Chapter 6
Black-Scholes Pricing and Hedging

The Black and Scholes (1973) PDE is a Partial Differential Equation that is used for the pricing of vanilla options under the absence of arbitrage and self-financing portfolio assumptions. In this chapter, we derive the Black-Scholes PDE and present its solution by the heat kernel method, with application to the pricing and hedging of European call and put options.

6.1 The Black-Scholes PDE

In this chapter, we work in a market based on a riskless asset with price $(A_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{A_{t+dt}-A_t}{A_t}=rdt, \qquad \frac{dA_t}{A_t}=rdt, \qquad \frac{dA_t}{dt}=rA_t, \quad t\geqslant 0,$$

with

$$A_t=A_0\,\mathrm{e}^{rt}, \qquad t\geqslant 0,$$

and a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ modeled using a geometric Brownian motion defined from the equation

$$\frac{dS_t}{S_t}=\mu dt+\sigma dB_t, \quad t\geqslant 0, \tag{6.1}$$

which admits the solution

$$S_t = S_0 \exp\left(\sigma B_t + \mu t - \frac{1}{2}\sigma^2 t\right), \qquad t \geqslant 0,$$

see Proposition 5.15.

```
library(quantmod); getSymbols("0005.HK",from="2016-02-15",to="2017-05-11",src="yahoo")
Market_Prices<-Ad(`0005.HK`); returns = (Market_Prices-lag(Market_Prices)) / lag(Market_Prices)
sigma=sd(as.numeric(returns[-1])); r=mean(as.numeric(returns[-1]))
N=length(Market_Prices); t <- 0:N; a=(1+r)*(1-sigma)-1;b=(1+r)*(1+sigma)-1
X <- matrix((a+b)/2+(b-a)*rnorm( N-1, 0, 1)/2, 1, N-1)
X <- as.numeric(Market_Prices[1])*cbind(0,t(apply((1+X),1,cumprod))); X[,1]=Market_Prices[1];
x=seq(100,100+N-1); dates <- index(Market_Prices)
Geometric_Brownian_Motion<-xts(x =X[1,], order.by = dates)
myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme();myTheme$col$line.col <- "blue"; myTheme$rylab <- FALSE;
chart_Series(Market_Prices,pars=myPars, theme = myTheme);
dexp<-as.numeric(Market_Prices[1])*exp(r*seq(1,305)); ddexp<-xts(x =dexp, order.by = dates)
dev.new(width=16,height=8); par(mfrow=c(1,2));
add_TA(exp(log(ddexp)), on=1, col="black",layout=NULL, lwd=4 ,legend=NULL)
graph <- chart_Series(Geometric_Brownian_Motion,theme=myTheme,pars=myPars); myylim <- graph$get_ylim()
graph <- add_TA(exp(log(ddexp)), on=1, col="black",layout=NULL, lwd=4 ,legend=NULL)
myylim[[2]] <- structure(c(min(Market_Prices),max(Market_Prices)), fixed=TRUE)
graph$set_ylim(myylim); graph
```

The adjusted close price Ad() is the closing price after adjustments for applicable splits and dividend distributions.

The next Figure 6.1 presents a graph of underlying asset price market data, which is compared to the geometric Brownian motion simulations of Figures 5.4 and 5.5.

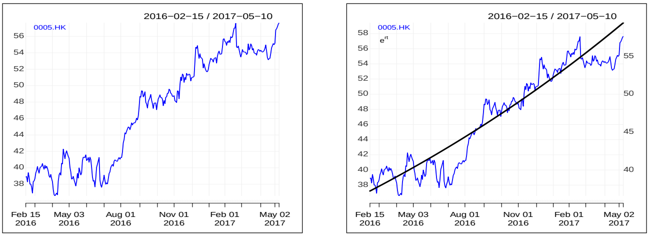

Fig. 6.1: Graph of underlying market prices.

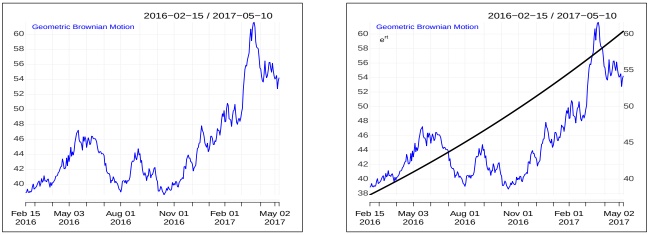

Fig. 6.2: Graph of simulated geometric Brownian motion.

The R package Sim.DiffProc can be used to estimate the coefficients of a geometric Brownian motion fitting observed market data.

```
library("Sim.DiffProc")
fx <- expression( theta[1]*x ); gx <- expression( theta[2]*x )
fitsde(data = as.ts(Market_Prices), drift = fx, diffusion = gx, start = list(theta1=0.01,
      theta2=0.01),pmle="euler")
```

In the sequel, we start by deriving the Black and Scholes (1973) Partial Differential Equation (PDE) for the value of a self-financing portfolio. Note that the drift parameter μ in (6.1) is absent in the PDE (6.2), and it does not appear as well in the Black and Scholes (1973) formula (6.12).

Proposition 6.1. *Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that*

(i) the portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the portfolio value $V_t := \eta_t A_t + \xi_t S_t$, takes the form

$$V_t = g(t, S_t), \qquad t \geqslant 0,$$

for some function $g \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ of t and S_t.

Then, the function $g(t,x)$ satisfies the Black and Scholes (1973) PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0, \quad (6.2)$$

and $\xi_t = \xi_t(S_t)$ is given by the partial derivative

$$\xi_t = \xi_t(S_t) = \frac{\partial g}{\partial x}(t, S_t), \qquad t \geqslant 0. \quad (6.3)$$

Proof. (*i*) First, we note that the self-financing condition (5.9) in Proposition 5.8 implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rg(t, S_t)dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \end{aligned} \tag{6.4}$$

$t \geqslant 0$. We now rewrite (5.20) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \geqslant 0,$$

as in (4.22), by taking

$$u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \qquad t \geqslant 0.$$

(ii) By (4.24), the application of Itô's formula Theorem 4.24 to $V_t = g(t, S_t)$ leads to

$$\begin{aligned} dV_t &= dg(t, S_t) \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + \frac{\partial g}{\partial x}(t, S_t)dS_t + \frac{1}{2}(dS_t)^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + v_t \frac{\partial g}{\partial x}(t, S_t)dt + u_t \frac{\partial g}{\partial x}(t, S_t)dB_t + \frac{1}{2}|u_t|^2 \frac{\partial^2 g}{\partial x^2}(t, S_t)dt \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t)dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t)dB_t. \end{aligned} \tag{6.5}$$

By respective identification of components in dB_t and dt in (6.4) and (6.5), we get

$$\begin{cases} rg(t, S_t)dt + (\mu - r)\xi_t S_t dt = \dfrac{\partial g}{\partial t}(t, S_t)dt + \mu S_t \dfrac{\partial g}{\partial x}(t, S_t)dt + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t, S_t)dt, \\ \xi_t S_t \sigma dB_t = S_t \sigma \dfrac{\partial g}{\partial x}(t, S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rg(t, S_t) = \dfrac{\partial g}{\partial t}(t, S_t) + rS_t \dfrac{\partial g}{\partial x}(t, S_t) + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t, S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t, S_t), \qquad 0 \leqslant t \leqslant T, \end{cases} \tag{6.6}$$

which yields (6.2) after substituting S_t with $x > 0$. □

As a byproduct of Proposition 6.1 and (6.5), we also find

$$V_t = V_0 + r\int_0^t V_s ds + \sigma \int_0^t S_s \frac{\partial g}{\partial x}(s, S_s)dB_s, \qquad t \geqslant 0,$$

which also yields the discounted expansion

$$\widetilde{V}_t = V_0 + \sigma \int_0^t \widetilde{S}_s \frac{\partial g}{\partial x}(s, S_s) dB_s, \qquad t \geqslant 0. \tag{6.7}$$

The derivative giving ξ_t in (6.3) is called the *Delta* of the option price, see Proposition 6.4 below. The amount invested on the riskless asset is

$$\eta_t A_t = V_t - \xi_t S_t = g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t),$$

and η_t is given by

$$\begin{aligned}\eta_t &= \frac{V_t - \xi_t S_t}{A_t} \\ &= \frac{1}{A_t}\left(g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t)\right) \\ &= \frac{1}{A_0 \, \mathrm{e}^{rt}}\left(g(t, S_t) - S_t \frac{\partial g}{\partial x}(t, S_t)\right).\end{aligned}$$

In the next Proposition 6.2 we add a terminal condition $g(T, x) = h(x)$ to the Black-Scholes PDE (6.2) in order to price a claim payoff C of the form $C = h(S_T)$. As in the discrete-time case, the arbitrage-free price $\pi_t(C)$ at time $t \in [0, T]$ of the claim payoff C is defined to be the value V_t of the self-financing portfolio hedging C.

Proposition 6.2. *Under the assumptions of Proposition 6.1, the arbitrage-free price $\pi_t(C)$ at time $t \in [0, T]$ of the (vanilla) option with claim payoff $C = h(S_T)$ is given by $\pi_t(C) = g(t, S_t)$ and the hedging allocation ξ_t is given by the partial derivative (6.3), where the function $g(t, x)$ is solution of the following Black-Scholes PDE:*

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ g(T,x) = h(x), \quad x > 0. \end{cases} \tag{6.8}$$

Proof. Proposition 6.1 shows that the solution $g(t, x)$ of (6.2), $g \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$, represents the value $V_t = \eta_t A_t + \xi_t S_t = g(t, S_t)$, $t \in \mathbb{R}_+$, of a self-financing portfolio strategy $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$. By Definition 3.1, $\pi_t(C) := V_t = g(t, S_t)$ is the arbitrage-free price at time $t \in [0, T]$ of the vanilla option with claim payoff $C = h(S_T)$. □

The absence of the drift parameter μ from the PDE (6.8) can be understood in the next forward contract example, in which the claim payoff can be hedged by leveraging on the value S_t of the underlying asset, independently of the trend parameter μ.

