# Chapter 1
# Assets, Portfolios, and Arbitrage

In this chapter, the concepts of portfolio, arbitrage, market completeness, pricing, and hedging, are introduced in a simplified single-step financial model with only two time instants $t = 0$ and $t = 1$. A binary asset price model is considered as an example in Section 1.6.



## 1.1 Portfolio Allocation and Short Selling

We will use the following notation. An element $\overline{x}$ of $\mathbb{R}^{d+1}$ is a vector

$$\overline{x} = \big(x^{(0)}, x^{(1)}, \ldots, x^{(d)}\big)$$

made of $d+1$ components. The scalar product $\overline{x} \cdot \overline{y}$ of two vectors $\overline{x}, \overline{y} \in \mathbb{R}^{d+1}$ is defined by

$$\overline{x} \cdot \overline{y} := x^{(0)}y^{(0)} + x^{(1)}y^{(1)} + \cdots + x^{(d)}y^{(d)}.$$

The vector

$$\overline{S}_0 = \big(S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(d)}\big)$$

denotes the prices at time $t = 0$ of $d+1$ assets. Namely, $S_0^{(i)} > 0$ is the price at time $t = 0$ of asset $n^o$ $i = 0, 1, \ldots, d$.




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The asset values $S_1^{(i)} > 0$ of assets No $i = 0, 1, \ldots, d$ at time $t = 1$ are represented by the vector

$$\overline{S}_1 = \big(S_1^{(0)}, S_1^{(1)}, \ldots, S_1^{(d)}\big),$$

whose components $\big(S_1^{(1)}, \ldots, S_1^{(d)}\big)$ are random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In addition we will assume that asset $n^o$ 0 is a riskless asset (of savings account type) that yields an interest rate $r > 0$, *i.e.* we have

$$S_1^{(0)} = (1 + r) S_0^{(0)}.$$

**Definition 1.1.** *A* portfolio *based on the assets* $0, 1, \ldots, d$ *is a vector*

$$\bar{\xi} = \big(\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}\big) \in \mathbb{R}^{d+1},$$

*in which* $\xi^{(i)}$ *represents the (possibly fractional) quantity of asset* $n^o$ $i$ *owned by an investor,* $i = 0, 1, \ldots, d$.

The *price* of such a portfolio, or the cost of the corresponding investment, is given by

$$\bar{\xi} \bullet \overline{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)} = \xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} + \cdots + \xi^{(d)} S_0^{(d)}$$

at time $t = 0$. At time $t = 1$, the *value* of the portfolio has evolved into

$$\bar{\xi} \bullet \overline{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} = \xi^{(0)} S_1^{(0)} + \xi^{(1)} S_1^{(1)} + \cdots + \xi^{(d)} S_1^{(d)}.$$

There are various ways to construct a portfolio allocation $\big(\xi^{(i)}\big)_{i=0,1,\ldots,d}$.

i) If $\xi^{(0)} > 0$, the investor puts the amount $\xi^{(0)} S_0^{(0)} > 0$ on a savings account with interest rate $r$.

ii) If $\xi^{(0)} < 0$, the investor borrows the amount $-\xi^{(0)} S_0^{(0)} > 0$ with the same interest rate $r$.

iii) For $i = 1, 2, \ldots, d$, if $\xi^{(i)} > 0$ then the investor purchases a (possibly fractional) quantity $\xi^{(i)} > 0$ of the asset $n^o$ $i$.

iv) If $\xi^{(i)} < 0$, the investor borrows a quantity $-\xi^{(i)} > 0$ of asset $i$ and sells it to obtain the amount $-\xi^{(i)} S_0^{(i)} > 0$.

In the latter case one says that the investor *short sells* a quantity $-\xi^{(i)} > 0$ of the asset $n^o$ $i$, which lowers the cost of the portfolio.

**Definition 1.2.** *The short selling ratio, or percentage of daily turnover activity related to short selling, is defined as as the ratio of the number of daily short sold shares divided by daily volume.*

Profits are usually made by first buying at a low price and then selling at a high price. Short sellers apply the same rule but in the reverse time order: first sell high, and then buy low if possible, by applying the following procedure.

1. Borrow the asset $n^o$ $i$.
2. At time $t = 0$, sell the asset $n^o$ $i$ on the market at the price $S_0^{(i)}$ and invest the amount $S_0^{(i)}$ at the interest rate $r > 0$.
3. Buy back the asset $n^o$ $i$ at time $t = 1$ at the price $S_1^{(i)}$, with hopefully $S_1^{(i)} < (1+r)S_0^{(i)}$.
4. Return the asset to its owner, with possibly a (small) fee $p > 0$.*

At the end of the operation the profit made on share $n^o$ $i$ equals

$$(1+r)S_0^{(i)} - S_1^{(i)} - p > 0, \qquad ☺$$

which is positive provided that $S_1^{(i)} < (1+r)S_0^{(i)}$ and $p > 0$ is sufficiently small.

## 1.2 Arbitrage

Arbitrage can be described as:

> "the purchase of currencies, securities, or commodities in one market for immediate resale in others in order to profit from unequal prices".†

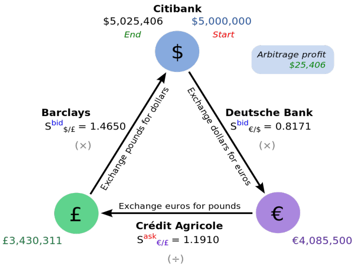
In other words, an arbitrage opportunity is the possibility to make a strictly positive amount of money starting from zero, or even from a negative amount. In a sense, the existence of an arbitrage opportunity can be seen as a way to "beat" the market.

For example, triangular arbitrage is a way to realize arbitrage opportunities based on discrepancies in the cross exchange rates of foreign currencies, as seen in Figure 1.1.‡

* The cost $p$ of short selling will not be taken into account in later calculations.

† https://www.collinsdictionary.com/dictionary/english/arbitrage

‡ https://en.wikipedia.org/wiki/Triangular_arbitrage. For an example of forex arbitrage, refer to the kimchi premium.




(a) (Wikipedia).

XE Live Exchange Rates

| USD | EUR | GBP |
|---|---|---|
| 1.00000 | 0.89347 | 0.76988 |
| 1.11923 | 1.00000 | 0.86167 |
| 1.29891 | 1.16054 | 1.00000 |

(b) Cross exchange rates.

Fig. 1.1: Examples of triangular arbitrage.

As an attempt to realize triangular arbitrage based on the data of Figure 1.1b, one could:

1. Change US$1.00 into €0.89347,
2. Change €0.89347 into £0.89347 × 0.86167 = £0.769876295,
3. Change back £0.769876295 into US$0.769876295 × 1.2981 = US$0.999376418,

which would actually result into a small loss. Alternatively, one could:

1. Change US$1.00 into £0.76988,
2. Change £0.76988 into €1.16054 × 0.76988 =€0.893476535,
3. Change back €0.893476535 into US$0.893476535 × 1.11923 = US$1.000005742,

which would result into a small gain, assuming the absence of transaction costs.

