat{V}_t = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\Big|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

Next, we show that the portfolio allocation $(\widehat{\phi}_t, \widehat{\eta}_t)_{t\in[0,T]}$ is self-financing. By *numéraire invariance*, cf. *e.g.* page 184 of Protter (2001), we have, using the relation $d\widehat{V}_t = \widehat{\phi}_t d\widehat{X}_t$ from (16.33),

$$\begin{aligned}
dV_t &= d\big(N_t \widehat{V}_t\big) \\
&= \widehat{V}_t dN_t + N_t d\widehat{V}_t + dN_t \bullet d\widehat{V}_t \\
&= \widehat{V}_t dN_t + N_t \widehat{\phi}_t d\widehat{X}_t + \widehat{\phi}_t dN_t \bullet d\widehat{X}_t \\
&= \widehat{\phi}_t \widehat{X}_t dN_t + N_t \widehat{\phi}_t d\widehat{X}_t + \widehat{\phi}_t dN_t \bullet d\widehat{X}_t + \big(\widehat{V}_t - \widehat{\phi}_t \widehat{X}_t\big) dN_t \\
&= \widehat{\phi}_t d\big(N_t \widehat{X}_t\big) + \widehat{\eta}_t dN_t \\
&= \widehat{\phi}_t dX_t + \widehat{\eta}_t dN_t.
\end{aligned}$$

□

We now consider an application to the forward Delta hedging of European-type options with payoff $C = N_T \widehat{g}\big(\widehat{X}_T\big)$ where $\widehat{g} : \mathbb{R} \longrightarrow \mathbb{R}$ and $\big(\widehat{X}_t\big)_{t\in\mathbb{R}_+}$ has the Markov property as in (16.28), where $\widehat{\sigma} : \mathbb{R}_+ \times \mathbb{R}$ is a *deterministic* function. Assuming that the function $\widehat{C}(t, x)$ defined by

$$\widehat{V}_t := \widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \,\big|\, \mathcal{F}_t\big] = \widehat{C}\big(t, \widehat{X}_t\big)$$

is $\mathcal{C}^2$ on $\mathbb{R}_+$, we have the following corollary of Proposition 16.16, which extends the Black-Scholes Delta hedging technique to the general change of numéraire setup.

Corollary 16.17. *Letting* $\widehat{\eta}_t = \widehat{C}\big(t, \widehat{X}_t\big) - \widehat{X}_t \dfrac{\partial \widehat{C}}{\partial x}\big(t, \widehat{X}_t\big)$, $0 \leqslant t \leqslant T$, *the portfolio allocation*

$$\left(\frac{\partial \widehat{C}}{\partial x}\big(t, \widehat{X}_t\big), \widehat{\eta}_t\right)_{t\in[0,T]}$$

with value

$$V_t = \widehat{\eta}_t N_t + X_t \frac{\partial \widehat{C}}{\partial x}\big(t, \widehat{X}_t\big), \qquad t \geqslant 0,$$

is self-financing and hedges the claim payoff $C = N_T \widehat{g}\big(\widehat{X}_T\big)$.

Proof. This result follows directly from Proposition 16.16 by noting that by Itô's formula, and the martingale property of $\widehat{V}_t$ under $\widehat{\mathbb{P}}$ the stochastic integral representation (16.33) is given by

$$\begin{aligned}
\widehat{V}_T &= \widehat{C} \\
&= \widehat{g}\big(\widehat{X}_T\big)
\end{aligned}$$

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$$
\begin{aligned}
&= \widehat{C}(0,\widehat{X}_0) + \int_0^T \frac{\partial \widehat{C}}{\partial t}(t,\widehat{X}_t)dt + \frac{1}{2}\int_0^T \frac{\partial^2 \widehat{C}}{\partial x^2}(t,\widehat{X}_t)|\widehat{\sigma}_t|^2 dt \\
&\quad + \int_0^T \frac{\partial \widehat{C}}{\partial x}(t,\widehat{X}_t)d\widehat{X}_t \\
&= \widehat{C}(0,\widehat{X}_0) + \int_0^T \frac{\partial \widehat{C}}{\partial x}(t,\widehat{X}_t)d\widehat{X}_t \\
&= \widehat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^T \widehat{\phi}_t d\widehat{X}_t, \qquad 0 \leqslant t \leqslant T,
\end{aligned}
$$

hence

$$
\widehat{\phi}_t = \frac{\partial \widehat{C}}{\partial x}(t,\widehat{X}_t), \qquad 0 \leqslant t \leqslant T.
$$