Example - Forward contracts

The holder of a long forward contract is committed to purchasing a risky asset at the price K at maturity time T, while the contract issuer has the obligation to hand in the asset priced S_T in exchange for the amount K at maturity time T.

Clearly, the contract has the claim payoff $C = S_T - K$, and it can be hedged by simply holding $\xi_t = 1$ asset in the portfolio at all times $t \in [0, T]$. Denoting by V_t the option price at time $t \in [0, T]$, the amount $S_t - V_t$ has to be borrowed at time t in order to purchase the asset. As the amount K received at maturity T should be used to refund the loan at time T, we should have $(S_t - V_t)\,\mathrm{e}^{(T-t)r} = K$, hence

$$V_t = S_t - K\,\mathrm{e}^{-(T-t)r}, \quad 0 \leqslant t \leqslant T. \tag{6.9}$$

We note that the riskless allocation $\eta_t = -K\,\mathrm{e}^{-rT}$ is also constant over time $t \in [0, T]$ due to self-financing.

More precisely, the forward contract can be realized by the option issuer via the following steps:

a) At time t, *receive* the option premium $V_t := S_t - \mathrm{e}^{-(T-t)r}K$ from the option buyer.
b) *Borrow* $\mathrm{e}^{-(T-t)r}K$ from the bank, to be refunded at maturity.
c) *Buy* the risky asset using the amount $S_t - \mathrm{e}^{-(T-t)r}K + \mathrm{e}^{-(T-t)r}K = S_t$.
d) *Hold* the risky asset until maturity (do nothing, constant portfolio strategy).
e) At maturity T, *hand in* the asset to the option holder, who will pay the amount K in return.
f) Use the amount $K = \mathrm{e}^{(T-t)r}\,\mathrm{e}^{-(T-t)r}K$ to *refund* the lender of $\mathrm{e}^{-(T-t)r}K$ borrowed at time t.

On the other hand, the payoff C of the long forward contract can be written as $C = S_T - K = h(S_T)$ where h is the (affine) payoff function $h(x) = x - K$, and the Black-Scholes PDE (6.8) admits the easy solution

$$g(t, x) = x - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad 0 \leqslant t \leqslant T, \tag{6.10}$$

showing that the price at time $t \in [0, T]$ of the long forward contract is

$$g(t, S_t) = S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T.$$

In addition, the Delta of the option price is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1, \qquad 0 \leqslant t \leqslant T,$$

which recovers the static, “hedge and forget” strategy, cf. Exercise 6.7.

Forward contracts can be used for physical delivery, *e.g.* for live cattle. In the case of European options, the basic "hedge and forget" constant strategy

$$\xi_t = 1, \quad \eta_t = \eta_0, \quad 0 \leqslant t \leqslant T,$$

will hedge the option only if

$$S_T + \eta_0 A_T \geqslant (S_T - K)^+,$$

i.e. if $-\eta_0 A_T \leqslant K \leqslant S_T$.

Futures contracts

For a futures contract expiring at time T, we take $K = S_0 \mathrm{e}^{rT}$ and the contract is usually quoted at time t in terms of the forward price

$$\mathrm{e}^{(T-t)r}\big(S_t - K\,\mathrm{e}^{-(T-t)r}\big) = \mathrm{e}^{(T-t)r}S_t - K = \mathrm{e}^{(T-t)r}S_t - S_0\,\mathrm{e}^{rT},$$

discounted at time T, or simply using $\mathrm{e}^{(T-t)r}S_t$. Futures contracts are *non-deliverable* forward contracts which are "marked to market" at each time step via a cash flow exchange between the two parties, ensuring that the absolute difference $\big|\mathrm{e}^{(T-t)r}S_t - K\big|$ is being credited to the buyer's account if $\mathrm{e}^{(T-t)r}S_t > K$, or to the seller's account if $\mathrm{e}^{(T-t)r}S_t < K$.

6.2 European Call Options

Recall that in the case of the European call option with strike price K the payoff function is given by $h(x) = (x-K)^+$ and the Black-Scholes PDE (6.8) reads

$$\begin{cases} r g_{\mathrm{c}}(t,x) = \dfrac{\partial g_{\mathrm{c}}}{\partial t}(t,x) + rx\dfrac{\partial g_{\mathrm{c}}}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g_{\mathrm{c}}}{\partial x^2}(t,x) \\ g_{\mathrm{c}}(T,x) = (x-K)^+. \end{cases} \tag{6.11}$$

The next proposition will be proved in Sections 6.5-6.6, see Proposition 6.11.

Proposition 6.3. *The solution of the PDE* (6.11) *is given by the* Black-Scholes *formula for call options:*

$$g_c(t,x) = \mathrm{Bl}(x,K,\sigma,r,T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \tag{6.12}$$

with

$$d_+(T-t) := \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} \tag{6.13}$$

and

$$d_-(T-t) := \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{6.14}$$

$0 \leqslant t < T$.

We note the relation

$$d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t}, \qquad 0 \leqslant t < T. \tag{6.15}$$

Here, "log" denotes the *natural logarithm* "ln", and

$$\Phi(x) := \mathbb{P}(X \leqslant x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

denotes the Cumulative Distribution Function (CDF) of a standard normal random variable $X \simeq \mathcal{N}(0,1)$, with the relation

$$\Phi(-x) = 1 - \Phi(x), \qquad x \in \mathbb{R}. \tag{6.16}$$

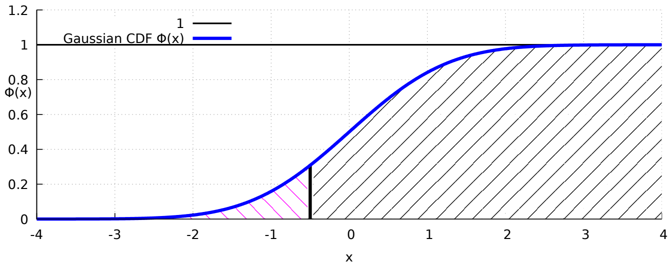

Fig. 6.3: Graph of the Gaussian Cumulative Distribution Function (CDF).

In other words, the European call option with strike price K and maturity T is priced at time $t \in [0,T]$ as

$$\begin{aligned} g_c(t,S_t) &= \mathrm{Bl}(S_t,K,\sigma,r,T-t) \\ &= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad 0 \leqslant t \leqslant T. \end{aligned}$$

The following R script implements the Black-Scholes formula for European call options in R.*

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
BSCall}
```

In comparison with the discrete-time Cox-Ross-Rubinstein (CRR) model of Section 2.6, the interest in the formula (6.12) is to provide an analytical solution that can be evaluated in a single step, which is computationally much more efficient.

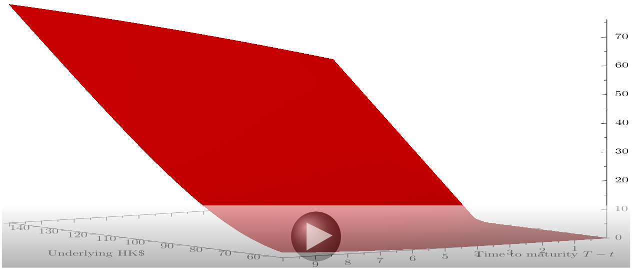

Fig. 6.4: Graph of the Black-Scholes call price map with strike price $K = 100$.†

Figure 6.4 presents an interactive graph of the Black-Scholes call price map, *i.e.* the solution

$$(t, x) \longmapsto g_c(t, x) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$$

of the Black-Scholes PDE (6.8) for a call option.

* Download the corresponding **IPython notebook** that can be run **here** or **here**.

† Right-click on the figure for interaction and "Full Screen Multimedia" view.

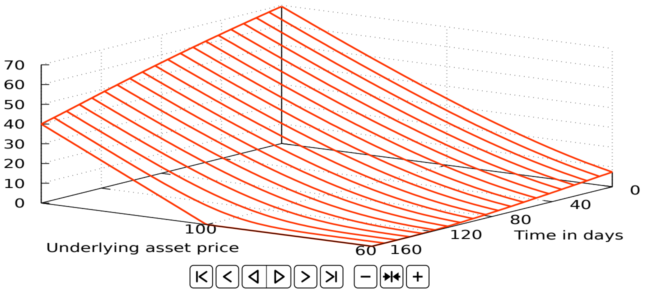

Fig. 6.5: Time-dependent solution of the Black-Scholes PDE (call option).*

The next proposition is proved by a direct differentiation of the Black-Scholes function, and will be recovered later using a probabilistic argument in Proposition 7.13 below.

Proposition 6.4. *The Black-Scholes Delta of the European call option is given by*

$$\xi_t = \xi_t(S_t) = \frac{\partial g_c}{\partial x}(t, S_t) = \Phi\big(d_+(T-t)\big) \in [0,1], \tag{6.17}$$

where

$$d_+(T-t) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}$$

is given by (6.13).

Proof. From Relation (6.15), we note that the derivative of the standard normal probability density function

$$\varphi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \qquad x \in \mathbb{R},$$

satisfies

$$\begin{aligned}
\varphi(d_+(T-t)) &= \varphi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} + |\sigma|\sqrt{T-t}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(d_-(T-t))^2 - (T-t)r - \log\frac{x}{K}\right)
\end{aligned}$$

* The animation works in Acrobat Reader on the entire pdf file.

$$= \frac{K}{x\sqrt{2\pi}} \mathrm{e}^{-(T-t)r} \exp\left(-\frac{1}{2}(d_-(T-t))^2\right)$$
$$= \frac{K}{x} \mathrm{e}^{-(T-t)r} \varphi(d_-(T-t)),$$

hence by (6.12) we have

$$\begin{aligned}
\frac{\partial g_c}{\partial x}(t,x) &= \frac{\partial}{\partial x}\left(x\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right) \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\left(\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right) \\
&= \Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad +x\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad +\frac{1}{|\sigma|\sqrt{T-t}}\varphi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad -\frac{K\,\mathrm{e}^{-(T-t)r}}{x|\sigma|\sqrt{T-t}}\varphi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \Phi(d_+(T-t)) + \frac{1}{|\sigma|\sqrt{T-t}}\varphi(d_+(T-t)) - \frac{K\,\mathrm{e}^{-(T-t)r}}{x|\sigma|\sqrt{T-t}}\varphi(d_-(T-t)) \\
&= \Phi(d_+(T-t)),
\end{aligned} \tag{6.18}$$

and we conclude from (6.3). □

As a consequence of Proposition 6.4, the Black-Scholes call price splits into a risky component $S_t\Phi\big(d_+(T-t)\big)$ and a riskless component $-K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$, as follows:

$$g_c(t,S_t) = \underbrace{S_t\Phi\big(d_+(T-t)\big)}_{\text{Risky investment (held)}} - \underbrace{K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),}_{\text{Risk free investment (borrowed)}} \quad 0 \leqslant t \leqslant T. \tag{6.19}$$

See Exercise 6.4 for a computation of the boundary values of $g_c(t,x)$, $t \in [0,T)$, $x > 0$. The following R script is an implementation of the Black-Scholes Delta for European call options.