Next, we state a mathematical formulation of the concept of arbitrage.

**Definition 1.3.** *A portfolio allocation $\bar{\xi} \in \mathbb{R}^{d+1}$ constitutes an arbitrage opportunity if the three following conditions are satisfied:*

*i)* $\bar{\xi} \cdot \overline{S}_0 \leqslant 0$ *at time* $t = 0$, [Start from a zero-cost portfolio, or with a debt.]

*ii)* $\bar{\xi} \cdot \overline{S}_1 \geqslant 0$ *at time* $t = 1$, [Finish with a nonnegative amount.]

*iii)* $\mathbb{P}\big(\bar{\xi} \cdot \overline{S}_1 > 0\big) > 0$ *at time* $t = 1$. [Profit is made with nonzero probability.]

Note that there exist multiple ways to break the assumptions of Definition 1.3 in order to achieve absence of arbitrage. For example, under absence of arbitrage, satisfying Condition $(i)$ means that either $\bar{\xi} \cdot \overline{S}_1$ cannot be almost surely* nonnegative (*i.e.*, potential losses cannot be avoided), or $\mathbb{P}\big(\bar{\xi} \cdot \overline{S}_1 > 0\big) = 0$, (*i.e.*, no strictly positive profit can be made).

* "Almost surely", or "*a.s.*", means "with probability one".

**Realizing arbitrage**

In the example below, we realize arbitrage by buying and holding an asset.

1. Borrow the amount $-\xi^{(0)}S_0^{(0)} > 0$ on the riskless asset $n^o$ 0.
2. Use the amount $-\xi^{(0)}S_0^{(0)} > 0$ to purchase a quantity $\xi^{(i)} = -\xi^{(0)}S_0^{(0)}/S_0^{(i)}$, of the risky asset $n^o$ $i \geqslant 1$ at time $t = 0$ and price $S_0^{(i)}$ so that the initial portfolio cost is
$$\xi^{(0)}S_0^{(0)} + \xi^{(i)}S_0^{(i)} = 0.$$
3. At time $t = 1$, sell the risky asset $n^o$ $i$ at the price $S_1^{(i)}$, with hopefully $S_1^{(i)} > (1+r)S_0^{(i)}$.
4. Refund the amount $-(1+r)\xi^{(0)}S_0^{(0)} > 0$ with interest rate $r > 0$.

At the end of the operation the profit made is

$$\begin{aligned}\xi^{(i)}S_1^{(i)} - \big(-(1+r)\xi^{(0)}S_0^{(0)}\big) &= \xi^{(i)}S_1^{(i)} + (1+r)\xi^{(0)}S_0^{(0)} \\ &= -\xi^{(0)}\frac{S_0^{(0)}}{S_0^{(i)}}S_1^{(i)} + (1+r)\xi^{(0)}S_0^{(0)} \\ &= -\xi^{(0)}\frac{S_0^{(0)}}{S_0^{(i)}}\big(S_1^{(i)} - (1+r)S_0^{(i)}\big) \\ &= \xi^{(i)}\big(S_1^{(i)} - (1+r)S_0^{(i)}\big) \\ &> 0,\end{aligned}$$

or $S_1^{(i)} - (1+r)S_0^{(i)}$ per unit of stock invested, which is positive provided that $S_1^{(i)} > S_0^{(i)}$ and $r$ is sufficiently small. Therefore, arbitrage has been realized if $\mathbb{P}\big(S_1^{(i)} > S_0^{(i)}\big) > 0$.

Arbitrage opportunities can be similarly realized using the short selling procedure described in Section 1.1.

| City | Currency | US$ |
|---|---:|---:|
| Tokyo | JPY 38,800 | $346 |
| Hong Kong | HK$2,956.67 | $381 |
| Seoul | KRW 378,533 | $400 |
| Taipei | NT$12,980 | $404 |
| New York | US$433 | $433 |
| Sydney | A$633.28 | $483 |
| Frankfurt | €399 | $513 |
| Paris | €399 | $513 |
| Rome | €399 | $513 |
| Brussels | €399.66 | $514 |
| London | £279.99 | $527 |
| Manila | PhP 29,500 | $563 |
| Jakarta | Rp 5,754,1676 | $627 |

Fig. 1.2: Arbitrage: Xbox Retail prices around the world.

There are many real-life examples of situations where arbitrage opportunities can occur, such as:

- assets with different returns (finance),
- servers with different speeds (queueing, networking, computing),
- highway lanes with different speeds (driving).

In the latter two examples, the absence of arbitrage is consequence of the fact that switching to a faster lane or server may result into congestion, thus annihilating the potential benefit of the shift.

六合彩投注換算表

**MARK SIX INVESTMENT TABLE**

| 複式 **Multiple** | | 一胆拖 **One Banker with** | | 两胆拖 **Two Bankers with** | | 三胆拖 **Three Bankers with** | | 四胆拖 **Four Bankers with** | | 五胆拖 **Five Bankers with** | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 所選號碼總數 No. of Selections | HK$ | 配腳數目 No. of Legs | HK$ | 配腳數目 No. of Legs | HK$ | 配腳數目 No. of Legs | HK$ | 配腳數目 No. of Legs | HK$ | 配腳數目 No. of Legs | HK$ |
| 7 | 35 | 6 | 30 | 5 | 25 | 4 | 20 | 3 | 15 | 2 | 10 |
| 8 | 140 | 7 | 105 | 6 | 75 | 5 | 50 | 4 | 30 | 3 | 15 |
| 9 | 420 | 8 | 280 | 7 | 175 | 6 | 100 | 5 | 50 | 4 | 20 |
| 10 | 1,050 | 9 | 630 | 8 | 350 | 7 | 175 | 6 | 75 | 5 | 25 |
| 11 | 2,310 | 10 | 1,260 | 9 | 630 | 8 | 280 | 7 | 105 | 6 | 30 |
| 12 | 4,620 | 11 | 2,310 | 10 | 1,050 | 9 | 420 | 8 | 140 | 7 | 35 |
| 13 | 8,580 | 12 | 3,960 | 11 | 1,650 | 10 | 600 | 9 | 180 | 8 | 40 |
| 14 | 15,015 | 13 | 6,435 | 12 | 2,475 | 11 | 825 | 10 | 225 | 9 | 45 |
| 15 | 25,025 | 14 | 10,010 | 13 | 3,575 | 12 | 1,100 | 11 | 275 | 10 | 50 |
| 49 | 69,919,080 | 48 | 8,561,520 | 47 | 891,825 | 46 | 75,900 | 45 | 4,950 | 44 | 220 |

Table 1.1: Absence of arbitrage - the Mark Six "Investment Table".

In the table of Figure 1.1 the absence of arbitrage opportunities is materialized by the fact that the price of each combination is found to be proportional to its probability, thus making the game fair and disallowing any opportunity or arbitrage that would result of betting on a more profitable combination.