□

In the case of an exchange option with payoff function

$$
C = (X_T - \kappa N_T)^+ = N_T\big(\widehat{X}_T - \kappa\big)^+
$$

on the geometric Brownian motion $\big(\widehat{X}_t\big)_{t\in[0,T]}$ under $\widehat{\mathbb{P}}$ with

$$
\widehat{\sigma}_t\big(\widehat{X}_t\big) = \widehat{\sigma}(t)\widehat{X}_t, \tag{16.35}
$$

where $\big(\widehat{\sigma}(t)\big)_{t\in[0,T]}$ is a deterministic volatility function of time, we have the following corollary on the hedging of exchange options based on the Margrabe (1978) formula (16.30).

Corollary 16.18. *The decomposition*

$$
\mathbb{E}^*\left[e^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \,\Big|\, \mathcal{F}_t \right] = X_t \Phi^0_+\big(t,\widehat{X}_t\big) - \kappa N_t \Phi^0_-\big(t,\widehat{X}_t\big)
$$

yields a self-financing portfolio allocation $\big(\Phi^0_+\big(t,\widehat{X}_t\big), -\kappa\Phi^0_-\big(t,\widehat{X}_t\big)\big)_{t\in[0,T]}$ *in the assets* (X_t,N_t)*, that hedges the claim payoff* $C = (X_T - \kappa N_T)^+$*.*

Proof. We apply Corollary 16.17 and the relation

$$
\frac{\partial \widehat{C}}{\partial x}(t,x) = \Phi^0_+(t,x), \qquad x \in \mathbb{R},
$$

for the function $\widehat{C}(t,x) = x\Phi^0_+(t,x) - \kappa\Phi^0_-(t,x)$, cf. Relation (6.17) in Proposition 6.4. □

Note that the Delta hedging method requires the computation of the function $\widehat{C}(t,x)$ and that of the associated finite differences, and may not apply to path-dependent claims.

Examples:

a) When the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function of time and $N_t = \mathrm{e}^{\int_t^T r(s)ds}$, Corollary 16.18 yields the usual Black-Scholes hedging strategy

$$\Big(\Phi_+\big(t,\widehat{X}_t\big), -\kappa\,\mathrm{e}^{\int_0^T r(s)ds}\Phi_-\big(t,X_t\big)\Big)_{t\in[0,T]}$$
$$= \big(\Phi_+^0\big(t,\,\mathrm{e}^{\int_t^T r(s)ds}\widehat{X}_t\big), -\kappa\,\mathrm{e}^{\int_0^T r(s)ds}\Phi_-^0\big(t,\,\mathrm{e}^{\int_t^T r(s)ds}X_t\big)\big)_{t\in[0,T]},$$

in the assets $\big(X_t,\,\mathrm{e}^{\int_0^t r(s)ds}\big)$, that hedges the claim payoff $C = (X_T-\kappa)^+$, with

$$\Phi_+(t,x) := \Phi\left(\frac{\log(x/\kappa)+\int_t^T r(s)ds+(T-t)\sigma^2/2}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) := \Phi\left(\frac{\log(x/\kappa)+\int_t^T r(s)ds-(T-t)\sigma^2/2}{\sigma\sqrt{T-t}}\right).$$

b) In case $N_t = P(t,T)$ and $X_t = P(t,S)$, $0\leqslant t\leqslant T<S$, Corollary 16.18 shows that when $\big(\widehat{X}_t\big)_{t\in[0,T]}$ is modeled as the geometric Brownian motion (16.35) under $\widehat{\mathbb{P}}$, the bond call option with payoff $(P(T,S)-\kappa)^+$ can be hedged as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+\,\big|\,\mathcal{F}_t\right] = P(t,S)\Phi_+\big(t,\widehat{X}_t\big)-\kappa P(t,T)\Phi_-\big(t,\widehat{X}_t\big)$$

by the self-financing portfolio allocation

$$\big(\Phi_+\big(t,\widehat{X}_t\big), -\kappa\Phi_-\big(t,\widehat{X}_t\big)\big)_{t\in[0,T]}$$

in the assets $(P(t,S),P(t,T))$, *i.e.* one needs to hold the quantity $\Phi_+\big(t,\widehat{X}_t\big)$ of the bond maturing at time S, and to short a quantity $\kappa\Phi_-\big(t,\widehat{X}_t\big)$ of the bond maturing at time T.