```
DeltaCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
DeltaCall = pnorm(d1);DeltaCall}
```

In Figure 6.6 we plot the Delta of the European call option as a function of the underlying asset price and of the time remaining until maturity.

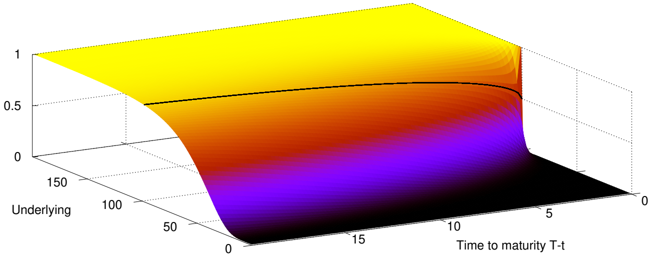

Fig. 6.6: Delta of a European call option with strike price $K = 100$, $r = 3\%$, $\sigma = 10\%$.

The *Gamma* of the European call option is defined as the first derivative or sensitivity of Delta with respect to the underlying asset price. It also represents the second derivative of the option price with respect to the underlying asset price. This gives

$$\begin{aligned}\gamma_t &= \frac{1}{S_t|\sigma|\sqrt{T-t}}\Phi'\big(d_+(T-t)\big)\\ &= \frac{1}{S_t|\sigma|\sqrt{2(T-t)\pi}}\exp\left(-\frac{1}{2}\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\right)\\ &\geqslant 0.\end{aligned}$$

In particular, a positive value of γ_t implies that the Delta $\xi_t = \xi_t(S_t)$ should increase when the underlying asset price S_t increases. In other words, the position ξ_t in the underlying asset should be increased by additional purchases if the underlying asset price S_t increases.

In Figure 6.7 we plot the (truncated) value of the Gamma of a European call option as a function of the underlying asset price and of time to maturity.

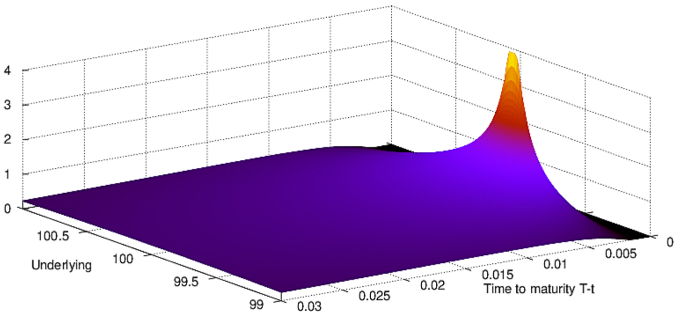

Fig. 6.7: Gamma of European call and put options with strike price $K = 100$.

As Gamma is always nonnegative, the Black-Scholes hedging strategy is to keep buying the risky underlying asset when its price increases, and to sell it when its price decreases, as can be checked from Figure 6.7.

Numerical example - Hedging a call option

In Figure 6.8 we consider the historical stock price of HSBC Holdings (0005.HK) over one year:

Fig. 6.8: Graph of the stock price of HSBC Holdings.

Consider the call option issued by Societe Generale on 31 December 2008 with strike price K=\$63.704, maturity T = October 05, 2009, and an entitlement ratio of 100, meaning that one option contract is divided into 100 *warrants*, cf. page 10. The next graph gives the time evolution of the Black-Scholes portfolio value

$$t \longmapsto g_{\mathrm{c}}(t, S_t)$$

driven by the market price $t \longmapsto S_t$ of the risky underlying asset as given in Figure 6.8, in which the number of days is counted from the origin and not from maturity.

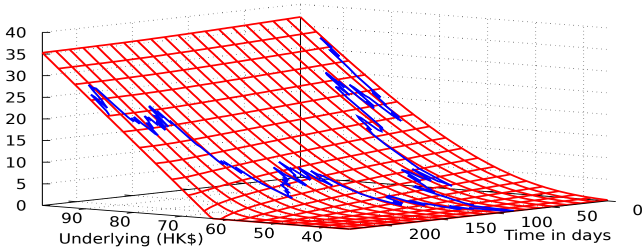

Fig. 6.9: Path of the Black-Scholes price for a call option on HSBC with $K = 63.70$.

As a consequence of (6.19), in the Black-Scholes call option hedging model, the amount invested in the risky asset is

$$\begin{aligned} S_t\xi_t &= S_t\Phi\big(d_+(T-t)\big) \\ &= S_t\Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\geqslant 0, \end{aligned}$$

which is always nonnegative, *i.e.* there is no short selling, and the amount invested on the riskless asset is

$$\begin{aligned} \eta_t A_t &= -K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\ &= -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\leqslant 0, \end{aligned}$$

which is always nonpositive, *i.e.* we are constantly borrowing money on the riskless asset, as noted in Figure 6.10.

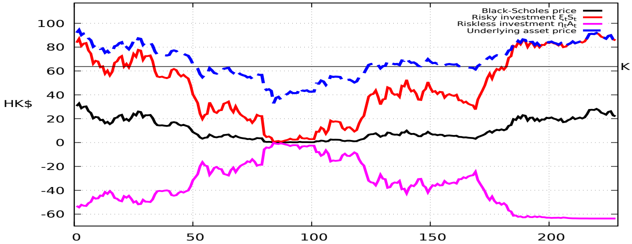

Fig. 6.10: Time evolution of a hedging portfolio for a call option on HSBC.

A comparison of Figure 6.10 with market data can be found in Figures 9.10a and 9.10b below.

Cash settlement. In the case of a cash settlement, the option issuer will satisfy the option contract by selling $\xi_T = 1$ stock at the price $S_T = \$83$, refund the $K = \$63$ risk-free investment, and hand in the remaining amount $C = (S_T - K)^+ = 83 - 63 = \$20$ to the option holder.

Physical delivery. In the case of physical delivery of the underlying asset, the option issuer will deliver $\xi_T = 1$ stock to the option holder in exchange for $K = \$63$, which will be used together with the portfolio value to refund the risk-free loan.

6.3 European Put Options

Similarly, in the case of the European put option with strike price K the payoff function is given by $h(x) = (K - x)^+$ and the Black-Scholes PDE (6.8) reads

$$\begin{cases} r g_{\mathrm{p}}(t,x) = \dfrac{\partial g_{\mathrm{p}}}{\partial t}(t,x) + rx\dfrac{\partial g_{\mathrm{p}}}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g_{\mathrm{p}}}{\partial x^2}(t,x), \\ g_{\mathrm{p}}(T,x) = (K - x)^+, \end{cases} \tag{6.20}$$

The next proposition can be proved from the call-put parity of Proposition 6.6 and Proposition 6.11, see Sections 6.5-6.6.

Proposition 6.5. *The solution of the PDE* (6.20) *is given by the* Black-Scholes *formula for put options:*

$$g_p(t,x) = K\,e^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big), \tag{6.21}$$

with

$$d_+(T-t) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} \tag{6.22}$$

and

$$d_-(T-t) = \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{6.23}$$

$0 \leqslant t < T$,

Figure 6.11 presents an interactive graph of the Black-Scholes put price map $(t,x) \mapsto g_p(t,x)$ given in (6.21).

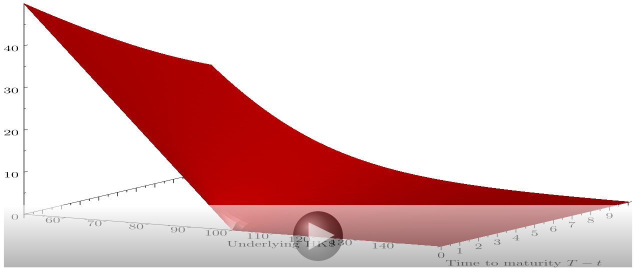

Fig. 6.11: Graph of the Black-Scholes put price function with strike price $K = 100$.*

In other words, the European put option with strike price K and maturity T is priced at time $t \in [0,T]$ as

$$g_p(t,S_t) = K\,e^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big), \quad 0 \leqslant t \leqslant T.$$

* Right-click on the figure for interaction and "Full Screen Multimedia" view.

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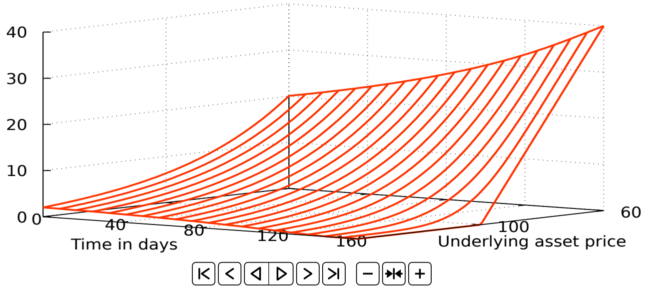

Fig. 6.12: Time-dependent solution of the Black-Scholes PDE (put option).*

The following R script is an implementation of the Black-Scholes formula for European put options in R.