In what follows, we consider that market agents are rational and that as a result, any arbitrage opportunity will be immediately exploited until the market is arbitrage free. In other words, we work under the assumption that arbitrage opportunities do not occur in practice, and we will rely on this hypothesis for the pricing of financial instruments.

### Example: share rights

Let us give a market example of pricing by absence of arbitrage.

From March 24 to 31, 2009, HSBC issued *rights* to buy shares at the price of \$28. This *right* behaves similarly to an option in the sense that it gives the right (with no obligation) to buy the stock at the discount price $K = \$28$. On March 24, the HSBC stock price closed at \$41.70.

The question is: how to value the price $\$R$ of the right to buy one share? This question can be answered by looking for arbitrage opportunities. Indeed, the underlying stock can be purchased in two different ways:

1. Buy the stock directly on the market at the price of \$41.70. Cost: \$41.70,

   or:

2. First, purchase the right at price $\$R$, and then the stock at price \$28. Total cost: \$R+\$28.

a) In case

$$\$R + \$28 < \$41.70, \tag{1.1}$$

arbitrage would be possible for an investor who owns no stock and no rights, by

   i) Buying the right at a price $\$R$, and then
   ii) Buying the stock at price \$28, and
   iii) Reselling the stock at the market price of \$41.70.

The profit made by this investor would equal

$$\$41.70 - (\$R + \$28) > 0.$$

b) On the other hand, in case

$$\$R + \$28 > \$41.70, \tag{1.2}$$

arbitrage would be possible for an investor who owns the rights, by:

   i) Buying the stock on the market at \$41.70,
   ii) Selling the right by contract at the price $\$R$, and then

iii) Selling the stock at \$28 to that other investor.

In this case, the profit made would equal

$$\$R + \$28 - \$41.70 > 0.$$

In the absence of arbitrage opportunities, the combination of (1.1) and (1.2) implies that $\$R$ should satisfy

$$\$R + \$28 - \$41.70 = 0,$$

*i.e.* the arbitrage-free price of the right is given by the equation

$$\$R = \$41.70 - \$28 = \$13.70. \tag{1.3}$$

Interestingly, the *market* price of the right was \$13.20 at the close of the session on March 24. The difference of \$0.50 can be explained by the presence of various market factors such as transaction costs, the time value of money, or simply by the fact that asset prices are constantly fluctuating over time. It may also represent a small arbitrage opportunity, which cannot be at all excluded. Nevertheless, the absence of arbitrage argument (1.3) prices the right at \$13.70, which is quite close to its market value. Thus the absence of arbitrage hypothesis appears as an accurate tool for pricing.

## 1.3 Risk-Neutral Probability Measures

In order to use absence of arbitrage in the general context of pricing financial derivatives, we will need the notion of *risk-neutral probability measure.*

The next definition says that under a risk-neutral probability measure, the risky assets $n^o$ $1, 2, \ldots, d$ have same *average* rate of return as the riskless asset $n^o$ 0.

**Definition 1.4.** *A probability measure* $\mathbb{P}^*$ *on* $\Omega$ *is called a* risk-neutral measure *if for all* $i = 2, \ldots, d$ *we have*

$$\mathbb{E}^*\big[S_1^{(i)}\big] = (1+r)S_0^{(i)}. \tag{1.4}$$

Here, $\mathbb{E}^*$ denotes the expectation under the probability measure $\mathbb{P}^*$. Note that for $i = 0$, we have $\mathbb{E}^*\big[S_1^{(0)}\big] = S_1^{(0)} = (1+r)S_0^{(0)}$ by definition.

In other words, $\mathbb{P}^*$ is called *risk-neutral* because taking risks under $\mathbb{P}^*$ by buying a stock $S_1^{(i)}$ has a neutral effect: on average the expected yield of the risky asset equals the risk-free interest rate obtained by investing on the savings account with interest rate $r$, *i.e.*, we have



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$$\mathbb{E}^*\left[\frac{S_1^{(i)} - S_0^{(i)}}{S_0^{(i)}}\right] = r.$$

On the other hand, under a "risk premium" probability measure $\mathbb{P}^\#$, the expected return (or net discounted gain) of the risky asset $S_1^{(i)}$ would be higher than $r$, *i.e.*, we would have

$$\mathbb{E}^\#\left[\frac{S_1^{(i)} - S_0^{(i)}}{S_0^{(i)}}\right] > r,$$

or

$$\mathbb{E}^\#\big[S_1^{(i)}\big] > (1+r)S_0^{(i)}, \qquad i = 1, 2, \ldots, d,$$

whereas under a "negative premium" measure $\mathbb{P}^\flat$, the expected return of the risky asset $S_1^{(i)}$ would be lower than $r$, *i.e.*, we would have

$$\mathbb{E}^\flat\left[\frac{S_1^{(i)} - S_0^{(i)}}{S_0^{(i)}}\right] < r,$$

or

$$\mathbb{E}^\flat\big[S_1^{(i)}\big] < (1+r)S_0^{(i)}, \qquad i = 1, 2, \ldots, d.$$

In the sequel we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

**Definition 1.5.** *A probability measure* $\mathbb{P}^*$ *on* $(\Omega, \mathcal{F})$ *is said to be* equivalent *to another probability measure* $\mathbb{P}$ *when*

$$\mathbb{P}^*(A) = 0 \quad \textit{if and only if} \quad \mathbb{P}(A) = 0, \quad \textit{for all} \quad A \in \mathcal{F}. \tag{1.5}$$

The following Theorem 1.6 can be used to check for the existence of arbitrage opportunities, and is known as the first fundamental theorem of asset pricing.

**Theorem 1.6.** *A market is* without *arbitrage opportunity if and only if it admits at least one risk-neutral probability measure* $\mathbb{P}^*$ equivalent *to* $\mathbb{P}$.

*Proof.* $(i)$ Sufficiency. Assume that there exists a risk-neutral probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$. Since $\mathbb{P}^*$ is risk-neutral, we have

$$\bar{\xi} \cdot \overline{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)} = \frac{1}{1+r} \sum_{i=0}^{d} \xi^{(i)} \mathbb{E}^*\big[S_1^{(i)}\big] = \frac{1}{1+r} \mathbb{E}^*\big[\bar{\xi} \cdot \overline{S}_1\big]. \tag{1.6}$$