Exercises

Exercise 16.1 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0 under the risk-neutral probability measure $\mathbb{P}^*$. Consider a numéraire $(N_t)_{t\in\mathbb{R}_+}$ given by

$$N_t := N_0\,\mathrm{e}^{\eta B_t-\eta^2 t/2}, \qquad t\geqslant 0,$$

and a risky asset price process $(X_t)_{t\in\mathbb{R}_+}$ given by

$$X_t := X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

in a market with risk-free interest rate $r = 0$. Let $\widehat{\mathbb{P}}$ denote the forward measure relative to the numéraire $(N_t)_{t\in\mathbb{R}_+}$, under which the process $\widehat{X}_t := X_t/N_t$ of forward prices is known to be a martingale.

a) Using the Itô formula, compute

$$d\widehat{X}_t = d\left(\frac{X_t}{N_t}\right) = \frac{X_0}{N_0} d\big(\mathrm{e}^{(\sigma-\eta)B_t - (\sigma^2-\eta^2)t/2}\big).$$

b) Explain why the exchange option price $\mathbb{E}^*[(X_T - \lambda N_T)^+]$ at time 0 has the Black-Scholes form

$$\mathbb{E}^*[(X_T - \lambda N_T)^+] = X_0 \Phi\left(\frac{\log\big(\widehat{X}_0/\lambda\big)}{\widehat{\sigma}\sqrt{T}} + \frac{\widehat{\sigma}\sqrt{T}}{2}\right) - \lambda N_0 \Phi\left(\frac{\log\big(\widehat{X}_0/\lambda\big)}{\widehat{\sigma}\sqrt{T}} - \frac{\widehat{\sigma}\sqrt{T}}{2}\right). \tag{16.36}$$

Hints:

(i) Use the change of numéraire identity

$$\mathbb{E}^*[(X_T - \lambda N_T)^+] = N_0 \widehat{\mathbb{E}}\big[\big(\widehat{X}_T - \lambda\big)^+\big].$$

(ii) The forward price $\widehat{X}_t$ is a martingale under the forward measure $\widehat{\mathbb{P}}$ relative to the numéraire $(N_t)_{t\in\mathbb{R}_+}$.

c) Give the value of $\widehat{\sigma}$ in terms of σ and η.

Exercise 16.2 Let $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ be correlated standard Brownian motions started at 0 under the risk-neutral probability measure $\mathbb{P}^*$, with correlation $\mathrm{Corr}\big(B_s^{(1)}, B_t^{(2)}\big) = \rho \min(s,t)$, *i.e.* $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$. Consider two asset prices $\big(S_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(S_t^{(2)}\big)_{t\in\mathbb{R}_+}$ given by the geometric Brownian motions

$$S_t^{(1)} := S_0^{(1)} \,\mathrm{e}^{rt + \sigma B_t^{(1)} - \sigma^2 t/2}, \quad \text{and} \quad S_t^{(2)} := S_0^{(2)} \,\mathrm{e}^{rt + \eta B_t^{(2)} - \eta^2 t/2}, \qquad t \geqslant 0.$$

Let $\widehat{\mathbb{P}}_2$ denote the forward measure with numéraire $(N_t)_{t\in\mathbb{R}_+} := \big(S_t^{(2)}\big)_{t\in\mathbb{R}_+}$ and Radon-Nikodym density

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_2}{\mathrm{d}\mathbb{P}^*} = \mathrm{e}^{-rT} \frac{S_T^{(2)}}{S_0^{(2)}} = \mathrm{e}^{\eta B_T^{(2)} - \eta^2 T/2}.$$