```
BSPut <- function(S, K, r, T, sigma)
{d1 = (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 = d1 - sigma * sqrt(T);
BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1);BSPut}
```

Call-put parity

Proposition 6.6. Call-put parity. *We have the relation*

$$S_t - K\,\mathrm{e}^{-(T-t)r} = g_{\mathrm{c}}(t, S_t) - g_{\mathrm{p}}(t, S_t), \qquad 0 \leqslant t \leqslant T, \tag{6.24}$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K\,\mathrm{e}^{-(T-t)r}$.

Proof. The call-put parity (6.24) is a consequence of the relation

$$x - K = (x-K)^+ - (K-x)^+$$

satisfied by the terminal call and put payoff functions in the linear Black-Scholes PDE (6.8), which is solved as

$$x - K\,\mathrm{e}^{-(T-t)r} = g_{\mathrm{c}}(t, x) - g_{\mathrm{p}}(t, x)$$

for $t \in [0, T]$, since $x - K\,\mathrm{e}^{-(T-t)r}$ is the pricing function of the long forward contract with payoff $S_T - K$, see (6.9). It can also be verified directly from (6.12) and (6.21) as

$$\begin{aligned} g_{\mathrm{c}}(t,x) - g_{\mathrm{p}}(t,x) &= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\ &\quad -\Big(K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big)\Big) \end{aligned}$$

* The animation works in Acrobat Reader on the entire pdf file.

$$= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$$
$$-K\,\mathrm{e}^{-(T-t)r}\big(1-\Phi\big(d_-(T-t)\big)\big) + x\big(1-\Phi\big(d_+(T-t)\big)\big)$$
$$= x - K\,\mathrm{e}^{-(T-t)r}.$$

□

The *Delta* of the Black-Scholes put option can be obtained by differentiation of the call-put parity relation (6.24) and Proposition 6.4.

Proposition 6.7. *The* Delta *of the Black-Scholes put option is given by*

$$\xi_t = -\big(1-\Phi\big(d_+(T-t)\big)\big) = -\Phi\big(-d_+(T-t)\big) \in [-1,0], \quad 0 \leqslant t \leqslant T.$$

Proof. By differentiating on both sides of the call-put parity relation (6.24) and applying Proposition 6.4, we have

$$\begin{aligned}\frac{\partial g_{\mathrm{p}}}{\partial x}(t,S_t) &= \frac{\partial g_{\mathrm{c}}}{\partial x}(t,S_t) - 1\\ &= \Phi\big(d_+(T-t)\big) - 1\\ &= -\Phi\big(-d_+(T-t)\big), \qquad 0 \leqslant t \leqslant T,\end{aligned}$$

where we applied (6.16). □

As a consequence of Proposition 6.7, the Black-Scholes put price splits into a risky component $-S_t\Phi\big(-d_+(T-t)\big)$ and a riskless component $K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)$, as follows:

$$g_{\mathrm{p}}(t,S_t) = \underbrace{K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)}_{\text{Risk-free investment (savings)}} - \underbrace{S_t\Phi\big(-d_+(T-t)\big)}_{\text{Risky investment (short)}}, \quad 0 \leqslant t \leqslant T. \tag{6.25}$$

```
DeltaPut <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T)); DeltaPut = -pnorm(-d1);DeltaPut}
```

In Figure 6.13 we plot the Delta of the European put option as a function of the underlying asset price and of the time remaining until maturity.

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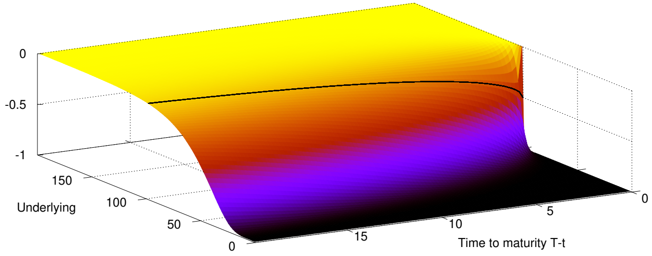


Fig. 6.13: Delta of a European put option with strike price $K = 100$, $r = 3\%$, $\sigma = 10\%$.

Numerical example - Hedging a put option

For one more example, we consider a put option issued by BNP Paribas on 04 November 2008 with strike price K=\$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593, cf. page 10. In the next Figure 6.14, the number of days is counted from the origin, not from maturity.

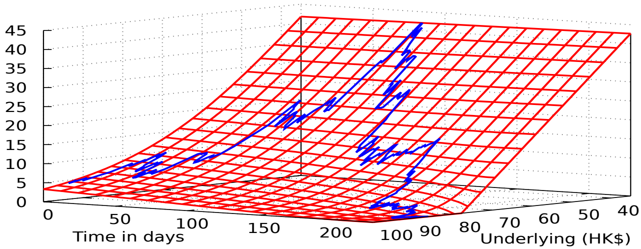


Fig. 6.14: Path of the Black-Scholes price for a put option on HSBC.

As a consequence of (6.25), the amount invested on the risky asset for the hedging of a put option is

$$-S_t\Phi\big(-d_+(T-t)\big) = -S_t\Phi\left(-\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)$$
$$\leqslant 0,$$

i.e. there is always short selling, and the amount invested on the riskless asset priced $A_t = \mathrm{e}^{rt}$, $t \in [0, T]$, is

$$\eta_t A_t = K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)$$

$$= K\,\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)$$
$$\geqslant 0,$$

which is always nonnegative, *i.e.* we are constantly saving money on the riskless asset, as noted in Figure 6.15.

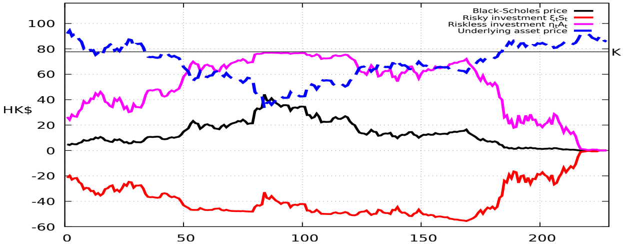


Fig. 6.15: Time evolution of the hedging portfolio for a put option on HSBC.

In the above example the put option finished out of the money (OTM), so that no cash settlement or physical delivery occurs. A comparison of Figure 6.10 with market data can be found in Figures 9.11a and 9.11b below.

6.4 Market Terms and Data

Table 6.1 provides a summary of formulas for the computation of Black-Scholes sensitivities, also called *Greeks*.*

		Call option	Put option
Option price	$g(t,S_t)$	$S_t\Phi(d_+(T-t)) - K\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))$	$K\,\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)) - S_t\Phi(-d_+(T-t))$
Delta (Δ)	$\frac{\partial g}{\partial x}(t,S_t)$	$\Phi(d_+(T-t)) \geqslant 0$	$-\Phi(-d_+(T-t)) \leqslant 0$
Gamma (Γ)	$\frac{\partial^2 g}{\partial x^2}(t,S_t)$	$\frac{\Phi'(d_+(T-t))}{S_t\|\sigma\|\sqrt{T-t}} \geqslant 0$	
Vega	$\frac{\partial g}{\partial \sigma}(t,S_t)$	$S_t\sqrt{T-t}\Phi'(d_+(T-t)) \geqslant 0$	
Theta (Θ)	$\frac{\partial g}{\partial t}(t,S_t)$	$-\frac{S_t\|\sigma\|\Phi'(d_+(T-t))}{2\sqrt{T-t}} - rK\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \leqslant 0$	$-\frac{S_t\|\sigma\|\Phi'(d_+(T-t))}{2\sqrt{T-t}} + rK\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))$
Rho (ρ)	$\frac{\partial g}{\partial r}(t,S_t)$	$(T-t)K\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \geqslant 0$	$-(T-t)K\,\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)) \leqslant 0$

Table 6.1: Black-Scholes Greeks (Wikipedia).

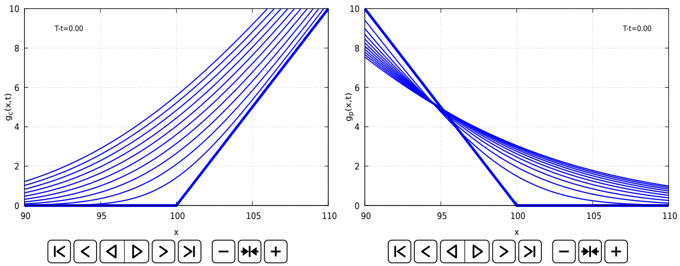
* "Every class feels like attending a Greek lesson" (AY2018-2019 student feedback).

From Table 6.1 we can conclude that call option prices are increasing functions of the underlying asset price S_t, of the interest rate r, and of the volatility parameter σ. Similarly, put option prices are decreasing functions of the underlying asset price S_t, of the interest rate r, and increasing functions of the volatility parameter σ, see also Exercise 6.14.

Parameter	Variation of call option prices	Variation of put option prices
Underlying S_t	Increasing ↗	Decreasing ↘
Volatility σ	Increasing ↗	Increasing ↗
Time t	Decreasing ↘ if $r \geqslant 0$	Depends on the underlying price level if $r > 0$
	Depends on the underlying price level if $r < 0$	Decreasing ↘ if $r \leqslant 0$
Interest rate r	Increasing ↗	Increasing ↘

Table 6.2: Variations of Black-Scholes prices.

The change of sign in the sensitivity Theta (Θ) with respect to time t can be verified in the following Figure 6.16 when $r > 0$.



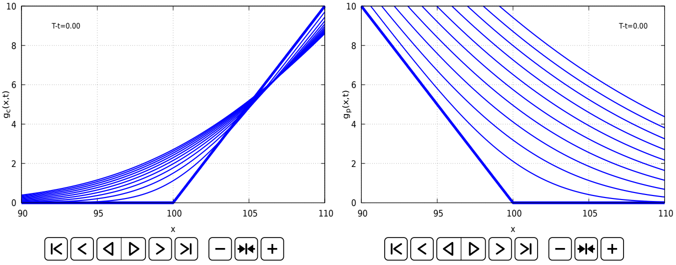
(a) Black-Scholes call price maps.

(b) Black-Scholes put price maps

Fig. 6.16: Time-dependent solutions of the Black-Scholes PDE with $r = +3\% > 0$.*

The next two figures show the variations of call and put option prices as functions of time when $r < 0$.

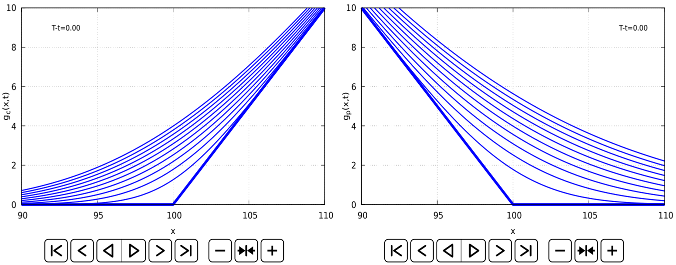
* The animation works in Acrobat Reader on the entire pdf file.



(a) Black-Scholes call price maps. (b) Black-Scholes put price maps

Fig. 6.17: Time-dependent solutions of the Black-Scholes PDE with $r = -3\% < 0$.*

The next two figures show the variations of call and put option prices as functions of time when $r = 0$.



(a) Black-Scholes call price maps. (b) Black-Scholes put price maps

Fig. 6.18: Time-dependent solutions of the Black-Scholes PDE with $r = 0$.†

Intrinsic value. The *intrinsic value* at time $t \in [0, T]$ of the option with claim payoff $C = h(S_T^{(1)})$ is given by the immediate exercise payoff $h(S_t^{(1)})$. The *extrinsic value* at time $t \in [0, T]$ of the option is the remaining difference $\pi_t(C) - h(S_t^{(1)})$ between the option price $\pi_t(C)$ and the immediate exercise payoff $h(S_t^{(1)})$. In general, the option price $\pi_t(C)$ decomposes as

$$\pi_t(C) = \underbrace{h(S_t^{(1)})}_{\text{Intrinsic value}} + \underbrace{\pi_t(C) - h(S_t^{(1)})}_{\text{Extrinsic value}}, \qquad 0 \leqslant t \leqslant T.$$

Moneyness. The moneyness is the ratio of the intrinsic value of the option *vs.* the current price of the underlying asset, *i.e.*

† The animation works in Acrobat Reader on the entire pdf file.

$$M_t(C) := \frac{h\big(S_t^{(1)}\big)}{S_t^{(1)}}, \qquad 0 \leqslant t \leqslant T.$$

The option is said to be "*in the money*" (ITM) when $M_t > 0$, "*at the money*" (ATM) when $M_t = 0$, and "*out of the money*" (OTM) when $M_t < 0$.

Break-even price. The *break-even* price BEP_t of the underlying asset is the value of S for which the intrinsic option value $h(S)$ equals the option price $\pi_t(C)$ at time $t \in [0, T]$. For European call options with payoff function $h(x) = (x - K)^+$, it is given by

$$\mathrm{BEP}_t := K + \pi_t(C) = K + g_{\mathrm{c}}(t, S_t), \qquad t = 0, 1, \ldots, N. \tag{6.26}$$

whereas for European put options with payoff function $h(x) = (K - x)^+$, it is given by

$$\mathrm{BEP}_t := K - \pi_t(C) = K - g_{\mathrm{p}}(t, S_t), \qquad 0 \leqslant t \leqslant T. \tag{6.27}$$

Premium. The option *premium* OP_t can be defined as the variation required from the underlying asset price in order to reach the break-even price, *i.e.* we have

$$\mathrm{OP}_t := \frac{\mathrm{BEP}_t - S_t}{S_t} = \frac{K + g(t, S_t) - S_t}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

for European call options, and

$$\mathrm{OP}_t := \frac{S_t - \mathrm{BEP}_t}{S_t} = \frac{S_t + g(t, S_t) - K}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

for European put options, see Figure 6.19 below. The term "premium" is sometimes also used to denote the arbitrage-free price $g(t, S_t)$ of the option.

Gearing. The *gearing* at time $t \in [0, T]$ of the option with claim payoff $C = h(S_T)$ is defined as the ratio

$$\mathrm{G}_t := \frac{S_t}{\pi_t(C)} = \frac{S_t}{g(t, S_t)}, \qquad 0 \leqslant t \leqslant T.$$

Effective gearing. The *effective gearing* at time $t \in [0, T]$ of the option with claim payoff $C = h(S_T)$ is defined as the ratio

$$\begin{aligned} \mathrm{EG}_t &:= \mathrm{G}_t \xi_t \\ &= \frac{\xi_t S_t}{\pi_t(C)} \end{aligned}$$

$$= \frac{S_t}{\pi_t(C)} \frac{\partial g}{\partial x}(t, S_t)$$
$$= \frac{S_t}{g(t, S_t)} \frac{\partial g}{\partial x}(t, S_t)$$
$$= S_t \frac{\partial}{\partial x} \log g(t, S_t), \qquad 0 \leqslant t \leqslant T.$$

The effective gearing

$$\mathrm{EG}_t = \frac{\xi_t S_t}{\pi_t(C)}$$

can be interpreted as the *hedge ratio*, *i.e.* the percentage of the portfolio which is invested on the risky asset. When written as

$$\frac{\Delta g(t, S_t)}{g(t, S_t)} = \mathrm{EG}_t \times \frac{\Delta S_t}{S_t},$$

the effective gearing gives the relative variation, or percentage change, $\Delta g(t, S_t)/g(t, S_t)$ of the option price $g(t, S_t)$ from the relative variation $\Delta S_t/S_t$ in the underlying asset price.

The ratio $\mathrm{EG}_t = S_t \partial \log g(t, S_t)/\partial x$ can also be interpreted as an *elasticity coefficient*.

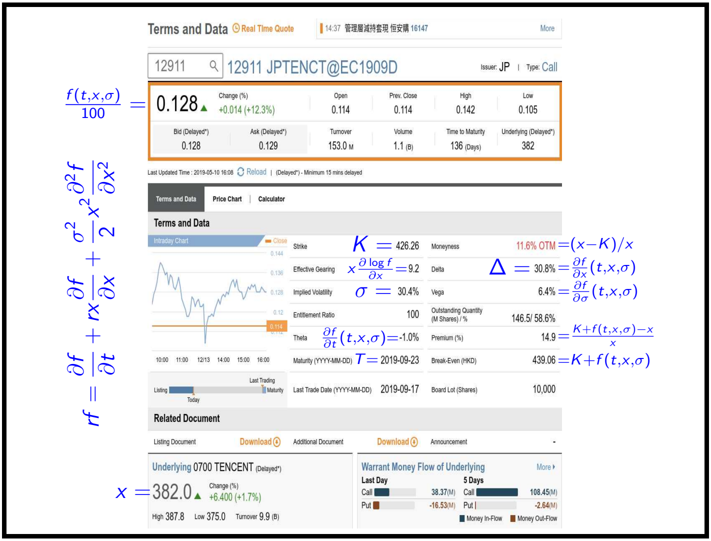


Fig. 6.19: Warrant terms and data.

The R package bizdays (requires to install QuantLib) can be used to compute calendar time *vs.* business time to maturity