We proceed by contradiction, and suppose that the market admits an arbitrage opportunity $\bar{\xi}$ according to Definition 1.3. In this case, Definition 1.3-$(ii)$ shows that $\bar{\xi} \cdot \overline{S}_1 \geqslant 0$, and Definition 1.3-$(iii)$ implies $\mathbb{P}\big(\bar{\xi} \cdot \overline{S}_1 > 0\big) > 0$, hence $\mathbb{P}^*(\bar{\xi} \cdot \overline{S}_1 > 0) > 0$ because $\mathbb{P}$ is equivalent to $\mathbb{P}^*$. Since by Relation (A.7) we have

$$\begin{aligned} 0 &< \mathbb{P}^*(\bar{\xi} \cdot \overline{S}_1 > 0) \\ &= \mathbb{P}^*\Big( \bigcup_{n \geqslant 1} \{\bar{\xi} \cdot \overline{S}_1 > 1/n\} \Big) \\ &= \lim_{n \to \infty} \mathbb{P}^*(\bar{\xi} \cdot \overline{S}_1 > 1/n) \\ &= \lim_{\varepsilon \searrow 0} \mathbb{P}^*(\bar{\xi} \cdot \overline{S}_1 > \varepsilon), \end{aligned}$$

there exists $\varepsilon > 0$ such that $\mathbb{P}^*(\bar{\xi} \cdot \overline{S}_1 \geqslant \varepsilon) > 0$, hence

$$\begin{aligned} \mathbb{E}^*[\bar{\xi} \cdot \overline{S}_1] &\geqslant \mathbb{E}^*[\bar{\xi} \cdot \overline{S}_1 \mathbb{1}_{\{\bar{\xi} \cdot \overline{S}_1 \geqslant \varepsilon\}}] \\ &\geqslant \varepsilon \mathbb{E}^*[\mathbb{1}_{\{\bar{\xi} \cdot \overline{S}_1 \geqslant \varepsilon\}}] \\ &= \varepsilon \mathbb{P}^*(\bar{\xi} \cdot \overline{S}_1 \geqslant \varepsilon) \\ &> 0, \end{aligned}$$

and by (1.6) we conclude that

$$\bar{\xi} \cdot \overline{S}_0 = \frac{1}{1+r} \mathbb{E}^*[\bar{\xi} \cdot \overline{S}_1] > 0,$$

which contradicts Definition 1.3-$(i)$. We conclude that the market is without arbitrage opportunities.

$(ii)$ The proof of necessity, see Theorem 1.6 in Föllmer and Schied (2004), relies on the theorem of separation of convex sets by hyperplanes, see Proposition 1.7 below. It can be briefly sketched as follows in the case $d = 2$ of a portfolio containing two risky assets priced $(S_i^{(1)}, S_i^{(2)})_{i=0,1}$ with *discounted* market returns

$$R^{(1)} := \frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} - r, \quad R^{(2)} := \frac{S_1^{(2)} - S_0^{(2)}}{S_0^{(2)}} - r.$$

Assume that the relation

$$\mathbb{E}_{\mathbb{Q}}[R^{(2)}] = \mathbb{E}_{\mathbb{Q}}[R^{(1)}] = 0 \tag{1.7}$$

does not hold for any $\mathbb{Q}$ in the family $\mathcal{P}$ of probability measures $\mathbb{Q}$ on $\Omega$ equivalent to $\mathbb{P}$, *i.e.* no risk-neutral probability measure exists. We now apply the convex separation theorem Proposition 1.7 below to the subset $\mathcal{C}$ of $\mathbb{R}^2$ defined as

$$\mathcal{C} := \{(\mathbb{E}_{\mathbb{Q}}[R^{(1)}], \mathbb{E}_{\mathbb{Q}}[R^{(2)}]) \ : \ \mathbb{Q} \in \mathcal{P}\},$$

which is convex since for any $\alpha \in [0,1]$ and

$$\big(\mathbb{E}_{\mathbb{Q}}\big[R^{(1)}\big],\ \mathbb{E}_{\mathbb{Q}}\big[R^{(2)}\big]\big),\big(\mathbb{E}_{\mathbb{Q}'}\big[R^{(1)}\big],\ \mathbb{E}_{\mathbb{Q}'}\big[R^{(2)}\big]\big)\in\mathcal{C},\quad \mathbb{Q},\mathbb{Q}'\in\mathcal{P},$$

we have

$$\begin{aligned}&\alpha\big(\mathbb{E}_{\mathbb{Q}}\big[R^{(1)}\big],\ \mathbb{E}_{\mathbb{Q}}\big[R^{(2)}\big]\big)+(1-\alpha)\big(\mathbb{E}_{\mathbb{Q}'}\big[R^{(1)}\big],\ \mathbb{E}_{\mathbb{Q}'}\big[R^{(2)}\big]\big)\\&\quad=\big(\mathbb{E}_{\alpha\mathbb{Q}+(1-\alpha)\mathbb{Q}'}\big[R^{(1)}\big],\ \mathbb{E}_{\alpha\mathbb{Q}+(1-\alpha)\mathbb{Q}'}\big[R^{(2)}\big]\big)\in\mathcal{C}\end{aligned}$$

since $\alpha\mathbb{Q}+(1-\alpha)\mathbb{Q}'\in\mathcal{P}$. If (1.7) does not hold under any $\mathbb{P}^*\in\mathcal{P}$ then $(0,0)\notin\mathcal{C}$, and Proposition 1.7 applied to the convex sets $\mathcal{C}$ and $\{(0,0)\}$ shows the existence of $c\in\mathbb{R}$ such that

$$\mathbb{E}_{\mathbb{Q}}\big[R^{(1)}+cR^{(2)}\big]=\mathbb{E}_{\mathbb{Q}}\big[R^{(1)}\big]+c\ \mathbb{E}_{\mathbb{Q}}\big[R^{(2)}\big]\geqslant 0 \text{ for all } \mathbb{Q}\in\mathcal{P}, \tag{1.8}$$

and the existence of $\mathbb{P}^*\in\mathcal{P}$ such that

$$\mathbb{E}_{\mathbb{P}^*}\big[R^{(1)}+cR^{(2)}\big]=\mathbb{E}_{\mathbb{P}^*}\big[R^{(1)}\big]+c\ \mathbb{E}_{\mathbb{P}^*}\big[R^{(2)}\big]>0, \tag{1.9}$$

up to a change of direction in both inequalities. As it holds for all $\mathbb{Q}\in\mathcal{P}$, the inequality (1.8) shows that $\mathbb{P}^*$-almost surely we have *

$$R^{(1)}+cR^{(2)}\geqslant 0,$$

while (1.9) implies

$$\mathbb{P}^*\big(R^{(1)}+cR^{(2)}>0\big)>0. \tag{1.10}$$

Next, choosing $a\in\mathbb{R}$ such that

$$aS_0^{(0)}+S_0^{(1)}+cS_0^{(2)}=0,$$

the portfolio allocation

$$\bar{\xi}:=\big(\xi^{(0)},\xi^{(1)},\xi^{(2)}\big)=\left(a,1,c\frac{S_0^{(1)}}{S_0^{(2)}}\right),$$

on the assets $\big(S_0^{(0)},S_0^{(1)},S_0^{(2)}\big)$ satisfies $\bar{\xi}\bullet\overline{S}_0=0$ and

$$\begin{aligned}\bar{\xi}\bullet\overline{S}_1&=\xi^{(0)}S_1^{(0)}+\xi^{(1)}S_1^{(1)}+\xi^{(2)}S_1^{(2)}\\&=(1+r)aS_0^{(0)}+\big(1+r+R^{(1)}\big)S_0^{(1)}+c\frac{S_0^{(1)}}{S_0^{(2)}}\big(1+r+R^{(2)}\big)S_0^{(2)}\\&=\big(R^{(1)}+cR^{(2)}\big)S_0^{(1)}\\&\geqslant 0,\qquad \mathbb{P}\text{-}a.s.,\end{aligned}$$

