Chapter 11
Barrier Options

Barrier options are financial derivatives whose payoffs depend on the crossing of a certain predefined barrier level by the underlying asset price process $(S_t)_{t\in[0,T]}$. In this chapter, we consider barrier options whose payoffs depend on an extremum of $(S_t)_{t\in[0,T]}$, in addition to the terminal value S_T. Barrier options are then priced by computing the discounted expected values of their claim payoffs, or by PDE arguments.

11.1 Options on Extrema

Vanilla options with payoff $C = \phi(S_T)$ can be priced as

$$\mathrm{e}^{-rT}\mathbb{E}^*[\phi(S_T)] = \mathrm{e}^{-rT}\int_0^\infty \phi(y)\varphi_{S_T}(y)dy$$

where $\varphi_{S_T}(y)$ is the (one parameter) *probability density* function of S_T, which satisfies

$$\mathbb{P}(S_T \leqslant y) = \int_0^y \varphi_{S_T}(v)dv, \qquad y > 0.$$

Recall that typically we have

$$\phi(x) = (x-K)^+ = \begin{cases} x - K & \text{if } x \geqslant K, \\ 0 & \text{if } x < K, \end{cases}$$

for the European call option with strike price K, and

$$\phi(x) = \mathbb{1}_{[K,\infty)}(x) = \begin{cases} \$1 & \text{if } x \geqslant K, \\ \\ 0 & \text{if } x < K, \end{cases}$$

for the binary call option with strike price K. On the other hand, exotic options, also called path-dependent options, are options whose payoff C may depend on the whole path

$$\{S_t \ : \ 0 \leqslant t \leqslant T\}$$

of the underlying asset price process via a "complex" operation such as averaging or computing a maximum. They are opposed to vanilla options whose payoff

$$C = \phi(S_T),$$

depends only on the terminal value S_T of the price process via a payoff function ϕ, and can be priced by the computation of path integrals, see Section 17.3.

For example, the payoff of an option on extrema may take the form

$$C := \phi\big(M_0^T, S_T\big),$$

where

$$M_0^T = \operatorname*{Max}_{t\in[0,T]} S_t$$

is the maximum of $(S_t)_{t\in\mathbb{R}_+}$ over the time interval $[0,T]$. In such situations the option price at time $t=0$ can be expressed as

$$\mathrm{e}^{-rT}\mathbb{E}^*\big[\phi\big(M_0^T, S_T\big)\big] = \mathrm{e}^{-rT}\int_0^\infty\int_0^\infty \phi(x,y)\varphi_{M_0^T,S_T}(x,y)dxdy$$

where $\varphi_{M_0^T,S_T}$ is the *joint probability density* function of (M_0^T, S_T), which satisfies

$$\mathbb{P}(M_0^T \leqslant x \text{ and } \ S_T \leqslant y) = \int_0^x\int_0^y \varphi_{M_0^T,S_T}(u,v)dudv, \qquad x,y \geqslant 0.$$

General case

Using the joint probability density function of $\widetilde{W}_T = W_T + \mu T$ and

$$\widehat{X}_0^T = \operatorname*{Max}_{t\in[0,T]} \widetilde{W} = \operatorname*{Max}_{t\in[0,T]} (W_t + \mu t),$$

see Proposition 10.2, we are able to price any exotic option with payoff $\phi(\widetilde{W}_T, \widehat{X}_0^T)$, as

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[\phi\big(\widehat{X}_0^T, \widetilde{W}_T\big) \,\big|\, \mathcal{F}_t\big],$$

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https://personal.ntu.edu.sg/nprivault/indext.html

with in particular, letting $a \vee b := \text{Max}(a, b)$,

$$\mathrm{e}^{-rT}\mathbb{E}^*\big[\phi\big(\widehat{X}_0^T, \widetilde{W}_T\big)\big] = \mathrm{e}^{-rT}\int_{-\infty}^{\infty}\int_{y\vee 0}^{\infty}\phi(x,y)\mathrm{d}\mathbb{P}^*\big(\widehat{X}_0^T \leqslant x, \widetilde{W}_T \leqslant y\big).$$

In this chapter, we work in a (continuous) geometric Brownian model, in which the asset price $(S_t)_{t\in[0,T]}$ has the dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad t \geqslant 0,$$

where $\sigma > 0$ and $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. In particular, by Lemma 5.14 the value V_t of a self-financing portfolio satisfies

$$V_T\,\mathrm{e}^{-rT} = V_0 + \sigma\int_0^T \xi_t S_t\,\mathrm{e}^{-rt}dW_t, \quad t \in [0,T].$$

In order to price barrier* options by the above probabilistic method, we will use the probability density function of the maximum

$$M_0^T = \underset{t\in[0,T]}{\text{Max}}\ S_t$$

of geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$ over a given time interval $[0,T]$ and the joint probability density function $\varphi_{M_0^T,S_T}(u,v)$ derived in Chapter 10 by the *reflection principle*.

Proposition 11.1. *An exotic option with integrable claim payoff of the form*

$$C = \phi\big(M_0^T, S_T\big) = \phi\left(\underset{t\in[0,T]}{\text{Max}}\ S_t, S_T\right)$$

can be priced at time $t = 0$ as

$$\begin{aligned}
&\mathrm{e}^{-rT}\mathbb{E}^*[C] \\
&= \frac{\mathrm{e}^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_0^\infty\int_y^\infty \phi\big(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\big)(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy \\
&\quad + \frac{\mathrm{e}^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^0\int_0^\infty \phi\big(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\big)(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy.
\end{aligned}$$

Proof. We have

$$S_T = S_0\,\mathrm{e}^{\sigma W_T - \sigma^2 T/2 + rT} = S_0\,\mathrm{e}^{(W_T+\mu T)\sigma} = S_0\,\mathrm{e}^{\sigma\widetilde{W}_T},$$

with

* "A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank" Lyuu (2021).

$$\mu := -\frac{\sigma}{2} + \frac{r}{\sigma} \quad \text{and} \quad \widetilde{W}_T = W_T + \mu T,$$

and

$$\begin{aligned} M_0^T &= \operatorname*{Max}_{t\in[0,T]} S_t = S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t - \sigma^2 t/2 + rt} \\ &= S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma \widetilde{W}_t} = S_0 \,\mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,T]} \widetilde{W}_t} \\ &= S_0 \,\mathrm{e}^{\sigma \widehat{X}_0^T}, \end{aligned}$$

since $\sigma > 0$. Hence,

$$C = \phi\big(S_T, M_0^T\big) = \phi\big(S_0 \,\mathrm{e}^{\sigma W_T - \sigma^2 T/2 + rT}, M_0^T\big) = \phi\big(S_0 \,\mathrm{e}^{\sigma \widetilde{W}_T}, S_0 \,\mathrm{e}^{\sigma \widehat{X}_0^T}\big),$$

and by Proposition 10.3 we have

$$\begin{aligned} &\mathrm{e}^{-rT} \mathbb{E}^*[C] = \mathrm{e}^{-rT} \mathbb{E}^*\big[\phi\big(S_0 \,\mathrm{e}^{\sigma \widetilde{W}_T}, S_0 \,\mathrm{e}^{\sigma \widehat{X}_0^T}\big)\big] \\ &= \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \int_{y\vee 0}^{\infty} \phi\big(S_0 \,\mathrm{e}^{\sigma y}, S_0 \,\mathrm{e}^{\sigma x}\big) d\mathbb{P}\big(\widehat{X}_0^T \leqslant x, \widetilde{W}_T \leqslant y\big) \\ &= \frac{\mathrm{e}^{-rT}}{T^{3/2}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \int_{y\vee 0}^{\infty} \phi\big(S_0 \,\mathrm{e}^{\sigma y}, S_0 \,\mathrm{e}^{\sigma x}\big) (2x - y) \,\mathrm{e}^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy \\ &= \frac{\mathrm{e}^{-rT}}{T^{3/2}} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \int_{y}^{\infty} \phi\big(S_0 \,\mathrm{e}^{\sigma y}, S_0 \,\mathrm{e}^{\sigma x}\big) (2x - y) \,\mathrm{e}^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy \\ &+ \frac{\mathrm{e}^{-rT} 1}{T^{3/2}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \int_{0}^{\infty} \phi\big(S_0 \,\mathrm{e}^{\sigma y}, S_0 \,\mathrm{e}^{\sigma x}\big) (2x - y) \,\mathrm{e}^{-\mu^2 T/2 + \mu y - (2x-y)^2/(2T)} dx dy. \end{aligned}$$

□

Pricing barrier options

The payoff of an up-and-out barrier put option on the underlying asset price S_t with exercise date T, strike price K and barrier level (or call level) B is

$$C = (K - S_T)^+ \mathbb{1}_{\left\{ \operatorname*{Max}_{0\leqslant t\leqslant T} S_t < B \right\}} = \begin{cases} (K - S_T)^+ & \text{if } \operatorname*{Max}_{0\leqslant t\leqslant T} S_t < B, \\ 0 & \text{if } \operatorname*{Max}_{0\leqslant t\leqslant T} S_t \geqslant B. \end{cases}$$

This option is also called a *Callable Bear Contract*, or a Bear CBBC with no residual value, or a turbo warrant with no rebate, in which the call level B usually satisfies $B \leqslant K$.