```
install.packages("bizdays")
library(bizdays)
load_quantlib_calendars('HongKong', from='2018-01-01', to='2018-12-31')
load_quantlib_calendars('Singapore', from='2018-01-01', to='2018-12-31')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/HongKong')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/Singapore')
```

6.5 The Heat Equation

In the next proposition we notice that the solution $f(t,x)$ of the Black-Scholes PDE (6.8) can be transformed into a solution $g(t,y)$ of the simpler *heat equation* by a change of variable and a time inversion $t \longmapsto T-t$ on the interval $[0,T]$, so that the terminal condition at time T in the Black-Scholes equation (6.28) becomes an initial condition at time $t=0$ in the heat equation (6.31). See also here for a related discussion on changes of variables for the Black-Scholes PDE.

Proposition 6.8. *Assume that $f(t,x)$ solves the Black-Scholes call pricing PDE*

$$\begin{cases} rf(t,x) = \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2x^2\dfrac{\partial^2 f}{\partial x^2}(t,x), \\ f(T,x) = (x-K)^+, \end{cases} \tag{6.28}$$

with terminal condition $h(x) = (x-K)^+$, $x > 0$. Then, the function $g(t,y)$ defined by

$$g(t,y) = \mathrm{e}^{rt} f\big(T-t,\, \mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}\big) \tag{6.29}$$

solves the heat equation (6.31) with initial condition

$$\psi(y) := h\big(\mathrm{e}^{|\sigma|y}\big), \qquad y \in \mathbb{R}, \tag{6.30}$$

i.e. we have

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ g(0,y) = h\big(\mathrm{e}^{|\sigma|y}\big). \end{cases} \tag{6.31}$$

Proposition 6.8 will be proved in Section 6.6. It will allow us to solve the Black-Scholes PDE (6.28) based on the solution of the heat equation (6.31) with initial condition $\psi(y) = h\big(\mathrm{e}^{|\sigma|y}\big)$, $y \in \mathbb{R}$, by inversion of Relation (6.29) with $s = T-t$, $x = \mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}$, *i.e.*

$$f(s,x) = \mathrm{e}^{-(T-s)r} g\left(T-s,\, \frac{-(\sigma^2/2-r)(T-s)+\log x}{|\sigma|}\right).$$

Next, we focus on the *heat equation*

$$\frac{\partial \varphi}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y) \tag{6.32}$$

which is used to model the diffusion of heat over time through solids. Here, the data of $g(x,t)$ represents the temperature measured at time t and point x. We refer the reader to Widder (1975) for a complete treatment of this topic.

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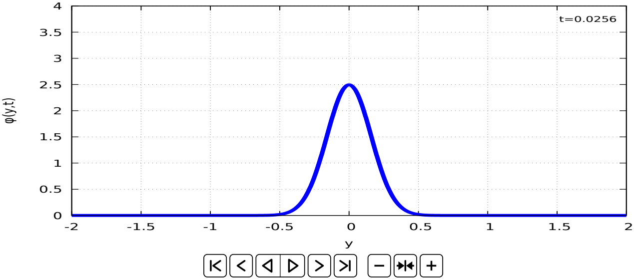

Fig. 6.20: Time-dependent solution of the heat equation.*

Proposition 6.9. *The fundamental solution of the heat equation* (6.32) *is given by the Gaussian probability density function*

$$\varphi(t,y) := \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-y^2/(2t)}, \qquad y \in \mathbb{R},$$

with variance $t > 0$.

Proof. The proof is done by a direct calculation, as follows:

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(t,y) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\right)\\ &= -\frac{\mathrm{e}^{-y^2/(2t)}}{2t^{3/2}\sqrt{2\pi}} + \frac{y^2}{2t^2}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\\ &= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right)\varphi(t,y),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y) &= -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{y}{t}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\right)\\ &= -\frac{\mathrm{e}^{-y^2/(2t)}}{2t\sqrt{2\pi t}} + \frac{y^2}{2t^2}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\\ &= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right)\varphi(t,y), \qquad t > 0, \ y \in \mathbb{R}.\end{aligned}$$

□

* The animation works in Acrobat Reader on the entire pdf file.

In Section 6.6 the heat equation (6.32) will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

Proposition 6.10. *The heat equation*

$$\begin{cases} \frac{\partial g}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y) \\ g(0,y) = \psi(y) \end{cases} \tag{6.33}$$

with bounded continuous initial condition

$$g(0,y) = \psi(y)$$

has the solution

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}, \qquad y \in \mathbb{R}, \quad t > 0. \tag{6.34}$$

Proof. We have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-(y-z)^2/(2t)}}{\sqrt{2\pi t}}\right) dz \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \left(\frac{(y-z)^2}{t^2} - \frac{1}{t}\right) \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial z^2} \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial y^2} \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y). \end{aligned}$$

On the other hand, it can be checked that at time $t = 0$ we have

$$\begin{aligned} \lim_{t\to 0}\int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} &= \lim_{t\to 0}\int_{-\infty}^{\infty} \psi(y+z)\, \mathrm{e}^{-z^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \psi(y), \qquad y \in \mathbb{R}, \end{aligned}$$

see also (6.35) below. □

The next Figure 6.21 shows the evolution of $g(t,x)$ with initial condition based on the European call payoff function $h(x) = (x-K)^+$, *i.e.*

$$g(0,y) = \psi(y) = h\big(\mathrm{e}^{|\sigma|y}\big) = \big(\mathrm{e}^{|\sigma|y} - K\big)^+, \qquad y \in \mathbb{R}.$$

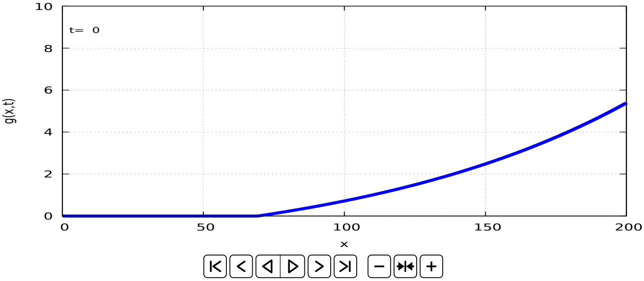

Fig. 6.21: Time-dependent solution of the heat equation.*

Let us provide a second proof of Proposition 6.10, this time using Brownian motion and stochastic calculus.

Proof of Proposition 6.10. First, we note that under the change of variable $x = z - y$ we have

$$\begin{aligned} g(t,y) &= \int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{\infty} \psi(y+x)\, \mathrm{e}^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}} \\ &= \mathbb{E}[\psi(y+B_t)] \\ &= \mathbb{E}[\psi(y-B_t)], \end{aligned}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion with $B_t \simeq \mathcal{N}(0,t)$, $t \geqslant 0$, under the initial condition

$$g(0,y) = \mathbb{E}[\psi(y+B_0)] = \mathbb{E}[\psi(y)] = \psi(y). \tag{6.35}$$

Applying Itô's formula to $\psi(y-B_t)$ and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, see Relation (4.17) in Proposition 4.21, we find

$$\begin{aligned} g(t,y) &= \mathbb{E}[\psi(y-B_t)] \\ &= \mathbb{E}\left[\psi(y) - \int_0^t \psi'(y-B_s)dB_s + \frac{1}{2}\int_0^t \psi''(y-B_s)ds\right] \end{aligned}$$

* The animation works in Acrobat Reader on the entire pdf file.

$$= \psi(y) - \mathbb{E}\left[\int_0^t \psi'(y-B_s)dB_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \psi''(y-B_s)ds\right]$$
$$= \psi(y) + \frac{1}{2}\int_0^t \mathbb{E}\left[\frac{\partial^2\psi}{\partial y^2}(y-B_s)\right]ds$$
$$= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\mathbb{E}\left[\psi(y-B_s)\right]ds$$
$$= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial y^2}(s,y)ds.$$

Hence we have

$$\frac{\partial g}{\partial t}(t,y) = \frac{\partial}{\partial t}\mathbb{E}[\psi(y-B_t)]$$
$$= \frac{1}{2}\frac{\partial^2}{\partial y^2}\mathbb{E}\left[\psi(y-B_t)\right]$$
$$= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).$$

Regarding the initial condition, we check that

$$g(0,y) = \mathbb{E}[\psi(y-B_0)] = \mathbb{E}[\psi(y)] = \psi(y).$$

□

The expression $g(t,y) = \mathbb{E}[\psi(y-B_t)]$ provides a probabilistic interpretation of the heat diffusion phenomenon based on Brownian motion. Namely, when $\psi_\varepsilon(y) := \mathbb{1}_{[-\varepsilon,\varepsilon]}(y)$, we find that

$$g_\varepsilon(t,y) = \mathbb{E}[\psi_\varepsilon(y-B_t)]$$
$$= \mathbb{E}[\mathbb{1}_{[-\varepsilon,\varepsilon]}(y-B_t)]$$
$$= \mathbb{P}\big(y-B_t \in [-\varepsilon,\varepsilon]\big)$$
$$= \mathbb{P}\big(y-\varepsilon \leqslant B_t \leqslant y+\varepsilon\big)$$