* "Almost surely", or "*a.s.*", means "with probability one".

hence Definition 1.3-$(i)$-$(ii)$ is satisfied, and (1.10) shows that

$$\mathbb{P}^*\big(\bar{\xi}\bullet\overline{S}_1\big)=\mathbb{P}^*\big(R^{(1)}+cR^{(2)}>0\big)>0,$$

hence Definition 1.3-$(iii)$ is satisfied and $\bar{\xi}$ is an arbitrage opportunity. Therefore, there exists $\mathbb{P}^*\in\mathcal{P}$ such that (1.7) is satisfied, *i.e.*

$$\mathbb{E}_{\mathbb{P}^*}\big[S_1^{(1)}\big]=\mathbb{E}_{\mathbb{P}^*}\big[\big(1+r+R^{(1)}\big)S_0^{(1)}\big]=(1+r)S_0^{(1)}$$

and

$$\mathbb{E}_{\mathbb{P}^*}\big[S_1^{(2)}\big]=\mathbb{E}_{\mathbb{P}^*}\big[\big(1+r+R^{(2)}\big)S_0^{(2)}\big]=(1+r)S_0^{(2)},$$

and $\mathbb{P}^*$ is a risk-neutral probability measure. □

Next is a version of the separation theorem for convex sets, used in the above proof, which relies on the more general Theorem 1.8 below.

**Proposition 1.7.** *Let $\mathcal{C}$ be a convex set in $\mathbb{R}^2$ such that $(0,0)\notin\mathcal{C}$. Then, there exists $c\in\mathbb{R}$ such that, e.g.,*

$$x+cy\geqslant 0,$$

*for all $(x,y)\in\mathcal{C}$, and there exists $(x^*,y^*)\in\mathcal{C}$ such that*

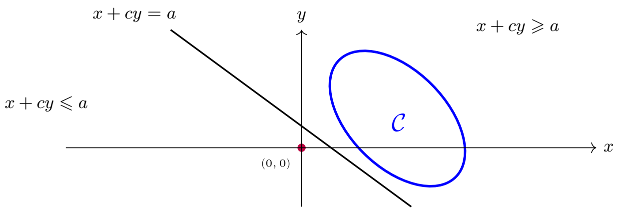
$$x^*+cy^*>0,$$

*up to a change of direction in both inequalities "$\geqslant$" and "$>$".*

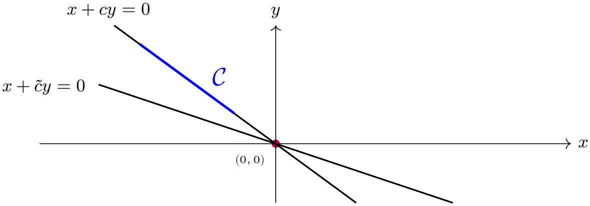
*Proof.* Theorem 1.8 below applied to $\mathcal{C}_1:=\{(0,0)\}$ and to $\mathcal{C}_2:=\mathcal{C}$ shows that for some numbers $a,c$ we have, *e.g.*,

$$0+0\times c=0\leqslant a\leqslant x+cy$$

for all $(x,y)\in\mathcal{C}$.




This allows us to conclude when $a>0$. When $a=0$, if $x+cy=0$ for all $(x,y)\in\mathcal{C}$ then the convex set $\mathcal{C}$ is an interval part of a straight line crossing $(0,0)$, for which there exists $\tilde{c}\in\mathbb{R}$ such that $x+\tilde{c}y\geqslant 0$ for all $(x,y)\in\mathcal{C}$ and $x^*+\tilde{c}y^*>0$ for some $(x^*,y^*)\in\mathcal{C}$, because $(0,0)\notin\mathcal{C}$.




□

The proof of Proposition 1.7 relies on the following result, see *e.g.* Theorem 4.14 in Hiriart-Urruty and Lemaréchal (2001).

**Theorem 1.8.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two disjoint convex sets in $\mathbb{R}^2$. Then there exists $a, c \in \mathbb{R}$ such that*

$$x + cy \leqslant a \qquad (x, y) \in \mathcal{C}_1,$$

*and*

$$a \leqslant x + cy, \qquad (x, y) \in \mathcal{C}_2,$$

*up to exchange of $\mathcal{C}_1$ and $\mathcal{C}_2$.*

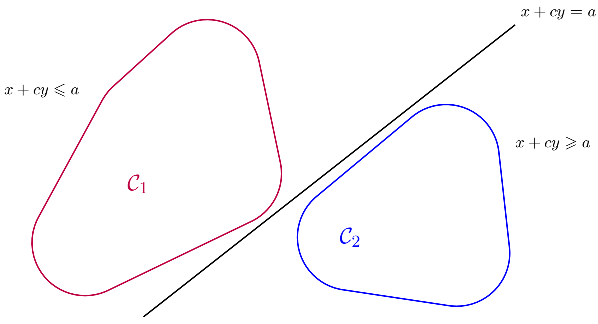



Fig. 1.3: Separation of convex sets by the linear equation $x + cy = a$.

## 1.4 Hedging of Contingent Claims

In this section we consider the notion of contingent claim. The adjective "contingent" means:

1. Subject to chance.
2. Occurring or existing only if (certain circumstances) are the case; dependent on.

More generally, we will work according to the following broad definition which covers contingent claims such as options, forward contracts etc.

**Definition 1.9.** *A contingent claim is a financial derivative whose payoff* $C : \Omega \longrightarrow \mathbb{R}$ *is a random variable depending on the realization(s) of uncertain event(s).*

In practice, the random variable $C$ will represent the payoff of an (option) contract at time $t = 1$.

Referring to Definition 2, the European call option with maturity $t = 1$ on the asset $n^o$ $i$ is a contingent claim whose payoff $C$ is given by

$$C = \big(S_1^{(i)} - K\big)^+ := \begin{cases} S_1^{(i)} - K & \text{if } S_1^{(i)} \geqslant K, \\ 0 & \text{if } S_1^{(i)} < K, \end{cases}$$

where $K$ is called the *strike price*. The claim payoff $C$ is called “contingent” because its value may depend on various market conditions, such as $S_1^{(i)} > K$. A contingent claim is also called a financial “derivative” for the same reason.