The payoff of a down-and-out barrier call option on the underlying asset price S_t with exercise date T, strike price K and barrier level B is

$$C = (S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}} = \begin{cases} (S_T - K)^+ & \text{if } \min\limits_{0 \leqslant t \leqslant T} S_t > B, \\ \\ 0 & \text{if } \min\limits_{0 \leqslant t \leqslant T} S_t \leqslant B. \end{cases}$$

This option is also called a *Callable Bull Contract*, or a Bull CBBC with no residual value, or a turbo warrant with no rebate, in which the call level B usually satisfies $B \geqslant K$. *

Category 'R' Callable Bull/Bear Contracts, or CBBCs, also called turbo warrants, involve a rebate or residual value computed as the payoff of a down-and-in lookback option. Category 'N' Callable Bull/Bear Contracts do not involve a residual value or rebate, and they usually satisfy $B = K$. See Eriksson and Persson (2006), Wong and Chan (2008) and Exercise 11.2 for the pricing of Category 'R' CBBCs with rebate.

Option type	CBBC	Behavior	Payoff		Price	Figure
Barrier call	Bull	down-and-out (knock-out)	$(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}}$	$B \leqslant K$	(11.10)	11.4a
				$B \geqslant K$	(11.11)	11.4b
		down-and-in (knock-in)	$(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t < B \right\}}$	$B \leqslant K$	(11.13)	11.7a
				$B \geqslant K$	(11.14)	11.7b
		up-and-out (knock-out)	$(S_T - K)^+ \mathbb{1}_{\left\{ \operatorname{Max}\limits_{0 \leqslant t \leqslant T} S_t < B \right\}}$	$B \leqslant K$	0	N.A.
				$B \geqslant K$	(11.5)	11.1
		up-and-in (knock-in)	$(S_T - K)^+ \mathbb{1}_{\left\{ \operatorname{Max}\limits_{0 \leqslant t \leqslant T} S_t > B \right\}}$	$B \leqslant K$	BSCall	6.4
				$B \geqslant K$	(11.15)	11.8
Barrier put		down-and-out (knock-out)	$(K - S_T)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}}$	$B \leqslant K$	(11.12)	11.6
				$B \geqslant K$	0	N.A.
		down-and-in (knock-in)	$(K - S_T)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t < B \right\}}$	$B \leqslant K$	(11.16)	11.9
				$B \geqslant K$	BSPut	6.11
	Bear	up-and-out (knock-out)	$(K - S_T)^+ \mathbb{1}_{\left\{ \operatorname{Max}\limits_{0 \leqslant t \leqslant T} S_t < B \right\}}$	$B \leqslant K$	(11.8)	11.2a
				$B \geqslant K$	(11.9)	11.2b
		up-and-in (knock-in)	$(K - S_T)^+ \mathbb{1}_{\left\{ \operatorname{Max}\limits_{0 \leqslant t \leqslant T} S_t > B \right\}}$	$B \leqslant K$	(11.17)	11.10a
				$B \geqslant K$	(11.18)	11.10b

Table 11.1: Barrier option types.

We can distinguish between eight different variations on barrier options, according to Table 11.1.

* Download this **code** for the pricing of Bull CBBCs (down-and-out barrier call options) with $B \geqslant K$ (right-click to save as attachment - may not work on).

In-out parity

We have the following parity relations between the prices of barrier options and vanilla call and put options:

$$\begin{cases} C_{\text{up-in}}(t) + C_{\text{up-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t], & (11.1) \\ \\ C_{\text{down-in}}(t) + C_{\text{down-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t], & (11.2) \\ \\ P_{\text{up-in}}(t) + P_{\text{up-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t], & (11.3) \\ \\ P_{\text{down-in}}(t) + P_{\text{down-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t], & (11.4) \end{cases}$$

where the price of the European call, resp. put option with strike price K are obtained from the Black-Scholes formula. Consequently, in what follows we will only compute the prices of the up-and-out barrier call and put options and of the down-and-out barrier call and put options.

Note that all knock-out barrier option prices vanish when $M_0^t > B$ or $m_0^t < B$, while the barrier up-and-out call, resp. the down-and-out barrier put option prices require $B > K$, resp. $B < K$, in order not to vanish.

11.2 Knock-Out Barrier

Up-and-out barrier call option

Let us consider an up-and-out barrier call option with maturity T, strike price K, barrier (or call level) B, and payoff

$$C = (S_T - K)^+ \mathbb{1}_{\left\{ \underset{0 \leqslant t \leqslant T}{\mathrm{Max}} S_t < B \right\}} = \begin{cases} S_T - K & \text{if } \underset{0 \leqslant t \leqslant T}{\mathrm{Max}} S_t \leqslant B, \\ \\ 0 & \text{if } \underset{0 \leqslant t \leqslant T}{\mathrm{Max}} S_t > B, \end{cases}$$

with $B \geqslant K$.

Proposition 11.2. *When $K \leqslant B$, the price*

$$\mathrm{e}^{-(T-t)r} \mathbb{1}_{\{M_0^t < B\}} \mathbb{E}^* \left[\left(x \frac{S_{T-t}}{S_0} - K \right)^+ \mathbb{1}_{\left\{ x \underset{0 \leqslant u \leqslant T-t}{\mathrm{Max}} \frac{S_u}{S_0} < B \right\}} \right]_{x = S_t}$$

of the up-and-out barrier call option with maturity T, strike price K and barrier level B is given by

$$
\begin{aligned}
& \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\,\Big|\,\mathcal{F}_t\right] \qquad (11.5)\\
&= S_t\mathbb{1}_{\left\{M_0^t<B\right\}}\left\{\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\right.\\
&\quad \left.-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\}\\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{M_0^t<B\right\}}\left\{\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\right.\\
&\quad \left.-\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\},
\end{aligned}
$$

where

$$
\delta_\pm^\tau(z)=\frac{1}{\sigma\sqrt{\tau}}\left(\log z+\left(r\pm\frac{\sigma^2}{2}\right)\tau\right), \qquad z>0. \tag{11.6}
$$

The price of the up-and-out barrier call option vanishes when $B \leqslant K$.

We also have

$$
\begin{aligned}
& \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathbb{1}_{\left\{M_0^t<B\right\}}\mathrm{Bl}(S_t,K,r,T-t,\sigma)-S_t\mathbb{1}_{\left\{M_0^t<B\right\}}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&- B\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\mathbb{1}_{\left\{M_0^t<B\right\}}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\
&+ \mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{M_0^t<B\right\}}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&+ \mathrm{e}^{-(T-t)r}K\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\mathbb{1}_{\left\{M_0^t<B\right\}}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right).
\end{aligned}
$$

The following R code implements the up-and-out pricing formula (11.5).

```
dp <- function( T , r, v, z ) { ( log(z) + ( r + v*v/2.0)*T )/v/sqrt(T) }
dm <- function( T , r , v, z ) { ( log(z) + ( r - v*v/2.0)*T )/v/sqrt(T) }
ind<-function(condition) ifelse(condition,1,0)
CBBC <- function(S,K,B,T,r,sig){ S*ind(S<B)*(pnorm(dp(T,r,sig,S/K))
      -pnorm(dp(T,r,sig,S/B)) -(B/S)**(1+2*r/sig**2)*(pnorm(dp(T,r,sig,B**2/K/S))
      -pnorm(dp(T,r,sig,B/S)))) -K*exp(-r*T)*ind(S<B)*((pnorm(dm(T,r,sig,S/K))
      -pnorm(dm(T,r,sig,S/B))) -(S/B)**(1-2*r/sig**2)*(pnorm(dm(T,r,sig,B**2/K/S))
      -pnorm(dm(T,r,sig,B/S)))))}
CBBC(S=90,K=100,B=120,T=1,r=0.01,sig=0.1)
library(devtools);
install_github("https://github.com/cran/fOptions")
install_github("https://github.com/cran/fExoticOptions")
library(fOptions); library(fExoticOptions);
StandardBarrierOption("cuo",90,100,120,0,1,0.01,0.01,0.1)
```

Note that taking $B = +\infty$ in the above identity (11.5) recovers the Black-Scholes formula

$$e^{-(T-t)r}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] = S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - e^{-(T-t)r}K\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)$$

for the price of European call options.