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a) Using the Girsanov Theorem 16.7, determine the shifts $\big(\widehat{B}_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(\widehat{B}_t^{(2)}\big)_{t\in\mathbb{R}_+}$ of $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ which are standard Brownian motions under $\widehat{\mathbb{P}}_2$.
b) Using the Itô formula, compute

$$d\widehat{S}_t^{(1)} = d\left(\frac{S_t^{(1)}}{S_t^{(2)}}\right) = \frac{S_0^{(1)}}{S_0^{(2)}} d\big(\mathrm{e}^{\sigma B_t^{(1)} - \eta B_t^{(2)} - (\sigma^2-\eta^2)t/2}\big),$$

and write the answer in terms of the martingales $d\widehat{B}_t^{(1)}$ and $d\widehat{B}_t^{(2)}$.
c) Using change of numéraire, explain why the exchange option price

$$\mathrm{e}^{-rT}\mathbb{E}^*\big[\big(S_T^{(1)} - \lambda S_T^{(2)}\big)^+\big]$$

at time 0 has the Black-Scholes form

$$\mathrm{e}^{-rT}\mathbb{E}^*\big[\big(S_T^{(1)} - \lambda S_T^{(2)}\big)^+\big] = S_0^{(1)}\Phi\left(\frac{\log\big(\widehat{S}_0^{(1)}/\lambda\big)}{\widehat{\sigma}\sqrt{T}} + \frac{\widehat{\sigma}\sqrt{T}}{2}\right) - \lambda S_0^{(2)}\Phi\left(\frac{\log\big(\widehat{S}_0^{(1)}/\lambda\big)}{\widehat{\sigma}\sqrt{T}} - \frac{\widehat{\sigma}\sqrt{T}}{2}\right),$$

where the value of $\widehat{\sigma}$ can be expressed in terms of σ and η.

Exercise 16.3 Consider two zero-coupon bond prices of the form $P(t,T) = F(t,r_t)$ and $P(t,S) = G(t,r_t)$, where $(r_t)_{t\in\mathbb{R}_+}$ is a short-term interest rate process. Taking $N_t := P(t,T)$ as a numéraire defining the forward measure $\widehat{\mathbb{P}}$, compute the dynamics of $(P(t,S))_{t\in[0,T]}$ under $\widehat{\mathbb{P}}$ using a standard Brownian motion $\big(\widehat{W}_t\big)_{t\in[0,T]}$ under $\widehat{\mathbb{P}}$.

Exercise 16.4 Forward contracts. Using a change of numéraire argument for the numéraire $N_t := P(t,T)$, $t\in[0,T]$, compute the price at time $t\in[0,T]$ of a forward (or future) contract with payoff $P(T,S) - K$ in a bond market with short-term interest rate $(r_t)_{t\in\mathbb{R}_+}$. How would you hedge this forward contract?

Exercise 16.5 (Question 2.7 page 17 of Downes et al. (2008)). Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by $dS_t = rS_t dt + \sigma S_t dB_t$ under the risk-neutral probability measure $\mathbb{P}^*$, where $r\in\mathbb{R}$ and $\sigma > 0$, and the option with payoff

$$S_T(S_T-K)^+ = \text{Max}(S_T(S_T-K),0)$$

at maturity T.

a) Show that the option payoff can be rewritten as

$$(S_T(S_T-K))^+ = N_T(S_T-K)^+$$

for a suitable choice of numéraire process $(N_t)_{t\in[0,T]}$.

b) Rewrite the option price $\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T(S_T-K))^+ \mid \mathcal{F}_t\big]$ using a forward measure $\widehat{\mathbb{P}}$ and a change of numéraire argument.

c) Find the dynamics of $(S_t)_{t\in\mathbb{R}_+}$ under the forward measure $\widehat{\mathbb{P}}$.

d) Price the option with payoff

$$S_T(S_T-K)^+ = \text{Max}(S_T(S_T-K),0)$$

at time $t\in[0,T]$ using the Black-Scholes formula.

Exercise 16.6 Consider the risk asset with dynamics $dS_t = rS_tdt + \sigma S_tdW_t$ with constant interest rate $r\in\mathbb{R}$ and volatility $\sigma>0$ under the risk-neutral measure $\mathbb{P}^*$. The power call option with payoff $(S_T^n - K^n)^+$, $n\geqslant 1$, is priced at time $t\in[0,T]$ as

$$\begin{aligned}&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\big(S_T^n-K^n\big)^+ \,\big|\, \mathcal{F}_t\big]\\ &\quad = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[S_T^n\mathbb{1}_{\{S_T\geqslant K\}} \,\big|\, \mathcal{F}_t\big] - K^n\,\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\mathbb{1}_{\{S_T\geqslant K\}} \,\big|\, \mathcal{F}_t\big]\end{aligned}$$

under the risk-neutral measure $\mathbb{P}^*$.

a) Write down the density $\dfrac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}$ using the numéraire process

$$N_t := S_t^n\,\mathrm{e}^{-(n-1)n\sigma^2t/2-(n-1)rt}, \qquad t\in[0,T].$$

b) Construct a standard Brownian motion $\widehat{W}_t$ under $\widehat{\mathbb{P}}$.

c) Compute the term $\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[S_T^n\mathbb{1}_{\{S_T\geqslant K\}}\big|\mathcal{F}_t\big]$ using change of numéraire.