Similarly, referring to Definition 1, the European put option with maturity $t = 1$ on the asset $n^o$ $i$ is a contingent claim with payoff

$$C = \big(K - S_1^{(i)}\big)^+ := \begin{cases} K - S_1^{(i)} & \text{if } S_1^{(i)} \leqslant K, \\ 0 & \text{if } S_1^{(i)} > K, \end{cases}$$

**Definition 1.10.** *A contingent claim with payoff* $C$ *is said to be attainable at time* $t = 1$ *if there exists a portfolio allocation* $\bar{\xi} = \big(\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}\big)$ *such that*

$$C = \bar{\xi} \cdot \overline{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} = \xi^{(0)} S_1^{(0)} + \xi^{(1)} S_1^{(1)} + \cdots + \xi^{(d)} S_1^{(d)},$$

*with* $\mathbb{P}$*-probability one.*

When a contingent claim with payoff $C$ is attainable, a trader will be able to:

1. at time $t = 0$, build a portfolio allocation $\bar{\xi} = \big(\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}\big) \in \mathbb{R}^{d+1}$,



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2. invest the amount

$$\bar{\xi} \bullet \overline{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)}$$

in this portfolio at time $t = 0$,

3. at time $t = 1$, obtain the equality

$$C = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)}$$

and pay the claim amount $C$ using the portfolio value

$$\bar{\xi} \bullet \overline{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} = \xi^{(0)} S_1^{(0)} + \xi^{(1)} S_1^{(1)} + \cdots + \xi^{(d)} S_1^{(d)}.$$

We note that in order to attain the claim payoff $C$, an initial investment $\bar{\xi} \bullet \overline{S}_0$ is needed at time $t = 0$. This amount, to be paid by the buyer to the issuer of the option (the option writer), is also called the *arbitrage-free price*, or option premium, of the contingent claim, and is denoted by

$$\pi_0(C) := \bar{\xi} \bullet \overline{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)}. \tag{1.11}$$

The action of allocating a portfolio allocation $\bar{\xi}$ such that

$$C = \bar{\xi} \bullet \overline{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} \tag{1.12}$$

is called *hedging*, or *replication*, of the contingent claim with payoff $C$.

**Definition 1.11.** *In case the portfolio value $\bar{\xi} \bullet \overline{S}_1$ at time $t = 1$ exceeds the amount of the claim, i.e. when*

$$\bar{\xi} \bullet \overline{S}_1 \geqslant C,$$

*we say that the portfolio allocation $\bar{\xi} = \left(\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}\right)$ is* super-hedging *the claim $C$.*

In this document we only focus on hedging, *i.e.* on *replication* of the contingent claim with payoff $C$, and we will not consider super-hedging.

As a simplified illustration of the principle of hedging, one may purchase an oil-related asset in order to hedge oneself against a potential price rise of gasoline. In this case, any increase in the price of gasoline that would result in a higher value of the financial derivative $C$ would be correlated to an increase in the underlying asset value, so that the equality (1.12) would be maintained.

## 1.5 Market Completeness

Market completeness is a strong property, stating that any contingent claim available on the market can be perfectly hedged.

**Definition 1.12.** *A market model is said to be* complete *if every contingent claim is attainable.*

The next result is the second fundamental theorem of asset pricing.

**Theorem 1.13.** *A market model without arbitrage opportunities is complete if and only if it admits only one* equivalent *risk-neutral probability measure* $\mathbb{P}^*$.

*Proof.* See the proof of Theorem 1.40 in Föllmer and Schied (2004). □

Theorem 1.13 will give us a concrete way to verify market completeness by searching for a unique solution $\mathbb{P}^*$ to Equation (1.4).

## 1.6 Example: Binary Market

In this section we work out a simple example that allows us to apply Theorem 1.6 and Theorem 1.13. We take $d = 1$, *i.e.* the portfolio is made of

- a riskless asset with interest rate $r$ and priced $(1 + r)S_0^{(0)}$ at time $t = 1$,
- and a risky asset priced $S_1^{(1)}$ at time $t = 1$.

We use the probability space

$$\Omega = \{\omega^-, \omega^+\},$$

which is the simplest possible nontrivial choice of probability space, made of only two possible outcomes with

$$\mathbb{P}(\{\omega^-\}) > 0 \quad \text{and} \quad \mathbb{P}(\{\omega^+\}) > 0,$$

in order for the setting to be nontrivial. In other words the behavior of the market is subject to only two possible outcomes, for example, one is expecting an important binary decision of "yes/no" type, which can lead to two distinct scenarios called $\omega^-$ and $\omega^+$.

In this context, the asset price $S_1^{(1)}$ is given by a random variable

$$S_1^{(1)} : \Omega \longrightarrow \mathbb{R}$$

whose value depends on whether the scenario $\omega^-$, resp. $\omega^+$, occurs.

Precisely, we set

$$S_1^{(1)}(\omega^-) = a, \qquad \text{and} \qquad S_1^{(1)}(\omega^+) = b,$$

*i.e.*, the value of $S_1^{(1)}$ becomes equal $a$ under the scenario $\omega^-$, and equal to $b$ under the scenario $\omega^+$, where $0 < a < b$. *

### Arbitrage

The first natural question is:

- *Arbitrage*: Does the market allow for arbitrage opportunities?

We will answer this question using Theorem 1.6, by searching for a risk-neutral probability measure $\mathbb{P}^*$ satisfying the relation

$$\mathbb{E}^*\big[S_1^{(1)}\big] = (1+r)S_0^{(1)}, \tag{1.13}$$

where $r > 0$ denotes the risk-free interest rate, cf. Definition 1.4.

In this simple framework, any measure $\mathbb{P}^*$ on $\Omega = \{\omega^-, \omega^+\}$ is characterized by the data of two numbers $\mathbb{P}^*(\{\omega^-\}) \in [0,1]$ and $\mathbb{P}^*(\{\omega^+\}) \in [0,1]$, such that

$$\mathbb{P}^*(\Omega) = \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^+\}) = 1. \tag{1.14}$$

Here, saying that $\mathbb{P}^*$ is *equivalent* to $\mathbb{P}$ simply means that

$$\mathbb{P}^*(\{\omega^-\}) > 0 \quad \text{and} \quad \mathbb{P}^*(\{\omega^+\}) > 0.$$

Although we should solve (1.13) for $\mathbb{P}^*$, at this stage it is not yet clear how $\mathbb{P}^*$ is involved in the equation. In order to make (1.13) more explicit we write the expected value as

$$\mathbb{E}^*\big[S_1^{(1)}\big] = a\mathbb{P}^*\big(S_1^{(1)} = a\big) + b\mathbb{P}^*\big(S_1^{(1)} = b\big),$$

hence Condition (1.13) for the existence of a risk-neutral probability measure $\mathbb{P}^*$ reads

$$a\mathbb{P}^*\big(S_1^{(1)} = a\big) + b\mathbb{P}^*\big(S_1^{(1)} = b\big) = (1+r)S_0^{(1)}.$$

Using the Condition (1.14) we obtain the system of two equations

$$\begin{cases} a\mathbb{P}^*(\{\omega^-\}) + b\mathbb{P}^*(\{\omega^+\}) = (1+r)S_0^{(1)} \\ \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \end{cases} \tag{1.15}$$

* The case $a = b$ leads to a trivial, constant market.

with *unique* risk-neutral solution

$$\begin{cases} p^* := \mathbb{P}^*(\{\omega^+\}) = \mathbb{P}^*\big(S_1^{(1)} = b\big) = \dfrac{(1+r)S_0^{(1)} - a}{b-a} \\ \\ q^* := \mathbb{P}^*(\{\omega^-\}) = \mathbb{P}^*\big(S_1^{(1)} = a\big) = \dfrac{b - (1+r)S_0^{(1)}}{b-a}. \end{cases} \quad (1.16)$$

In order for a solution $\mathbb{P}^*$ to exist as a probability measure, the numbers $\mathbb{P}^*(\{\omega^-\})$ and $\mathbb{P}^*(\{\omega^+\})$ must be nonnegative. In addition, for $\mathbb{P}^*$ to be equivalent to $\mathbb{P}$ they should be strictly positive from (1.5).