represents the probability of finding B_t within a neighborhood $[y-\varepsilon, y+\varepsilon]$ of the point $y \in \mathbb{R}$.

6.6 Solution of the Black-Scholes PDE

In this section we solve the Black-Scholes PDE by the kernel method of Section 6.5 and a change of variables. This solution method uses the change of variables (6.29) of Proposition 6.8 and a time inversion from which the terminal condition at time T in the Black-Scholes equation becomes an initial condition at time $t = 0$ in the heat equation.

Next, we provide the proof Proposition 6.8.

Proof of Proposition 6.8. Letting $s = T - t$ and $x = \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}$ and using Relation (6.29), *i.e.*

$$g(t, y) = \mathrm{e}^{rt} f\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big),$$

we have

$$\begin{aligned}
\frac{\partial g}{\partial t}(t, y) &= r \, \mathrm{e}^{rt} f\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big) - \mathrm{e}^{rt} \frac{\partial f}{\partial s}\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big) \\
&\quad + \left(\frac{\sigma^2}{2} - r\right) \mathrm{e}^{rt} \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t} \frac{\partial f}{\partial x}\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big) \\
&= r \, \mathrm{e}^{rt} f(T - t, x) - \mathrm{e}^{rt} \frac{\partial f}{\partial s}(T - t, x) + \left(\frac{\sigma^2}{2} - r\right) \mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x) \\
&= \frac{1}{2} \mathrm{e}^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T - t, x) + \frac{\sigma^2}{2} \mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x), \qquad (6.36)
\end{aligned}$$

where on the last step we used the Black-Scholes PDE. On the other hand we have

$$\frac{\partial g}{\partial y}(t, y) = |\sigma| \, \mathrm{e}^{rt} \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t} \frac{\partial f}{\partial x}\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big)$$

and

$$\begin{aligned}
\frac{1}{2} \frac{\partial g^2}{\partial y^2}(t, y) &= \frac{\sigma^2}{2} \mathrm{e}^{rt} \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t} \frac{\partial f}{\partial x}\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big) \\
&\quad + \frac{\sigma^2}{2} \mathrm{e}^{rt} \, \mathrm{e}^{2|\sigma| y + 2(\sigma^2/2 - r)t} \frac{\partial^2 f}{\partial x^2}\big(T - t, \, \mathrm{e}^{|\sigma| y + (\sigma^2/2 - r)t}\big) \\
&= \frac{\sigma^2}{2} \mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x) + \frac{\sigma^2}{2} \mathrm{e}^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T - t, x). \qquad (6.37)
\end{aligned}$$

We conclude by comparing (6.36) with (6.37), which shows that $g(t, x)$ solves the heat equation (6.33) with initial condition

$$g(0, y) = f\big(T, \, \mathrm{e}^{|\sigma| y}\big) = h\big(\mathrm{e}^{|\sigma| y}\big).$$

□

In the next proposition we provide a proof of Proposition 6.3 by deriving the Black-Scholes formula (6.12) from the solution of the PDE (6.28). The Black-Scholes formula will also be recovered by a probabilistic argument via the computation of an expected value in Proposition 7.6.

Proposition 6.11. *When $h(x) = (x - K)^+$, the solution of the Black-Scholes PDE (6.28) is given by*

$$f(t,x) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \qquad x>0,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x\in\mathbb{R},$$

and

$$\begin{cases} d_+(T-t) := \dfrac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \\ d_-(T-t) := \dfrac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \end{cases}$$

$x>0$, $t\in[0,T)$.

Proof. By inversion of Relation (6.29) with $s=T-t$ and $x=\mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}$, we get

$$f(s,x) = \mathrm{e}^{-(T-s)r}g\left(T-s, \frac{-(\sigma^2/2-r)(T-s)+\log x}{|\sigma|}\right)$$

and

$$h(x) = \psi\left(\frac{\log x}{|\sigma|}\right), \quad x>0, \quad \text{or} \quad \psi(y) = h\big(\mathrm{e}^{|\sigma|y}\big), \quad y\in\mathbb{R}.$$

Hence, using the solution (6.34) and Relation (6.30), we get

$$\begin{aligned}
f(t,x) &= \mathrm{e}^{-(T-t)r}g\left(T-t, \frac{-(\sigma^2/2-r)(T-t)+\log x}{|\sigma|}\right) \\
&= \mathrm{e}^{-(T-t)r}\int_{-\infty}^{\infty}\psi\left(\frac{-(\sigma^2/2-r)(T-t)+\log x}{|\sigma|}+z\right)\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= \mathrm{e}^{-(T-t)r}\int_{-\infty}^{\infty} h\big(x\,\mathrm{e}^{|\sigma|z-(\sigma^2/2-r)(T-t)}\big)\,\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= \mathrm{e}^{-(T-t)r}\int_{-\infty}^{\infty}\big(x\,\mathrm{e}^{|\sigma|z-(\sigma^2/2-r)(T-t)}-K\big)^+\,\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= \mathrm{e}^{-(T-t)r} \\
&\quad\times\int_{\frac{(-r+\sigma^2/2)(T-t)+\log(K/x)}{|\sigma|}}^{\infty}\big(x\,\mathrm{e}^{|\sigma|z-(\sigma^2/2-r)(T-t)}-K\big)\,\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{|\sigma|z-(\sigma^2/2-r)(T-t)}\,\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{|\sigma|z-(T-t)\sigma^2/2-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}}
\end{aligned}$$

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$$
\begin{aligned}
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}}\\
&= x\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-(z-(T-t)|\sigma|)^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}}\\
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}}\\
&= x\int_{-d_-(T-t)\sqrt{T-t}-(T-t)|\sigma|}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}}\\
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}}\\
&= x\int_{-d_-(T-t)-|\sigma|\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/2}\frac{dz}{\sqrt{2\pi}} - K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)}^{\infty}\mathrm{e}^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\\
&= x\big(1-\Phi\big(-d_+(T-t)\big)\big) - K\,\mathrm{e}^{-(T-t)r}\big(1-\Phi\big(-d_-(T-t)\big)\big)\\
&= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),
\end{aligned}
$$

where we used the relation (6.16), *i.e.*

$$
1-\Phi(a) = \Phi(-a), \qquad a \in \mathbb{R}.
$$

□

Exercises

Exercise 6.1 Bachelier (1900) model. Consider a market made of a riskless asset valued $A_t = A_0$, $t \geqslant 0$, with vanishing risk-free interest rate $r = 0$, and a risky asset whose price S_t is modeled by a standard Brownian motion as $S_t = B_t$, $t \geqslant 0$.

a) Show that the price $g(t, B_t)$ of the option with claim payoff $C = (B_T)^2$ satisfies the heat equation

$$
-\frac{\partial g}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y), \qquad (t,y) \in [0,T]\times\mathbb{R},
$$

with terminal condition $g(T,x) = x^2$.

b) Find the function $g(t,x)$ by solving the PDE of Question (a).

Hint: Try a solution of the form $g(t,x) = x^2 + f(t)$, $t \in [0,T]$.

c) Find the risky asset allocation ξ_t hedging the claim payoff $C = (B_T)^2$, and the amount $\eta_t A_t = \eta_t A_0$ invested in the riskless asset, for $t \in [0,T]$.

See Exercises 6.12 and 7.17-7.18 for extensions to nonzero interest rates.

Exercise 6.2 Consider a risky asset price $(S_t)_{t\in\mathbb{R}}$ modeled in the Cox et al. (1985) (CIR) model as

$$dS_t = \beta(\alpha - S_t)dt + \sigma\sqrt{S_t}dB_t, \qquad \alpha, \beta, \sigma > 0, \tag{6.38}$$

and let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, takes the form $V_t = g(t, S_t)$, $t \geqslant 0$. Figure 6.22 presents a random simulation of the solution to (6.38) with $\alpha = 0.025$, $\beta = 1$, and $\sigma = 1.3$.

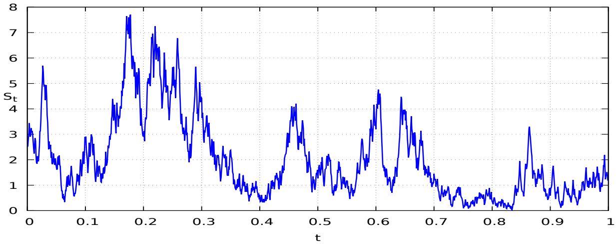

Fig. 6.22: Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N;a=0.025; b=2; sigma=0.05;
dB <- rnorm(N,mean=0,sd=sqrt(dt));R <- rep(0,N);R[1]=0.01
for (j in 2:N){R[j]=max(0,R[j-1]+(a-b*R[j-1])*dt+sigma*sqrt(R[j-1])*dB[j])}
plot(t, R, xlab = "t", ylab = "", type = "l", ylim = c(0,0.02), col = "blue")
abline(h=0,col="black",lwd=2)
```