The graph of Figure 11.1 represents the up-and-out barrier call option price given the value S_t of the underlying asset and the time $t \in [0, T]$ with $T = 220$ days.

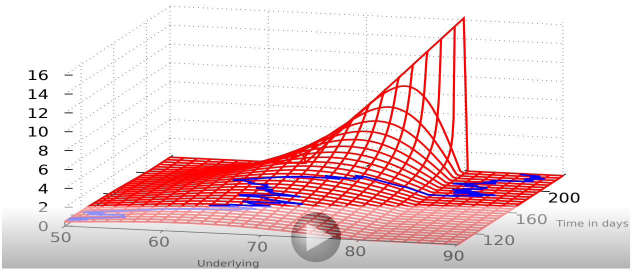


Fig. 11.1: Graph of the up-and-out call option price with $B = 80 > K = 65$.*

Proof of Proposition 11.2. We have $C = \phi(S_T, M_0^T)$ with

$$\phi(x, y) = (x - K)^+ \mathbb{1}_{\{y<B\}} = \begin{cases} (x-K)^+ & \text{if } y < B, \\ 0 & \text{if } y \geqslant B, \end{cases}$$

hence

* Right-click on the figure for interaction and "Full Screen Multimedia" view.

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{1}_{\left\{M_t^T<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[(S_T-K)^+\,\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}}\,S_r<B\right\}}\,\Bigg|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[\left(x\frac{S_T}{S_t}-K\right)^+\mathbb{1}_{\left\{x\underset{t\leqslant r\leqslant T}{\mathrm{Max}}\frac{S_r}{S_t}>B\right\}}\right]_{x=S_t}\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[\left(x\frac{S_{T-t}}{S_0}-K\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\mathrm{Max}}\frac{S_r}{S_0}<B\right\}}\right]_{x=S_t}\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[\left(x\,\mathrm{e}^{\sigma\widetilde{W}_{T-t}}-K\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\mathrm{Max}}\mathrm{e}^{\sigma\widetilde{W}_r}<B\right\}}\right]_{x=S_t}.
\end{aligned}$$

It then suffices to compute, using (10.11),

$$\begin{aligned}
&\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\right]\\
&= \mathbb{E}^*\left[\left(S_0\,\mathrm{e}^{\sigma\widetilde{W}_T}-K\right)\mathbb{1}_{\left\{S_0\,\mathrm{e}^{\sigma\widetilde{W}_T}>K\right\}}\mathbb{1}_{\left\{S_0\,\mathrm{e}^{\sigma\widehat{X}_0^T}<B\right\}}\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma y}>K\}}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma x}<B\}}d\mathbb{P}\left(\widehat{X}_0^T\leqslant x,\widetilde{W}_T\leqslant y\right)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{\sigma y>\log(K/S_0)\}}\mathbb{1}_{\{\sigma x<\log(B/S_0)\}}\varphi_{\widehat{X}_T,\widetilde{W}_T}(x,y)dxdy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{\sigma y>\log(K/S_0)\}}\mathbb{1}_{\{\sigma x<\log(B/S_0)\}}\mathbb{1}_{\{y\vee 0<x\}}\varphi_{\widehat{X}_T,\widetilde{W}_T}(x,y)dxdy\\
&= \frac{1}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\\
&\quad\int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy\\
&= \frac{\mathrm{e}^{-\mu^2T/2}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\,\mathrm{e}^{\mu y-y^2/(2T)}\\
&\qquad\times\int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}(2x-y)\,\mathrm{e}^{2x(y-x)/T}dxdy,
\end{aligned}$$

if $B\geqslant K$ and $B\geqslant S_0$ (otherwise the option price is 0), with $\mu:=r/\sigma-\sigma/2$ and $y\vee 0=\mathrm{Max}(y,0)$. Letting $a:=y\vee 0$ and $b:=\sigma^{-1}\log(B/S_0)$, we have

$$\int_a^b(2x-y)\,\mathrm{e}^{2x(y-x)/T}dx=\int_a^b(2x-y)\,\mathrm{e}^{2x(y-x)/T}dx$$

$$
\begin{aligned}
&= -\frac{T}{2}\left[\mathrm{e}^{2x(y-x)/T}\right]_{x=a}^{x=b} \\
&= \frac{T}{2}\left(\mathrm{e}^{2a(y-a)/T} - \mathrm{e}^{2b(y-b)/T}\right) \\
&= \frac{T}{2}\left(\mathrm{e}^{2(y\vee 0)(y-y\vee 0)/T} - \mathrm{e}^{2b(y-b)/T}\right) \\
&= \frac{T}{2}\left(1 - \mathrm{e}^{2b(y-b)/T}\right),
\end{aligned}
$$

hence, letting $c := \sigma^{-1}\log(K/S_0)$, we obtain

$$
\begin{aligned}
&\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\right] \\
&\quad = \frac{\mathrm{e}^{-\mu^2T/2}}{\sqrt{2\pi T}}\int_c^b \left(S_0\,\mathrm{e}^{\sigma y}-K\right)\,\mathrm{e}^{\mu y - y^2/(2T)}\left(1-\mathrm{e}^{2b(y-b)/T}\right)dy \\
&\quad = S_0\,\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\sigma+\mu)y-y^2/(2T)}\left(1-\mathrm{e}^{2b(y-b)/T}\right)dy \\
&\qquad -K\,\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{\mu y-y^2/(2T)}\left(1-\mathrm{e}^{2b(y-b)/T}\right)dy \\
&\quad = S_0\,\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\sigma+\mu)y-y^2/(2T)}dy \\
&\qquad -S_0\,\mathrm{e}^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\sigma+\mu+2b/T)y-y^2/(2T)}dy \\
&\qquad -K\,\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{\mu y-y^2/(2T)}dy \\
&\qquad +K\,\mathrm{e}^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{(\mu+2b/T)y-y^2/(2T)}dy.
\end{aligned}
$$

Using Relation (10.23), we find

$$
\begin{aligned}
&\mathrm{e}^{-rT}\mathbb{E}^*\left[\,(S_T-K)^+\,\mathbb{1}_{\left\{M_0^T<B\right\}}\right] \\
&= S_0\,\mathrm{e}^{-(r+\mu^2/2)T+(\sigma+\mu)^2T/2}\left(\Phi\left(\frac{-c+(\sigma+\mu)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\sigma+\mu)T}{\sqrt{T}}\right)\right) \\
&\quad -S_0\,\mathrm{e}^{-(r+\mu^2/2)T-2b^2/T+(\sigma+\mu+2b/T)^2T/2} \\
&\qquad \times\left(\Phi\left(\frac{-c+(\sigma+\mu+2b/T)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\sigma+\mu+2b/T)T}{\sqrt{T}}\right)\right) \\
&\quad -K\,\mathrm{e}^{-rT}\left(\Phi\left(\frac{-c+\mu T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+\mu T}{\sqrt{T}}\right)\right) \\
&\quad +K\,\mathrm{e}^{-(r+\mu^2/2)T-2b^2/T+(\mu+2b/T)^2T/2} \\
&\qquad \times\left(\Phi\left(\frac{-c+(\mu+2b/T)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\mu+2b/T)T}{\sqrt{T}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -S_0\,\mathrm{e}^{-(r+\mu^2/2)T-2b^2/T+(\sigma+\mu+2b/T)^2T/2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&\quad -K\,\mathrm{e}^{-rT}\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad +K\,\mathrm{e}^{-(r+\mu^2/2)T-2b^2T+(\mu+2b/T)^2T/2}\left(\Phi\left(\delta_-\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-\left(\frac{B}{S_0}\right)\right)\right),
\end{aligned}
$$