We deduce that if $a, b$ and $r$ satisfy the condition

$$a < (1+r)S_0^{(1)} < b, \quad (1.17)$$

then there exists a risk-neutral *equivalent* probability measure $\mathbb{P}^*$ which is unique, hence by Theorems 1.6 and 1.13 the market is without arbitrage and complete.

**Remark 1.14.** *i) If $a = (1+r)S_0^{(1)}$, resp. $b = (1+r)S_0^{(1)}$, then $\mathbb{P}^*(\{\omega^+\}) = 0$, resp. $\mathbb{P}^*(\{\omega^-\}) = 0$, and $\mathbb{P}^*$ is not equivalent to $\mathbb{P}$ in the sense of Definition 1.5.*

*Therefore, we check from (1.16) that the condition*

$$a < b \leqslant (1+r)S_0^{(1)} \quad \text{or} \quad (1+r)S_0^{(1)} \leqslant a < b, \quad (1.18)$$

*do not imply existence of an* equivalent *risk-neutral probability measure and absence of arbitrage opportunities in general.*

*ii) If $a = b = (1+r)S_0^{(1)}$ then (1.4) admits an infinity of solutions, hence the market is without arbitrage, but it is not complete. More precisely, in this case both the riskless and risky assets yield a deterministic return rate $r$ and the portfolio value becomes*

$$\bar{\xi} \cdot \overline{S}_1 = (1+r)\bar{\xi} \cdot \overline{S}_0,$$

*at time $t = 1$, hence the terminal value $\bar{\xi} \cdot \overline{S}_1$ is deterministic and this* single *value can not always match the value of a contingent claim with (random) payoff $C$, that could be allowed to take* two *distinct values $C(\omega^-)$ and $C(\omega^+)$. Therefore, market completeness does not hold when $a = b = (1+r)S_0^{(1)}$.*

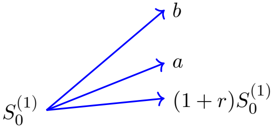
Let us give a financial interpretation of Condition (1.18).

1. If $(1+r)S_0^{(1)} \leqslant a < b$, let $\xi^{(1)} := 1$ and choose $\xi^{(0)}$ such that

$$\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} = 0$$

according to Definition 1.3-$(i)$, *i.e.*

$$\xi^{(0)} = -\xi^{(1)} \frac{S_0^{(1)}}{S_0^{(0)}} < 0.$$




In particular, Condition (i) of Definition 1.3 is satisfied, and the investor borrows the amount $-\xi^{(0)} S_0^{(0)} > 0$ on the riskless asset and uses it to buy one unit $\xi^{(1)} = 1$ of the risky asset. At time $t = 1$ he sells the risky asset $S_1^{(1)}$ at a price at least equal to $a$ and refunds the amount $-(1+r)\xi^{(0)} S_0^{(0)} > 0$ that he borrowed, with interest. The profit realized in this operation is

$$\begin{aligned} \bar{\xi} \cdot \overline{S}_1 &= (1+r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_1^{(1)} \\ &\geqslant (1+r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} a \\ &= -(1+r)\xi^{(1)} S_0^{(1)} + \xi^{(1)} a \\ &= \xi^{(1)} \big( -(1+r) S_0^{(1)} + a \big) \\ &\geqslant 0, \end{aligned}$$

☺

which satisfies Condition (ii) of Definition 1.3. In addition, Condition (iii) of Definition 1.3 is also satisfied because

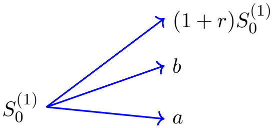
$$\mathbb{P}\big(\bar{\xi} \cdot \overline{S}_1 > 0\big) = \mathbb{P}\big(S_1^{(1)} = b\big) = \mathbb{P}(\{\omega^+\}) > 0.$$

2. If $a < b \leqslant (1+r)S_0^{(1)}$, let $\xi^{(0)} > 0$ and choose $\xi^{(1)}$ such that

$$\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} = 0,$$

according to Definition 1.3-$(i)$, *i.e.*

$$\xi^{(1)} = -\xi^{(0)} \frac{S_0^{(0)}}{S_0^{(1)}} < 0.$$




This means that the investor borrows a (possibly fractional) quantity $-\xi^{(1)} > 0$ of the risky asset, sells it for the amount $-\xi^{(1)} S_0^{(1)}$, and invests this money on the riskless account for the amount $\xi^{(0)} S_0^{(0)} > 0$. As mentioned in Section 1.1, in this case one says that the investor *shortsells* the risky asset. At time $t = 1$ she obtains $(1+r)\xi^{(0)} S_0^{(0)} > 0$ from the riskless asset, spends at most $b$ to buy back the risky asset, and returns it to its original owner. The profit realized in this operation is

$$\begin{aligned}
\bar{\xi} \cdot \overline{S}_1 &= (1+r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_1^{(1)} \\
&\geqslant (1+r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} b \\
&= -(1+r)\xi^{(1)} S_0^{(1)} + \xi^{(1)} b \\
&= \xi^{(1)} \big( -(1+r) S_0^{(1)} + b \big) \\
&\geqslant 0,
\end{aligned}$$

since $\xi^{(1)} < 0$. Note that here, $a \leqslant S_1^{(1)} \leqslant b$ became

$$\xi^{(1)} b \leqslant \xi^{(1)} S_1^{(1)} \leqslant \xi^{(1)} a$$

because $\xi^{(1)} < 0$. We can check as in Part 1 above that Conditions (i)-(iii) of Definition 1.3 are satisfied.

3. Finally if $a = b \neq (1+r)S_0^{(1)}$, then (1.4) admits no solution as a probability measure $\mathbb{P}^*$ hence arbitrage opportunities exist and can be constructed by the same method as above.

Under Condition (1.17) there is absence of arbitrage and Theorem 1.6 shows that no portfolio strategy can yield both $\bar{\xi} \cdot \overline{S}_1 \geqslant 0$ and $\mathbb{P}\big(\bar{\xi} \cdot \overline{S}_1 > 0\big) > 0$ starting from $\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} \leqslant 0$, however, this is less simple to show directly.

### Market completeness

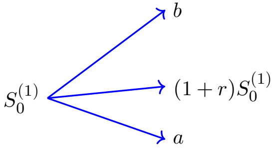
The second natural question is:

- *Completeness*: Is the market complete, *i.e.*, are all contingent claims attainable?