Based on the self-financing condition written as

$$\begin{aligned} dV_t &= rV_t dt - r\xi_t S_t dt + \xi_t dS_t \\ &= rV_t dt - r\xi_t S_t dt + \beta(\alpha - S_t)\xi_t dt + \sigma\xi_t\sqrt{S_t}dB_t, \quad t \geqslant 0, \end{aligned} \tag{6.39}$$

derive the PDE satisfied by the function $g(t, x)$ using the Itô formula.

Exercise 6.3 Black-Scholes PDE with dividends reinvested (Exercise 3.20 continued). Consider a riskless asset with price $A_t = A_0\,\mathrm{e}^{rt}$, $t \geqslant 0$, and an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate. By absence of arbitrage, the payment of a dividend entails a drop in the stock price by the same amount occurring generally on the *ex-dividend date*, on which the purchase of the security no longer entitles the investor to the dividend amount. The list of investors entitled to dividend payment is consolidated on the *date of record*, and payment is made on the *payable date*.

```
library(quantmod)
getSymbols("9983.HK",from="2010-01-01",to=Sys.Date(),src="yahoo")
getDividends("9983.HK",from="2010-01-01",to=Sys.Date(),src="yahoo")
```

a) Assuming that the portfolio with value $V_t = \xi_t S_t + \eta_t A_t$ at time t is self-financing and that dividends are continuously reinvested, write down the portfolio variation dV_t.
b) Assuming that the portfolio value V_t takes the form $V_t = g(t, S_t)$ at time t, derive the Black-Scholes PDE for the function $g(t,x)$ with its terminal condition.
c) Compute the price at time $t \in [0,T]$ of the European call option with strike price K by solving the corresponding Black-Scholes PDE.
d) Compute the Delta of the option.

Exercise 6.4

a) Check that the Black-Scholes formula (6.12) for European call options

$$g_{\mathrm{c}}(t,x) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

satisfies the following boundary conditions:

i) at $x = 0$, $g_{\mathrm{c}}(t,0) = 0$,
ii) at maturity $t = T$,

$$g_{\mathrm{c}}(T,x) = (x-K)^+ = \begin{cases} x - K, & x > K \\ 0, & x \leqslant K, \end{cases}$$

and

$$\lim_{t \nearrow T} \Phi(d_+(T-t)) = \begin{cases} 1, & x > K \\ \dfrac{1}{2}, & x = K \\ 0, & x < K, \end{cases}$$

iii) as time to maturity tends to infinity,

$$\lim_{T\to\infty} \mathrm{Bl}(x, K, \sigma, r, T-t) = x, \qquad t \geqslant 0.$$

b) Check that the Black-Scholes formula (6.21) for European put options

$$g_{\mathrm{p}}(t,x) = K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big)$$

satisfies the following boundary conditions:

i) at $x=0$, $g_{\rm p}(t,0) = K\,{\rm e}^{-(T-t)r}$,
ii) as x tends to infinity, $g_{\rm p}(t,\infty) = 0$ for all $t \in [0,T)$,
iii) at maturity $t = T$,

$$g_{\rm p}(T,x) = (K-x)^+ = \begin{cases} 0, & x > K \\ K-x, & x \leqslant K, \end{cases}$$

and

$$-\lim_{t \nearrow T} \Phi(-d_+(T-t)) = \begin{cases} 0, & x > K \\ -\dfrac{1}{2}, & x = K \\ -1, & x < K, \end{cases}$$

iv) as time to maturity tends to infinity,

$$\lim_{T\to\infty} {\rm Bl}_{\rm p}(S_t, K, \sigma, r, T-t) = 0, \qquad t \geqslant 0.$$

Exercise 6.5 Power options (Exercise 3.16 continued). Power options can be used for the pricing of realized variance and volatility swaps, see § 8.2. Let $(S_t)_{t\in\mathbb{R}_+}$ denote a geometric Brownian motion solution of

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $\mu \in \mathbb{R}$ and $\sigma > 0$.

a) Let $r \geqslant 0$. Solve the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t) \tag{6.40}$$

with terminal condition $g(x,T) = x^2$, $x > 0$, $t \in [0,T]$.

Hint: Try a solution of the form $g(x,t) = x^2 f(t)$, and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = A_0\,{\rm e}^{rt}$ in the portfolio with value

$$V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t, \qquad 0 \leqslant t \leqslant T,$$

hedging the contract with claim payoff $C = (S_T)^2$ at maturity.