$0 \leqslant x \leqslant B$, where $\delta_\pm^T(z)$ is defined in (11.6). Given the relations

$$
-T\left(r+\frac{\mu^2}{2}\right)-2\frac{b^2}{T}+\frac{T}{2}\left(\sigma+\mu+\frac{2b}{T}\right)^2 = 2b\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right) = \left(1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0},
$$

and

$$
-T\left(r+\frac{\mu^2}{2}\right)-2\frac{b^2}{T}+\frac{T}{2}\left(\mu+\frac{2b}{T}\right)^2 = -rT+2\mu b = -rT+\left(-1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0},
$$

this yields

$$
\begin{aligned}
&\mathrm{e}^{-rT}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\right] \qquad (11.7)\\
&= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -\mathrm{e}^{-rT}K\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -B\left(\frac{B}{S_0}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&\quad +\mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right)\\
&= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -S_0\left(\frac{B}{S_0}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&\quad -\mathrm{e}^{-rT}K\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad +\mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right),
\end{aligned}
$$

and this yields the result of Proposition 11.2, cf. § 7.3.3 pages 304-307 of Shreve (2004) for a different approach to this calculation. This concludes the proof of Proposition 11.2. □

Up-and-out barrier put option

This option is also called a *Callable Bear Contract*, or a Bear CBBC with no residual value, or a turbo warrant with no rebate, in which B denotes the call level*. The price

$$\mathrm{e}^{-(T-t)r}\mathbb{1}_{\{M_0^t<B\}}\mathbb{E}^*\left[\left(K-x\frac{S_{T-t}}{S_0}\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\operatorname{Max}}\frac{S_r}{S_0}<B\right\}}\right]_{x=S_t}$$

of the up-and-out barrier put option with maturity T, strike price K and barrier level B is given, if $B\leqslant K$, by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{M_0^T<B\}}\,\Big|\,\mathcal{F}_t\right]\\
&=S_t\mathbb{1}_{\{M_0^t<B\}}\left(\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)-1-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)-1\right)\right)\\
&-\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{M_0^t<B\}}\left(\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)-1-\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)-1\right)\right)\\
&=S_t\mathbb{1}_{\{M_0^t<B\}}\left(-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)+\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\
&-K\,\mathrm{e}^{-(T-t)r}\\
&\quad\times\mathbb{1}_{\{M_0^t<B\}}\left(-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)+\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right).
\end{aligned}\tag{11.8}$$

and, if $B\geqslant K$, by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{M_0^T<B\}}\,\Big|\,\mathcal{F}_t\right]\\
&=S_t\mathbb{1}_{\{M_0^t<B\}}\left(\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-1-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-1\right)\right)\\
&-\mathrm{e}^{-(T-t)r}K
\end{aligned}$$

* Download this R **code** for the pricing of Bear CBBCs (up-and-out barrier put options) with $B\leqslant K$ (right-click to save as attachment - may not work on 🍎).

$$\times \mathbb{1}_{\{M_0^t<B\}}\left(\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)-1-\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-1\right)\right)$$

$$= S_t\mathbb{1}_{\{M_0^t<B\}}\left(-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)+\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\right)$$

$$-K\,\mathrm{e}^{-(T-t)r}$$

$$\times \mathbb{1}_{\{M_0^t<B\}}\left(-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)+\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\right),$$

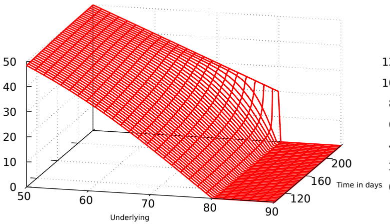
$$= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(K-x\frac{S_{T-t}}{S_0}\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\mathrm{Max}}\frac{S_r}{S_0}<B\right\}}\right]_{x=S_t}$$

$$= -S_t\mathbb{1}_{\{M_0^t<B\}}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)+S_t\mathbb{1}_{\{M_0^t<B\}}\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)$$

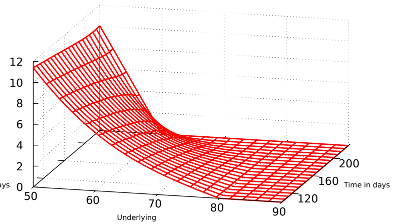
$$+K\mathbb{1}_{\{M_0^t<B\}}\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)-K\,\mathrm{e}^{-(T-t)r}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)$$

$$= \mathbb{1}_{\{M_0^t<B\}}\mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma)+S_t\mathbb{1}_{\{M_0^t<B\}}\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)$$

$$-K\mathbb{1}_{\{M_0^t<B\}}\mathrm{e}^{-(T-t)r}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right). \tag{11.9}$$



(a) Case $K=100>B=80$.



(b) Case $B=80>K=60$.

Fig. 11.2: Graphs of the up-and-out put option prices (11.8)-(11.9).

The following Figure 11.3 shows the market pricing data of an up-and-out barrier put option on BHP Billiton Limited ASX:BHP with $B=K=\$28$ for half a share, priced at 1.79.

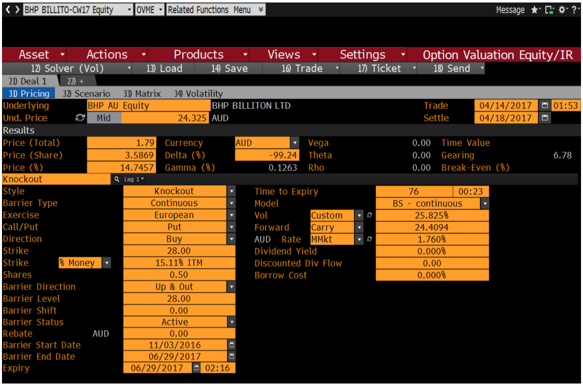


Fig. 11.3: Pricing data for an up-and-out put option with $K = B = \$28$.

The attached code performs an implied volatility calculation for up-and-out barrier put option (or Bear CBBC) prices with $B < K$, based on this **market data** set.

Down-and-out barrier call option

Let us now consider a down-and-out barrier call option on the underlying asset price S_t with exercise date T, strike price K, barrier level B, and payoff

$$C = (S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}} = \begin{cases} S_T - K & \text{if } \min\limits_{0 \leqslant t \leqslant T} S_t > B, \\ \\ 0 & \text{if } \min\limits_{0 \leqslant t \leqslant T} S_t \leqslant B, \end{cases}$$

with $0 \leqslant B \leqslant K$. The down-and-out barrier call option is also called a *Callable Bull Contract*, or a Bull CBBC with no residual value, or a turbo warrant with no rebate, in which B denotes the call level.* When $B \leqslant K$, we have

$$\mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}} \,\Big|\, \mathcal{F}_t \right] \tag{11.10}$$

$$= S_t \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K} \right) \right) - \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K} \right) \right)$$

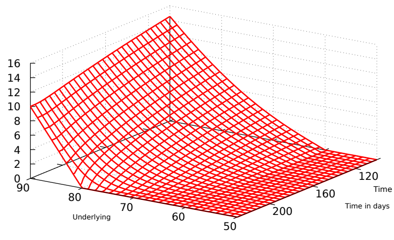
* Download this code for the pricing of Bull CBBC (down-and-out barrier call options) with $B \geqslant K$ (right-click to save as attachment - may not work on).

$$
\begin{aligned}
&-B\mathbb{1}_{\{m_0^t>B\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
&+\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
={}&\mathbb{1}_{\{m_0^t>B\}}\mathrm{Bl}(S_t,K,r,T-t,\sigma)\\
&-B\mathbb{1}_{\{m_0^t>B\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
&+\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
={}&\mathbb{1}_{\{m_0^t>B\}}\mathrm{Bl}(S_t,K,r,T-t,\sigma)\\
&-S_t\mathbb{1}_{\{m_0^t>B\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\mathrm{Bl}\left(\frac{B}{S_t},\frac{K}{B},r,T-t,\sigma\right),
\end{aligned}
$$

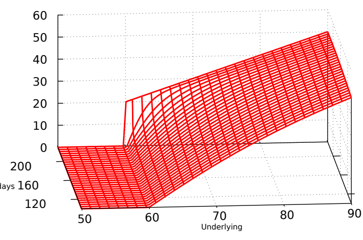
$0 \leqslant t \leqslant T$. When $B \geqslant K$, we find

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T}S_t>B\right\}}\,\middle|\,\mathcal{F}_t\right] \qquad (11.11)\\
={}&S_t\mathbb{1}_{\{m_0^t>B\}}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)-\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&-B\mathbb{1}_{\{m_0^t>B\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\\
&+\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right),
\end{aligned}
$$

$S_t > B$, $0 \leqslant t \leqslant T$, see Exercise 11.1 below.