Hint: We have

$$\begin{aligned}\mathbb{P}^*(S_T\geqslant K\mid\mathcal{F}_t) &= \mathbb{E}^*\big[\mathbb{1}_{\{S_T\geqslant K\}}\,\big|\,\mathcal{F}_t\big]\\ &= \Phi\left(\frac{\log(S_t/K)+(r^l-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

d) Price the power call option with payoff $(S_T^n-K^n)^+$ at time $t\in[0,T]$.

Exercise 16.7 Bond options. Consider two bonds with maturities T and S, with prices $P(t,T)$ and $P(t,S)$ given by

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$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \zeta_t^T dW_t,$$

and

$$\frac{dP(t,S)}{P(t,S)} = r_t dt + \zeta_t^S dW_t,$$

where $(\zeta^T(s))_{s\in[0,T]}$ and $(\zeta^S(s))_{s\in[0,S]}$ are deterministic volatility functions of time.

a) Show, using Itô's formula, that

$$d\left(\frac{P(t,S)}{P(t,T)}\right) = \frac{P(t,S)}{P(t,T)}(\zeta^S(t) - \zeta^T(t))d\widehat{W}_t,$$

where $\big(\widehat{W}_t\big)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$.

b) Show that

$$P(T,S) = \frac{P(t,S)}{P(t,T)}\exp\left(\int_t^T(\zeta^S(s) - \zeta^T(s))d\widehat{W}_s - \frac{1}{2}\int_t^T|\zeta^S(s) - \zeta^T(s)|^2 ds\right).$$

Let $\widehat{\mathbb{P}}$ denote the forward measure associated to the numéraire

$$N_t := P(t,T), \qquad 0 \leqslant t \leqslant T.$$

c) Show that for all $S,T > 0$ the price at time t

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S) - \kappa)^+ \,\Big|\, \mathcal{F}_t\right]$$

of a bond call option on $P(T,S)$ with payoff $(P(T,S) - \kappa)^+$ is equal to

$$\begin{aligned}&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S) - \kappa)^+ \,\Big|\, \mathcal{F}_t\right] \qquad (16.37)\\ &= P(t,S)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{\kappa P(t,T)}\right) - \kappa P(t,T)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{\kappa P(t,T)}\right),\end{aligned}$$

where

$$v^2 = \int_t^T|\zeta^S(s) - \zeta^T(s)|^2 ds.$$

d) Compute the self-financing hedging strategy that hedges the bond option using a portfolio based on the assets $P(t,T)$ and $P(t,S)$.

Exercise 16.8 Consider two risky assets S_1 and S_2 modeled by the geometric Brownian motions

$$S_1(t) = \mathrm{e}^{\sigma_1 W_t + \mu t} \qquad \text{and} \qquad S_2(t) = \mathrm{e}^{\sigma_2 W_t + \mu t}, \qquad t \geqslant 0, \qquad (16.38)$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$.

a) Find a condition on r, μ and σ_2 so that the discounted price process $\mathrm{e}^{-rt} S_2(t)$ is a martingale under $\mathbb{P}$.
b) Assume that $r - \mu = \sigma_2^2/2$, and let

$$X_t = \mathrm{e}^{(\sigma_2^2 - \sigma_1^2)t/2} S_1(t), \qquad t \geqslant 0.$$

Show that the discounted process $\mathrm{e}^{-rt} X_t$ is a martingale under $\mathbb{P}$.
c) Taking $N_t = S_2(t)$ as numéraire, show that the forward process $\widehat{X}(t) = X_t / N_t$ is a martingale under the forward measure $\widehat{\mathbb{P}}$ defined by the Radon-Nikodym density

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \mathrm{e}^{-rT} \frac{N_T}{N_0}.$$

Recall that

$$\widehat{W}_t := W_t - \sigma_2 t$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$.
d) Using the relation

$$\mathrm{e}^{-rT} \mathbb{E}^*[(S_1(T) - S_2(T))^+] = N_0 \widehat{\mathbb{E}} \left[\frac{(S_1(T) - S_2(T))^+}{N_T} \right],$$

compute the price

$$\mathrm{e}^{-rT} \mathbb{E}^*[(S_1(T) - S_2(T))^+]$$

of the exchange option on the assets S_1 and S_2.