Exercise 6.6 On December 18, 2007, a call option has been issued by Fortis Bank on the stock price S of the MTR Corporation with maturity $T = 23/12/2008$, strike price $K = $ HK\$ 36.08 and entitlement ratio=10, cf. page 10. Recall that in the Black-Scholes model, the price at time t of the European claim on the underlying asset priced S_t with strike price K, maturity T, interest rate r and volatility $\sigma > 0$ is given by the Black-Scholes formula as

$$f(t, S_t) = S_t \Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r} \Phi\big(d_-(T-t)\big),$$

where

$$\begin{cases} d_-(T-t) = \dfrac{(r-\sigma^2/2)(T-t) + \log(S_t/K)}{|\sigma|\sqrt{T-t}}, \\ \\ d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t} = \dfrac{(r+\sigma^2/2)(T-t) + \log(S_t/K)}{|\sigma|\sqrt{T-t}}. \end{cases}$$

Recall that by Proposition 6.4 we have

$$\frac{\partial f}{\partial x}(t, S_t) = \Phi\big(d_+(T-t)\big), \qquad 0 \leqslant t \leqslant T.$$

a) Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time t =November 07, 2008 with S_t = HK\$ 17.200, assuming a volatility $\sigma = 90\% = 0.90$ and an *annual* risk-free interest rate $r = 4.377\% = 0.04377$,
b) Still using the Gaussian cumulative distribution function, compute the quantity of the risky asset required in your portfolio at time t =November 07, 2008 in order to hedge one such option at maturity $T = 23/12/2008$.
c) Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

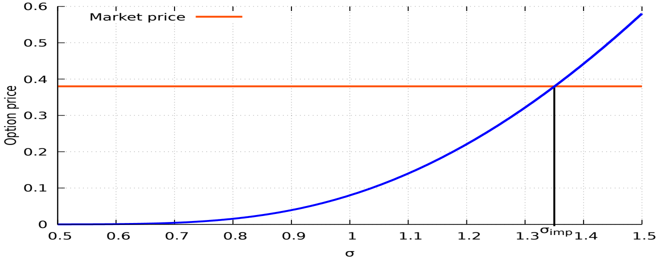

Fig. 6.23: Option price as a function of the volatility $\sigma > 0$.

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2);BSCall}
sigma <- seq(0.5,1.5, length=100);
plot(sigma,BSCall(17.2,36.08,0.04377,46/365,sigma) , type="l",lty=1, xlab="Sigma",
      ylab="Black-Scholes Call Price", ylim = c(0,0.6),col="blue",lwd=3);grid()
abline(h=0.23,col="red",lwd=3)
```

Knowing that the closing price of the warrant on November 07, 2008 was HK$ 0.023, which value can you infer for the implied volatility σ at this date?*

Exercise 6.7 Forward contracts. Recall that the price $\pi_t(C)$ of a claim payoff $C = h(S_T)$ of maturity T can be written as $\pi_t(C) = g(t, S_t)$, where the function $g(t, x)$ satisfies the *Black-Scholes PDE*

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ g(T,x) = h(x), \qquad\qquad\qquad\qquad\qquad (1) \end{cases}$$

with terminal condition $g(T, x) = h(x)$, $x > 0$.

a) Assume that C is a forward contract with payoff

$$C = S_T - K,$$

at time T. Find the function $h(x)$ in (1).

b) Find the solution $g(t, x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0, T]$.
Hint: search for a solution of the form $g(t, x) = x - \alpha(t)$ where $\alpha(t)$ is a function of t to be determined.

c) Compute the quantities

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t)$$

and η_t of risky and riskless assets in a self-financing portfolio hedging C, assuming $A_0 = \$1$.

d) Repeat the above questions with the terminal condition $g(T, x) = x$.

Exercise 6.8 (Exercise 3.8 continued). We consider a range forward contract having the payoff

$$S_T - F + (K_1 - S_T)^+ - (S_T - K_2)^+,$$

* Download the corresponding **code** or the **IPython notebook** that can be run **here** or **here** (right-click to save as attachment, may not work on).

on an underlying asset priced S_T at maturity T, where $0 < K_1 < F < K_2$, and the price process $(S_t)_{t\in\mathbb{R}_+}$ is modeled as the geometric Brownian motion

$$S_t = S_0\,\mathrm{e}^{rt+\sigma B_t-\sigma^2 t/2}, \qquad t \geqslant 0,$$

under the risk-neutral measure $\mathbb{P}^*$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

a) Give the value of $\mathbb{E}^*[S_T \mid \mathcal{F}_t]$, $0 \leqslant t \leqslant T$.
b) Price the range forward contract at time $t \in [0, T]$.

Hint. The Black-Scholes call pricing formula reads

$$g_{\mathrm{c}}(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

with

$$\begin{aligned}\frac{\big(d_-(T-t)\big)^2}{2} &= \frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\\ &= \frac{\big(d_+(T-t)\big)^2}{2} - (T-t)r - \log\frac{x}{K}.\end{aligned}$$

Exercise 6.9

a) Solve the Black-Scholes PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(t,x) \tag{6.41}$$

with terminal condition $g(T,x) = 1$, $x > 0$.

Hint: Try a solution of the form $g(t,x) = f(t)$ and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = A_0\,\mathrm{e}^{rt}$ in the portfolio with value

$$V_t = g(t,S_t) = \xi_t S_t + \eta_t A_t$$

hedging the contract with claim payoff $C = \$1$ at maturity.

Exercise 6.10 Log contracts can be used for the pricing and hedging of realized variance swaps, see § 8.2 and Exercise 8.6.

a) Solve the PDE

$$0 = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t) \tag{6.42}$$

with the terminal condition $g(x,T) := \log x$, $x > 0$.

Hint: Try a solution of the form $g(x,t) = f(t) + \log x$, and find $f(t)$.

b) Solve the Black-Scholes PDE

$$rh(x,t) = \frac{\partial h}{\partial t}(x,t) + rx\frac{\partial h}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 h}{\partial x^2}(x,t) \tag{6.43}$$

with the terminal condition $h(x,T) := \log x$, $x > 0$.

Hint: Try a solution of the form $h(x,t) = u(t)g(x,t)$, and find $u(t)$.

c) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = A_0 \mathrm{e}^{rt}$ in the portfolio with value

$$V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t$$

hedging a log contract with claim payoff $C = \log S_T$ at maturity.

Exercise 6.11 Binary options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral probability measure $\mathbb{P}^*$. The binary (or digital) *call* option is a contract with maturity T, strike price K, and payoff

$$C_d := \mathbb{1}_{[K,\infty)}(S_T) = \begin{cases} \$1 & \text{if } S_T \geqslant K, \\ 0 & \text{if } S_T < K. \end{cases}$$

a) Derive the Black-Scholes PDE satisfied by the pricing function $C_d(t,S_t)$ of the binary call option, together with its terminal condition.

b) Show that the solution $C_d(t,x)$ of the Black-Scholes PDE of Question (a) is given by

$$\begin{aligned} C_d(t,x) &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{|\sigma|\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)), \end{aligned}$$

where

$$d_-(T-t) := \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}, \qquad 0 \leqslant t < T.$$

Exercise 6.12

a) Bachelier (1900) model. Solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t \tag{6.44}$$

in terms of $\alpha, \sigma \in \mathbb{R}$, and the initial condition S_0.

b) Write down the Bachelier PDE satisfied by the function $C(t, x)$, where $C(t, S_t)$ is the price at time $t \in [0, T]$ of the contingent claim with claim payoff $C = \phi(S_T) = \exp(S_T)$, and identify the process Delta $(\xi_t)_{t\in[0,T]}$ that hedges this claim.

c) Solve the Bachelier PDE of Question (b) with the terminal condition $C(T, x) = \phi(x) = \mathrm{e}^x$, $x \in \mathbb{R}$.

Hint: Search for a solution of the form

$$C(t, x) = \exp\left(-(T-t)r + xh(t) + \frac{\sigma^2}{4r}(h^2(t) - 1)\right), \qquad (6.45)$$

where $h(t)$ is a function to be determined, with $h(T) = 1$.

d) Compute the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ that hedges the contingent claim with claim payoff $C = \exp(S_T)$.

Exercise 6.13 Let $\sigma > 0$, $K > 0$, $r \in \mathbb{R}$, and let Φ denote the standard Gaussian cumulative distribution function.

a) Show that for every fixed value of S, the function

$$d \longmapsto h(S, d) := S\Phi\big(d + |\sigma|\sqrt{T}\big) - K\,\mathrm{e}^{-rT}\Phi(d)$$

reaches its maximum at $d_*(S) := \dfrac{\log(S/K) + (r - \sigma^2/2)T}{|\sigma|\sqrt{T}}$.

Hint: The maximum is reached when the partial derivative $\dfrac{\partial h}{\partial d}$ vanishes.

b) By the differentiation rule

$$\frac{d}{dS}h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d_*'(S)\frac{\partial h}{\partial d}(S, d_*(S)),$$

recover the value of the Black-Scholes Delta.

Exercise 6.14

a) Compute the Black-Scholes call and put *Vega* by differentiation of the Black-Scholes function

$$g_c(t, x) = \mathrm{Bl}(x, K, \sigma, r, T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

with respect to the volatility parameter $\sigma > 0$, knowing that

$$-\frac{1}{2}\big(d_-(T-t)\big)^2 = -\frac{1}{2}\left(\frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2$$

$$= -\frac{1}{2}\left(d_+(T-t)\right)^2 + (T-t)r + \log\frac{x}{K}. \qquad (6.46)$$

How do the Black-Scholes call and put prices behave when subjected an increase in the volatility parameter σ?

b) Compute the Black-Scholes Rho by differentiation of the Black-Scholes function $g_c(t,x)$. How do the Black-Scholes call and put prices behave when subjected an increase in the interest rate parameter r?

Exercise 6.15 Consider the backward induction relation (3.14), *i.e.*

$$\widetilde{v}(t,x) = (1-p_N^*)\widetilde{v}\left(t+1, x(1+a_N)\right) + p_N^*\widetilde{v}\left(t+1, x(1+b_N)\right),$$

using the renormalizations $r_N := rT/N$ and

$$a_N := (1+r_N)(1-|\sigma|\sqrt{T/N}) - 1, \quad b_N := (1+r_N)(1+|\sigma|\sqrt{T/N}) - 1,$$

of Section 3.6, $N \geqslant 1$, with

$$p_N^* = \frac{r_N - a_N}{b_N - a_N} \quad \text{and} \quad p_N^* = \frac{b_N - r_N}{b_N - a_N}.$$

a) Show that the Black-Scholes PDE (6.2) of Proposition 6.1 can be recovered from the induction relation (3.14) when the number N of time steps tends to infinity.

b) Show that the expression of the Delta $\xi_t = \frac{\partial g_c}{\partial x}(t, S_t)$ can be similarly recovered from the finite difference relation (3.19), *i.e.*

$$\xi_t^{(1)}(S_{t-1}) = \frac{v\left(t, (1+b_N)S_{t-1}\right) - v\left(t, (1+a_N)S_{t-1}\right)}{S_{t-1}(b_N - a_N)}$$

as N tends to infinity.

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