(a) Case $B = 60 < K = 80$.



(b) Case $K = 40 < B = 60$.

Fig. 11.4: Graphs of the down-and-out call option price (11.10)-(11.11).

In the next Figure 11.5 we plot* the down-and-out barrier call option price (11.11) as a function of volatility with $B = 349.2 > K = 346.4$, $r = 0.03$, $T = 99/365$, and $S_0 = 360$.

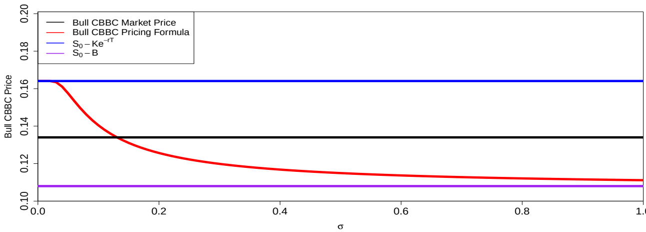


Fig. 11.5: Down-and-out call option price as a function of σ.

We note that with such parameters, the down-and-out barrier call option price (11.11) is upper bounded by the forward contract price $S_0 - K\,\mathrm{e}^{-rT}$ in the limit as σ tends to zero, and that it decreases to $S_0 - B$ in the limit as σ tends to infinity.

Down-and-out barrier put option

When $K \geqslant B$, the price

$$\mathrm{e}^{-(T-t)r}\mathbb{1}_{\{m_0^t > B\}}\mathbb{E}^*\left[\left(K - x\frac{S_{T-t}}{S_0}\right)^+ \mathbb{1}_{\left\{x \min\limits_{0\leqslant r\leqslant T-t} S_r/S_0 > B\right\}}\right]_{x=S_t}$$

of the down-and-out barrier put option with maturity T, strike price K and barrier level B is given by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{m_0^T > B\}}\,\Big|\,\mathcal{F}_t\right] \\
&= S_t\mathbb{1}_{\{m_0^t > B\}}\left\{\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\right. \\
&\qquad \left. - \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\} \\
&\quad - \mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t > B\}}\left\{\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\right.
\end{aligned}$$

* Download this R **code** for the pricing of down-and-out barrier call options (right-click to save as attachment - may not work on 🍎).

$$
\begin{aligned}
&\qquad -\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\Bigg\}\\
&= S_t \mathbb{1}_{\{m_0^t>B\}}\Bigg\{\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)\\
&\qquad -\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\Bigg\}\\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\Bigg\{\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)\\
&\qquad -\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\Bigg\}\\
&= \mathbb{1}_{\{m_0^t>B\}}\mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma)+S_t\mathbb{1}_{\{m_0^t>B\}}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \qquad (11.12)\\
&\quad -B\mathbb{1}_{\{m_0^t>B\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&\quad +\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right),
\end{aligned}
$$

while the corresponding price vanishes when $K \leqslant B$.

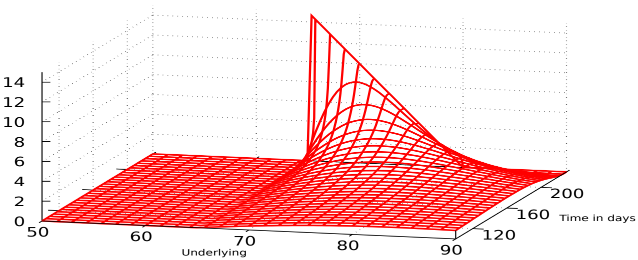


Fig. 11.6: Graph of the down-and-out put option price (11.12) with $K = 80 > B = 65$.

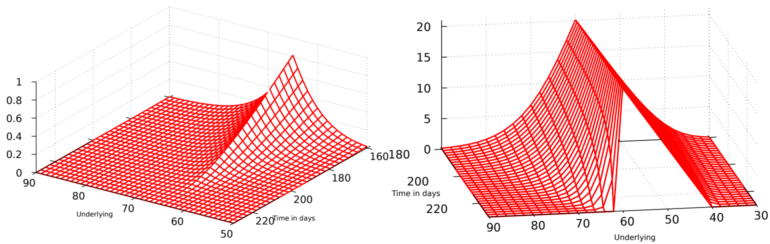
Note that although Figures 11.2b and 11.4a, resp. 11.2a and 11.4b, appear to share some symmetry property, the functions themselves are not exactly symmetric. Regarding Figures 11.1 and 11.6, the pricing function is actually the same, but the conditions $B < K$ and $B > K$ play opposite roles.

11.3 Knock-In Barrier

Down-and-in barrier call option

When $B \leqslant K$, the price of the down-and-in barrier call option is given from the down-and-out barrier call option price (11.10) and the down-in-out call parity relation (11.2) as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{m_0^T<B\right\}}\,\middle|\,\mathcal{F}_t\right] \qquad (11.13)\\
&= \mathbb{1}_{\left\{m_0^t\leqslant B\right\}}\mathrm{Bl}(S_t,K,r,T-t,\sigma)\\
&\quad +S_t\mathbb{1}_{\left\{m_0^t>B\right\}}\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{m_0^t>B\right\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right).
\end{aligned}$$



(a) Case $K = 80 > B = 65$. (b) Case $K = 40 < B = 60$.

Fig. 11.7: Graphs of the down-and-in call option price (11.13)-(11.14).

When $B \geqslant K$, the price of the down-and-in barrier call option is given from the down-and-out barrier call option price (11.11) and the down-in-out call parity relation (11.2) as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{m_t^T<B\right\}}\,\middle|\,\mathcal{F}_t\right] \qquad (11.14)\\
&= \mathrm{Bl}(S_t,K,r,T-t,\sigma)\\
&\quad -S_t\mathbb{1}_{\left\{m_0^t>B\right\}}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)+\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{m_0^t>B\right\}}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&\quad +\mathbb{1}_{\left\{m_0^t>B\right\}}S_t\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)
\end{aligned}$$

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$$-\,\mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right), \qquad 0 \leqslant t \leqslant T.$$

Up-and-in barrier call option

When $B \geqslant K$, the price of the up-and-in barrier call option is given from (11.5) and the up-in-out call parity relation (11.1) as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{M_0^T > B\}} \,\Big|\, \mathcal{F}_t\right] \qquad (11.15)\\
&= \mathbb{1}_{\{M_0^t \geqslant B\}} \mathrm{Bl}(S_t, K, r, T-t, \sigma) + S_t \mathbb{1}_{\{M_0^t < B\}} \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&\quad + B \mathbb{1}_{\{M_0^t < B\}} \left(\frac{B}{S_t}\right)^{2r/\sigma^2} \left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{K S_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\
&\quad - \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{M_0^t < B\}} \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&\quad - \mathrm{e}^{-(T-t)r} K \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{K S_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right).
\end{aligned}$$

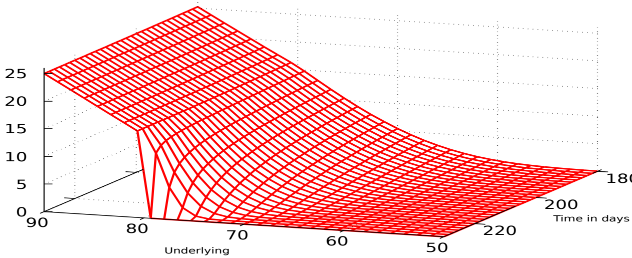


Fig. 11.8: Graph of the up-and-in call option price (11.15) with $B = 80 > K = 65$.

When $B \leqslant K$, the price of the up-and-in barrier call option is given from the Black-Scholes formula and the up-in-out call parity relation (11.1) as

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{M_0^T > B\}} \,\Big|\, \mathcal{F}_t\right] = \mathrm{Bl}(S_t, K, r, T-t, \sigma).$$

Down-and-in barrier put option

When $B \leqslant K$, the price of the down-and-in barrier put option is given from (11.12) and the down-in-out put parity relation (11.4) as

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{m_t^T<B\}}\,\middle|\,\mathcal{F}_t\right] \qquad (11.16) \\ &= \mathbb{1}_{\{m_0^t\leqslant B\}}\mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma) - S_t\mathbb{1}_{\{m_0^t>B\}}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \\ &+ B\mathbb{1}_{\{m_0^t>B\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\ &+ \mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right) \\ &- \mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right), \end{aligned}$$

$0 \leqslant t \leqslant T$.