Exercise 16.9 Compute the price $\mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\mathbb{1}_{\{R_T \geqslant \kappa\}} \,\middle|\, \mathcal{F}_t \right]$ at time $t \in [0, T]$ of a cash-or-nothing "binary" foreign exchange call option with maturity T and strike price κ on the foreign exchange rate process $(R_t)_{t \in \mathbb{R}_+}$ given by

$$dR_t = (r^l - r^{\mathrm{f}}) R_t dt + \sigma R_t dW_t,$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.
Hint: We have the relation

$$\mathbb{P}^* \left(x \,\mathrm{e}^{X + \mu} \geqslant \kappa \right) = \Phi \left(\frac{\mu - \log(\kappa / x)}{\sqrt{\mathrm{Var}[X]}} \right)$$

for $X \simeq \mathcal{N}(0, \mathrm{Var}[X])$ a centered Gaussian random variable.

Exercise 16.10 Extension of Proposition 16.8 to correlated Brownian motions. Assume that $(S_t)_{t \in \mathbb{R}_+}$ and $(N_t)_{t \in \mathbb{R}_+}$ satisfy the stochastic differential equations

$$dS_t = r_t S_t dt + \sigma_t^S S_t dW_t^S, \quad \text{and} \quad dN_t = \eta_t N_t dt + \sigma_t^N N_t dW_t^N,$$

where $(W_t^S)_{t\in\mathbb{R}_+}$ and $(W_t^N)_{t\in\mathbb{R}_+}$ have the correlation

$$dW_t^S \bullet dW_t^N = \rho dt,$$

where $\rho \in [-1, 1]$.

a) Show that $(W_t^N)_{t\in\mathbb{R}_+}$ can be written as

$$W_t^N = \rho W_t^S + \sqrt{1-\rho^2} W_t, \qquad t \geqslant 0,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, independent of $(W_t^S)_{t\in\mathbb{R}_+}$.

b) Letting $X_t = S_t/N_t$, show that dX_t can be written as

$$dX_t = (r_t - \eta_t + (\sigma_t^N)^2 - \rho\sigma_t^N\sigma_t^S)X_t dt + \widehat{\sigma}_t X_t dW_t^X,$$

where $(W_t^X)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$ and $\widehat{\sigma}_t$ is to be computed.

Exercise 16.11 Quanto options (Exercise 9.5 in Shreve (2004)). Consider an asset priced S_t at time t, with

$$dS_t = r S_t dt + \sigma^S S_t dW_t^S,$$

and an exchange rate $(R_t)_{t\in\mathbb{R}_+}$ given by

$$dR_t = (r - r^{\text{f}})R_t dt + \sigma^R R_t dW_t^R,$$

from (16.20) in Proposition 16.10, where $(W_t^R)_{t\in\mathbb{R}_+}$ is written as

$$W_t^R = \rho W_t^S + \sqrt{1-\rho^2} W_t, \qquad t \geqslant 0,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, independent of $(W_t^S)_{t\in\mathbb{R}_+}$, *i.e.*, we have

$$dW_t^R \bullet dW_t^S = \rho dt,$$

where ρ is a correlation coefficient.

a) Let

$$a = r^l - r^{\text{f}} + \rho\sigma^R\sigma^S - (\sigma^R)^2$$

and $X_t = \mathrm{e}^{at} S_t/R_t$, $t \geqslant 0$, and show by Exercise 16.10 that dX_t can be written as

$$dX_t = r X_t dt + \widehat{\sigma} X_t dW_t^X,$$

where $(W_t^X)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$ and $\widehat{\sigma}$ is to be determined.

b) Compute the price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{S_T}{R_T}-\kappa\right)^+ \Bigg| \mathcal{F}_t\right]$$

of the quanto option at time $t \in [0,T]$.

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