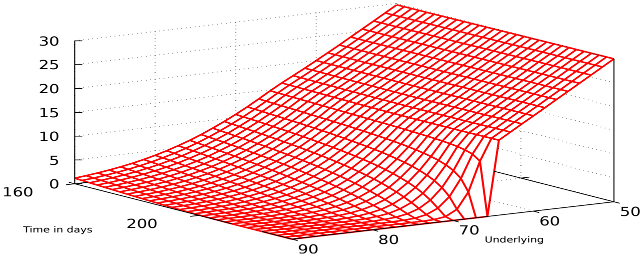


Fig. 11.9: Graph of the down-and-in put option price (11.16) with $K = 80 > B = 65$.

When $B \geqslant K$, the price of the down-and-in barrier put option is given from the Black-Scholes put function and the down-in-out put parity relation (11.4) as

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{m_t^T<B\}}\,\middle|\,\mathcal{F}_t\right] = \mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma),$$

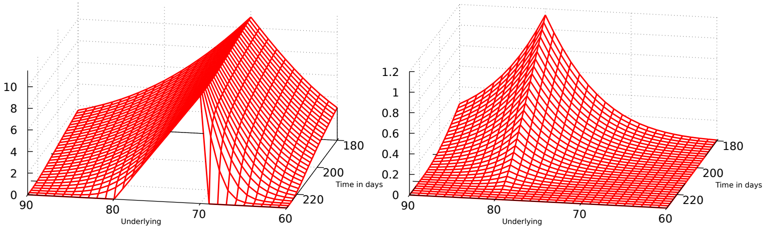
$0 \leqslant t \leqslant T$.

Up-and-in barrier put option

When $B \leqslant K$, the price of the down-and-in barrier put option is given from (11.8) and the up-in-out put parity relation (11.3) as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{M_0^T>B\}}\,\Big|\,\mathcal{F}_t\right] \qquad (11.17)\\
&= \mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma)\\
&\quad -S_t\mathbb{1}_{\{M_0^t<B\}}\left(\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\right)\\
&\quad +K\,\mathrm{e}^{-(T-t)r}\\
&\qquad \times\mathbb{1}_{\{M_0^t<B\}}\left(\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\right).
\end{aligned}$$

$0 \leqslant t \leqslant T$.



(a) $K = 80 > B = 70$.

(b) Case $K = 70 < B = 80$.

Fig. 11.10: Graphs of the up-and-in put option price (11.17)-(11.18).

By (11.9) and the up-in-out put parity relation (11.3), the price of the up-and-in barrier put option is given when $B \geqslant K$ by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{M_0^T>B\}}\,\Big|\,\mathcal{F}_t\right] \qquad (11.18)\\
&= \mathbb{1}_{\{M_0^t\geqslant B\}}\mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma)\\
&\quad -S_t\mathbb{1}_{\{M_0^t<B\}}\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
&\quad +K\mathbb{1}_{\{M_0^t<B\}}\mathrm{e}^{-(T-t)r}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right).
\end{aligned}$$

11.4 PDE Method

The up-and-out barrier call option price has been evaluated by probabilistic arguments in the previous sections. In this section we complement this approach with the derivation of a Partial Differential Equation (PDE) for this option price function.

The up-and-out barrier call option price can be written as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{0\leqslant r\leqslant t}{\mathrm{Max}}\,S_r<B\right\}}\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}}\,S_r<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{\underset{0\leqslant r\leqslant t}{\mathrm{Max}}\,S_r<B\right\}}\mathbb{E}^*\left[(S_T-K)^+\,\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}}\,S_r<B\right\}}\,\Bigg|\,\mathcal{F}_t\right]\\
&= \mathbb{1}_{\left\{M_0^t<B\right\}}g(t,S_t),
\end{aligned}$$

where the function $g(t,x)$ of t and S_t is given by

$$g(t,x) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\,\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}}\,S_r<B\right\}}\,\Bigg|\,S_t=x\right]. \qquad (11.19)$$

Next, by the same argument as in the proof of Proposition 6.1 we derive the Black-Scholes partial differential equation (PDE) satisfied by $g(t,x)$, and written for the value of a self-financing portfolio.

Proposition 11.3. *Let $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that*

(i) $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the portfolio value $V_t := \eta_t A_t + \xi_t S_t$, $t \geqslant 0$, is given as in (11.19) by

$$V_t = \mathbb{1}_{\left\{M_0^t<B\right\}}g(t,S_t), \qquad t \geqslant 0.$$

Then, the function $g(t,x)$ pricing the up-and-out barrier call option satisfies the Black-Scholes PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,x), \qquad (11.20)$$

$t>0$, $0<x<B$, and ξ_t is given by

$$\xi_t = \frac{\partial g}{\partial x}(t,S_t), \qquad 0\leqslant t\leqslant T, \qquad (11.21)$$

provided that $M_0^t<B$.

Proof. By (11.19) the price at time t of the up-and-out barrier call option discounted to time 0 is given by

$$\mathrm{e}^{-rt}\mathbb{1}_{\left\{M_0^t<B\right\}}g(t,S_t)$$

$$= \mathrm{e}^{-rT}\mathbb{1}_{\{M_0^t<B\}}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}} S_r < B\right\}} \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathrm{e}^{-rT}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{M_0^t<B\}}\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}} S_r < B\right\}} \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathrm{e}^{-rT}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{0\leqslant r\leqslant T}{\mathrm{Max}} S_r < B\right\}} \,\middle|\, S_t\right],$$

which is a martingale indexed by $t \geqslant 0$. Next, applying the Itô formula to $t \longmapsto \mathrm{e}^{-rt}g(t,S_t)$ "on $\{M_0^t \leqslant B,\ 0 \leqslant t \leqslant T\}$", we have

$$\begin{aligned}
d(\mathrm{e}^{-rt}g(t,S_t)) &= -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}dg(t,S_t) \\
&= -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial g}{\partial t}(t,S_t)dt \\
&\quad + r\,\mathrm{e}^{-rt}S_t\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}\,\mathrm{e}^{-rt}\sigma^2S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\
&\quad + \mathrm{e}^{-rt}\sigma S_t\frac{\partial g}{\partial x}(t,S_t)dW_t.
\end{aligned} \tag{11.22}$$

In order to derive (11.21) we note that, as in the proof of Proposition 6.1, the self-financing condition (5.9) implies

$$\begin{aligned}
&d(\mathrm{e}^{-rt}V_t) = -r\,\mathrm{e}^{-rt}V_tdt + \mathrm{e}^{-rt}dV_t \\
&= -r\,\mathrm{e}^{-rt}V_tdt + \eta_t\,\mathrm{e}^{-rt}dA_t + \xi_t\,\mathrm{e}^{-rt}dS_t \\
&= -r(\eta_tA_t+\xi_tS_t)\,\mathrm{e}^{-rt}dt + r\eta_tA_t\,\mathrm{e}^{-rt}dt + r\xi_tS_t\,\mathrm{e}^{-rt}dt + \sigma\xi_tS_t\,\mathrm{e}^{-rt}dW_t \\
&= \sigma\xi_tS_t\,\mathrm{e}^{-rt}dW_t, \qquad t \geqslant 0,
\end{aligned} \tag{11.23}$$

and (11.21) follows by identification of (11.22) with (11.23) which shows that the sum of components in factor of dt have to vanish, hence

$$-rg(t,S_t) + \frac{\partial g}{\partial t}(t,S_t) + rS_t\frac{\partial g}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t) = 0.$$

□

In the next proposition we add a boundary condition to the Black-Scholes PDE (11.20) in order to hedge the up-and-out barrier call option with maturity T, strike price K, barrier (or call level) B, and payoff

$$C = (S_T-K)^+\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B\right\}} = \begin{cases} S_T - K & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t \leqslant B, \\ \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t > B, \end{cases}$$

with $B \geqslant K$.

Proposition 11.4. *The value $V_t = \mathbb{1}_{\{M_0^t < B\}} g(t, S_t)$ of the self-financing portfolio hedging the up-and-out barrier call option satisfies the Black-Scholes PDE*

$$\begin{cases} rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,x), & (11.24a) \\ g(t,x) = 0, \qquad x \geqslant B, \quad t \in [0,T], & (11.24b) \\ g(T,x) = (x-K)^+ \mathbb{1}_{\{x<B\}}, & (11.24c) \end{cases}$$

on the time-space domain $[0,T] \times [0,B]$ with terminal condition

$$g(T,x) = (x-K)^+ \mathbb{1}_{\{x<B\}}$$

and lateral boundary condition

$$g(t,x) = 0, \qquad x \geqslant B. \tag{11.25}$$

Condition (11.25) holds since the price of the claim at time t is 0 whenever $S_t = B$. When $K \leqslant B$, the closed-form solution of the PDE (11.24a) under the boundary conditions (11.24b)-(11.24c) is given from (11.5) in Proposition 11.2 as

$$\begin{aligned} g(t,x) &= x\left(\Phi\left(\delta_+^{T-t}\left(\frac{x}{K}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right)\right) \qquad (11.26) \\ &-x\left(\frac{x}{B}\right)^{-1-2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{x}\right)\right)\right) \\ &-K\,\mathrm{e}^{-(T-t)r}\left(\Phi\left(\delta_-^{T-t}\left(\frac{x}{K}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{x}{B}\right)\right)\right) \\ &+K\,\mathrm{e}^{-(T-t)r}\left(\frac{x}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right)\right), \end{aligned}$$

$0 < x \leqslant B$, $0 \leqslant t \leqslant T$,

see Figure 11.1. We note that the expression (11.26) can be rewritten using the standard Black-Scholes formula

$$\mathrm{Bl}(S,K,r,T,\sigma) = S\Phi\left(\delta_+^T\left(\frac{S}{K}\right)\right) - K\,\mathrm{e}^{-rT}\Phi\left(\delta_-^T\left(\frac{S}{K}\right)\right)$$

for the price of the European call option, as

$$g(t,x) = \mathrm{Bl}(x,K,r,T-t,\sigma) - x\Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right) + \mathrm{e}^{-(T-t)r}K\Phi\left(\delta_-^{T-t}\left(\frac{x}{B}\right)\right)$$
$$-B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{x}\right)\right)\right)$$
$$+\mathrm{e}^{-(T-t)r}K\left(\frac{x}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right)\right),$$

$0 < x \leqslant B$, $0 \leqslant t \leqslant T$.

Table 11.2 summarizes the boundary conditions satisfied for barrier option pricing in the Black-Scholes PDE.

Option type	CBBC	Behavior		Boundary conditions	
				Maturity T	Barrier B
Barrier call	Bull	down-and-out	$B \leqslant K$	$(x-K)^+$	0
		(knock-out)	$B \geqslant K$	$(x-K)^+\mathbb{1}_{\{x>B\}}$	0
		down-and-in	$B \leqslant K$	0	$\mathrm{Bl}(B,K,r,T-t,\sigma)$
		(knock-in)	$B \geqslant K$	$(x-K)^+\mathbb{1}_{\{x<B\}}$	$\mathrm{Bl}(B,K,r,T-t,\sigma)$
		up-and-out	$B \leqslant K$	0	0
		(knock-out)	$B \geqslant K$	$(x-K)^+\mathbb{1}_{\{x<B\}}$	0
		up-and-in	$B \leqslant K$	$(x-K)^+$	0
		(knock-in)	$B \geqslant K$	$(x-K)^+\mathbb{1}_{\{x>B\}}$	$\mathrm{Bl}(B,K,r,T-t,\sigma)$
Barrier put		down-and-out	$B \leqslant K$	$(K-x)^+\mathbb{1}_{\{x>B\}}$	0
		(knock-out)	$B \geqslant K$	0	0
		down-and-in	$B \leqslant K$	$(K-x)^+\mathbb{1}_{\{x<B\}}$	$\mathrm{Bl}_{\mathrm{p}}(B,K,r,T-t,\sigma)$
		(knock-in)	$B \geqslant K$	$(K-x)^+$	0
	Bear	up-and-out	$B \leqslant K$	$(K-x)^+\mathbb{1}_{\{x<B\}}$	0
		(knock-out)	$B \geqslant K$	$(K-x)^+$	0
		up-and-in	$B \leqslant K$	$(K-x)^+\mathbb{1}_{\{x>B\}}$	$\mathrm{Bl}_{\mathrm{p}}(B,K,r,T-t,\sigma)$
		(knock-in)	$B \geqslant K$	0	$\mathrm{Bl}_{\mathrm{p}}(B,K,r,T-t,\sigma)$

Table 11.2: Boundary conditions for barrier option prices.

11.5 Hedging Barrier Options

Figure 11.11 represents the value of Delta obtained from (11.21) for the up-and-out barrier call option in Exercise 11.1-(a).

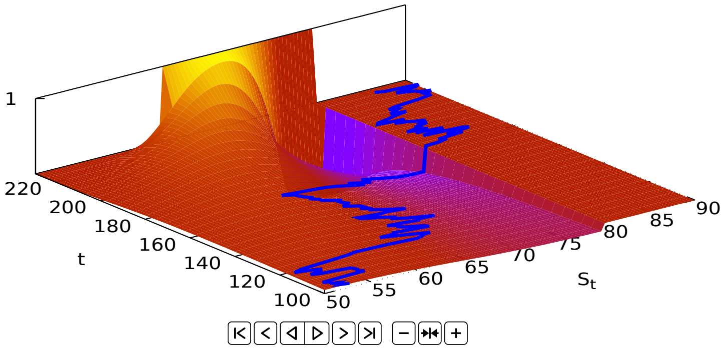


Fig. 11.11: Delta of the up-and-out barrier call with $B = 80 > K = 55$.*

Down-and-out barrier call option

Similarly, the price $g(t, S_t)$ at time t of the down-and-out barrier call option satisfies the Black-Scholes PDE

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(t,B) = 0, \quad t \in [0,T], \\ \\ g(T,x) = (x-K)^+\mathbb{1}_{\{x>B\}}, \end{cases}$$

on the time-space domain $[0,T] \times [0,B]$ with terminal condition $g(T,x) = (x-K)^+\mathbb{1}_{\{x>B\}}$ and the additional boundary condition

$$g(t,x) = 0, \qquad x \leqslant B,$$

since the price of the claim at time t is 0 whenever $S_t \leqslant B$, see (11.10) and Figure 11.4a when $B \leqslant K$, and (11.11) and Figure 11.4b when $B \geqslant K$.

Exercises

Exercise 11.1 Barrier options.

a) Compute the delta hedging strategies of the up-and-out barrier call and put options on the underlying asset price S_t with exercise date T, strike price K and barrier level B, with $B \geqslant K$.

* The animation works in Acrobat Reader on the entire pdf file.

b) Compute the joint probability density function

$$\varphi_{Y_T,W_T}(a,b)=\frac{d\mathbb{P}(Y_T\leqslant a \text{ and } W_T\leqslant b)}{dadb},\qquad a,b\in\mathbb{R},$$

of standard Brownian motion W_T and its *minimum*

$$Y_T=\min_{t\in[0,T]}W_t.$$

c) Compute the joint probability density function

$$\varphi_{\breve{Y}_T,\widetilde{W}_T}(a,b)=\frac{d\mathbb{P}(\breve{Y}_T\leqslant a \text{ and } \widetilde{W}_T\leqslant b)}{dadb},\qquad a,b\in\mathbb{R},$$

of *drifted* Brownian motion $\widetilde{W}_T=W_T+\mu T$ and its *minimum*

$$\breve{Y}_T=\min_{t\in[0,T]}\widetilde{W}_t=\min_{t\in[0,T]}(W_t+\mu t).$$

d) Compute the price at time $t\in[0,T]$ of the down-and-out barrier call option on the underlying asset price S_t with exercise date T, strike price K, barrier level B, and payoff

$$C=(S_T-K)^+\,\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T}S_t>B\right\}}=\begin{cases}S_T-K & \text{if } \min\limits_{0\leqslant t\leqslant T}S_t>B,\\[6pt] 0 & \text{if } \min\limits_{0\leqslant t\leqslant T}S_t\leqslant B,\end{cases}$$

in cases $0<B<K$ and $B\geqslant K$.

Exercise 11.2 Pricing Category 'R' CBBC rebates. Given $\tau>0$, consider an asset price $(S_t)_{t\in[\tau,\infty)}$, given by

$$S_{\tau+t}=S_\tau\,\mathrm{e}^{rt+\sigma W_t-\sigma^2t/2},\qquad t\geqslant 0,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r\geqslant 0$ and $\sigma>0$. In what follows, $\Delta\tau$ is the *deterministic* length of the Mandatory Call Event (MCE) valuation period which commences from the time upon which a MCE occurs up to the end of the following trading session.

a) Compute the expected rebate (or residual) $\mathbb{E}\left[\left(\min\limits_{s\in[0,\Delta\tau]}S_{\tau+s}-K\right)^+\,\Big|\,\mathcal{F}_\tau\right]$ of a Category 'R' Bull CBBC Contract (down-and-out barrier call option) having expired at a given time $\tau<T$, knowing that $S_\tau=B>K>0$, with $r>0$.

b) Compute the expected rebate $\mathbb{E}\Big[\Big(\min_{s\in[0,\Delta\tau]} S_{\tau+s} - K\Big)^+ \ \Big|\ \mathcal{F}_\tau\Big]$ of a Category 'R' Bull CBBC Contract having expired at a given time $\tau < T$, knowing that $S_\tau = B > K > 0$, with $r = 0$.

c) Find the expression of the probability density function of the first hitting time

$$\tau_B = \inf\{t \geqslant 0 \ : \ S_t = B\}$$

of the level $B > 0$ by the process $(S_t)_{t\in\mathbb{R}_+}$.

d) Price the CBBC rebate

$$\begin{aligned}
&\mathrm{e}^{-r\Delta\tau}\mathbb{E}\Big[\mathrm{e}^{-r\tau_B}\mathbb{1}_{[0,T]}(\tau_B)\Big(\min_{t\in[\tau_B,\tau_B+\Delta\tau]} S_t - K\Big)^+\Big] \\
&= \mathrm{e}^{-r\Delta\tau_B}\mathbb{E}\Big[\mathrm{e}^{-r\tau_B}\mathbb{1}_{[0,T]}(\tau_B)\mathbb{E}\Big[\Big(\min_{t\in[\tau_B,\tau_B+\Delta\tau]} S_t - K\Big)^+ \ \Big|\ \mathcal{F}_{\tau_B}\Big]\Big].
\end{aligned}$$

Exercise 11.3 Barrier forward contracts. Compute the price at time t of the following barrier forward contracts on the underlying asset price S_t with exercise date T, strike price K, barrier level B, and the following payoffs. In addition, compute the corresponding hedging strategies.

a) Up-and-in barrier long forward contract. Take

$$C = (S_T - K)\,\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t > B\right\}} = \begin{cases} S_T - K & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t > B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t \leqslant B. \end{cases}$$

b) Up-and-out barrier long forward contract. Take

$$C = (S_T - K)\,\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B\right\}} = \begin{cases} S_T - K & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t \geqslant B. \end{cases}$$

c) Down-and-in barrier long forward contract. Take

$$C = (S_T - K)\,\mathbb{1}_{\left\{\min_{0\leqslant t\leqslant T} S_t < B\right\}} = \begin{cases} S_T - K & \text{if } \min_{0\leqslant t\leqslant T} S_t < B, \\ 0 & \text{if } \min_{0\leqslant t\leqslant T} S_t \geqslant B. \end{cases}$$

d) Down-and-out barrier long forward contract. Take

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$$C = (S_T - K)\,\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t > B\right\}} = \begin{cases} S_T - K & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t > B, \\ 0 & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t \leqslant B. \end{cases}$$

e) Up-and-in barrier short forward contract. Take

$$C = (K - S_T)\,\mathbb{1}_{\left\{\mathrm{Max}_{0\leqslant t\leqslant T} S_t > B\right\}} = \begin{cases} K - S_T & \text{if } \operatorname*{Max}\limits_{0\leqslant t\leqslant T} S_t > B, \\ 0 & \text{if } \operatorname*{Max}\limits_{0\leqslant t\leqslant T} S_t \leqslant B. \end{cases}$$

f) Up-and-out barrier short forward contract. Take

$$C = (K - S_T)\,\mathbb{1}_{\left\{\mathrm{Max}_{0\leqslant t\leqslant T} S_t < B\right\}} = \begin{cases} K - S_T & \text{if } \operatorname*{Max}\limits_{0\leqslant t\leqslant T} S_t < B, \\ 0 & \text{if } \operatorname*{Max}\limits_{0\leqslant t\leqslant T} S_t \geqslant B. \end{cases}$$

g) Down-and-in barrier short forward contract. Take

$$C = (K - S_T)\,\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t < B\right\}} = \begin{cases} K - S_T & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t < B, \\ 0 & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t \geqslant B. \end{cases}$$

h) Down-and-out barrier short forward contract. Take

$$C = (K - S_T)\,\mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t > B\right\}} = \begin{cases} K - S_T & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t > B, \\ 0 & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t \leqslant B. \end{cases}$$

Exercise 11.4 Compute the Vega of the down-and-out and down-and-in barrier call option prices, *i.e.* compute the sensitivity of down-and-out and down-and-in barrier option prices with respect to the volatility parameter σ.

Exercise 11.5 Stability warrants. Price the up-and-out binary barrier option with payoff

$$C := \mathbb{1}_{\{S_T > K\}}\mathbb{1}_{\left\{M_0^T < B\right\}} = \mathbb{1}_{\left\{S_T > K \text{ and } M_0^T \leqslant B\right\}}$$

at time $t = 0$, with $K \leqslant B$.

Exercise 11.6 Check that the function $g(t,x)$ in (11.26) satisfies the boundary conditions

$$\begin{cases} g(t,B)=0, & t\in[0,T], \\ \\ g(T,x)=0, & x\leqslant K<B, \\ \\ g(T,x)=x-K, & K\leqslant x<B, \\ \\ g(T,x)=0, & x>B. \end{cases}$$

Exercise 11.7 Given $(S_t)_{t\in\mathbb{R}_+}$ the geometric Brownian motion

$$S_t = S_0\,\mathrm{e}^{\sigma W_t + rt - \sigma^2 t/2}, \qquad t\geqslant 0,$$

compute the up-knock-in forward option price

$$\mathrm{e}^{-rT}\mathbb{E}^*\big[(S_{\tau_B}-S_T)\mathbf{1}_{\{\tau_B\leqslant T\}}\big],$$

with barrier level $B\geqslant S_0$, where τ_B is the barrier hitting time $\tau_B := \inf\{t\geqslant 0 \ : \ S_t = B\}$.

Hint: Follow https://quant.stackexchange.com/questions/76538/hedge-up-knock-in-forward-option by applying FN6905-Proposition 10.5 and then the Girsanov theorem, see also Exercise 16.5.

Exercise 11.8 European knock-in/knock-out barrier options. Price the following vanilla options by computing their conditional discounted expected payoffs:

a) European knock-out barrier call option with payoff $(S_T-K)^+\mathbb{1}_{\{S_T\leqslant B\}}$,
b) European knock-in barrier put option with payoff $(K-S_T)^+\mathbb{1}_{\{S_T\leqslant B\}}$,
c) European knock-in barrier call option with payoff $(S_T-K)^+\mathbb{1}_{\{S_T\geqslant B\}}$,
d) European knock-out barrier put option with payoff $(K-S_T)^+\mathbb{1}_{\{S_T\geqslant B\}}$,

Exercise 11.9 Consider the correlated geometric Brownian motions $\big(S_t^{(1)}\big)_{t\geqslant 0}$ $\big(S_t^{(2)}\big)_{t\geqslant 0}$ defined as

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = r\,dt + \sigma_i\,dB_t^{(i)}, \qquad i=1,2,$$

where $\big(B_t^{(1)}\big)_{t\geqslant 0}$, $\big(B_t^{(2)}\big)_{t\geqslant 0}$ are standard Brownian motions with correlation $\rho\in(-1,1)$ under a risk-neutral measure $\mathbb{P}^*$.

a) Price the two-asset correlation call option with payoff

$$\mathbb{1}_{\{S_T^{(1)} > K_1\}} \big(S_T^{(2)} - K_2 \big)^+ .$$

b) Price the two-asset correlation put option with payoff

$$\mathbb{1}_{\{S_T^{(1)} < K_1\}} \big(K_2 - S_T^{(2)} \big)^+ .$$