In what follows we work under the condition

$$a < (1+r)S_0^{(1)} < b,$$

under which Theorems 1.6 and 1.13 show that the market is without arbitrage and complete since the risk-neutral probability measure $\mathbb{P}^*$ exists and is unique.




Let us recover this fact by elementary calculations. For any contingent claim with payoff $C$ we need to show that there exists a portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)})$ such that $C = \bar{\xi} \cdot \overline{S}_1$, *i.e.*

$$\begin{cases} (1+r)\xi^{(0)}S_0^{(0)} + \xi^{(1)}b = C(\omega^+) \\ \\ (1+r)\xi^{(0)}S_0^{(0)} + \xi^{(1)}a = C(\omega^-). \end{cases} \tag{1.19}$$

These equations can be solved as

$$\xi^{(0)} = \frac{bC(\omega^-) - aC(\omega^+)}{S_0^{(0)}(1+r)(b-a)} \quad \text{and} \quad \xi^{(1)} = \frac{C(\omega^+) - C(\omega^-)}{b-a}. \tag{1.20}$$

In this case, we say that the portfolio allocation $(\xi^{(0)}, \xi^{(1)})$ *hedges* the contingent claim with payoff $C$. In other words, any contingent claim is attainable, and the market is indeed complete. Here, the quantity

$$\xi^{(0)}S_0^{(0)} = \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)}$$

represents the amount invested on the riskless asset. Note that if $C(\omega^+) \geqslant C(\omega^-)$, then $\xi^{(1)} \geqslant 0$ and there is not short selling.

When $C$ has the form $C = h(S_1^{(1)})$, we have

$$\begin{aligned} \xi^{(1)} &= \frac{C(\omega^+) - C(\omega^-)}{b-a} \\ &= \frac{h(S_1^{(1)}(\omega^+)) - h(S_1^{(1)}(\omega^-))}{b-a} \\ &= \frac{h(b) - h(a)}{b-a}. \end{aligned}$$

Hence when $x \longmapsto h(x)$ is a non-decreasing function we have $\xi^{(1)} \geqslant 0$ and there is no short selling. This applies in particular to European call options with strike price $K$, for which the function $h(x) = (x - K)^+$ is non-decreasing.

In case $h$ is a non-increasing function, which is the case in particular for European put options with payoff function $h(x) = (K - x)^+$, we will similarly find that $\xi^{(1)} \leqslant 0$, *i.e.* short selling always occurs in this case.

### Arbitrage-free price

**Definition 1.15.** *The* arbitrage-free price $\pi_0(C)$ *of the contingent claim with payoff $C$ is defined in (1.11) as the initial value at time $t = 0$ of the portfolio hedging $C$, i.e.*

$$\pi_0(C) = \bar{\xi} \cdot \overline{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)}, \tag{1.21}$$

*where $\left(\xi^{(0)}, \xi^{(1)}\right)$ are given by (1.20).*

Arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value (mark to market).* Note that $\pi_0(C)$ cannot be 0 since this would entail the existence of an arbitrage opportunity according to Definition 1.3.

The next proposition shows that the arbitrage-free price $\pi_0(C)$ of the claim can be computed as the expected value of its payoff $C$ under the risk-neutral probability measure, after discounting by the factor $1 + r$ in order to account for the time value of money.

**Proposition 1.16.** *The arbitrage-free price $\pi_0(C) = \bar{\xi} \cdot \overline{S}_0$ of the contingent claim with payoff $C$ is given by*

$$\pi_0(C) = \frac{1}{1+r} \mathbb{E}^*[C]. \tag{1.22}$$

*Proof.* Using the expressions (1.16) for the risk-neutral probabilities $\mathbb{P}^*(\{\omega^-\})$, $\mathbb{P}^*(\{\omega^+\})$, and (1.20) for the portfolio allocation $\left(\xi^{(0)}, \xi^{(1)}\right)$, we have

$$\begin{aligned}
\pi_0(C) &= \bar{\xi} \cdot \overline{S}_0 \\
&= \xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} \\
&= \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)} + S_0^{(1)} \frac{C(\omega^+) - C(\omega^-)}{b-a}
\end{aligned}$$

* Not to be confused with "market making".

$$
\begin{aligned}
&= \frac{1}{1+r}\left( C(\omega^-)\frac{b - S_0^{(1)}(1+r)}{b-a} + C(\omega^+)\frac{(1+r)S_0^{(1)} - a}{b-a} \right) \\
&= \frac{1}{1+r}\big( C(\omega^-)\mathbb{P}^*\big(S_1^{(1)} = a\big) + C(\omega^+)\mathbb{P}^*\big(S_1^{(1)} = b\big) \big) \\
&= \frac{1}{1+r}\mathbb{E}^*[C].
\end{aligned}
$$

□

In the case of a European call option with strike price $K \in [a, b]$, we have $C = \big(S_1^{(1)} - K\big)^+$ and

$$
\begin{aligned}
\pi_0\big(\big(S_1^{(1)} - K\big)^+\big) &= \frac{1}{1+r}\mathbb{E}^*\big[\big(S_1^{(1)} - K\big)^+\big] \\
&= \frac{1}{1+r}(b-K)\mathbb{P}^*\big(S_1^{(1)} = b\big) \\
&= \frac{1}{1+r}(b-K)\frac{(1+r)S_0^{(1)} - a}{b-a}. \\
&= \frac{b-K}{b-a}\left( S_0^{(1)} - \frac{a}{1+r} \right).
\end{aligned}
$$

In the case of a European put option, we have $C = \big(K - S_1^{(1)}\big)^+$ and

$$
\begin{aligned}
\pi_0\big(\big(K - S_1^{(1)}\big)^+\big) &= \frac{1}{1+r}\mathbb{E}^*\big[\big(K - S_1^{(1)}\big)^+\big] \\
&= \frac{1}{1+r}(K-a)\mathbb{P}^*\big(S_1^{(1)} = a\big) \\
&= \frac{1}{1+r}(K-a)\frac{b - (1+r)S_0^{(1)}}{b-a}. \\
&= \frac{K-a}{b-a}\left( \frac{b}{1+r} - S_0^{(1)} \right).
\end{aligned}
$$

Here, $\big(S_0^{(1)} - K\big)^+$, resp. $\big(K - S_0^{(1)}\big)^+$ is called the *intrinsic value* at time 0 of the call, resp. put option.

The simple setting described in this chapter raises several questions and remarks.

### Remarks

1. The fact that $\pi_0(C)$ can be obtained by two different methods, *i.e.* an algebraic method via (1.20) and (1.21) and a probabilistic method from (1.22), is not a simple coincidence. It is actually a simple example of the deep connection that exists between probability and analysis.



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In a continuous-time setting, (1.20) will be replaced with a *partial differential equation* (PDE), and (1.22) will be computed via the *Monte Carlo* method. In practice, the quantitative analysis departments of major financial institutions may be split into a "*PDE team*" and a "*Monte Carlo* team", often trying to determine the same option prices by two different methods.

2. What if we have three possible scenarios, *i.e.* $\Omega = \{\omega^-, \omega^o, \omega^+\}$ and the random asset $S_1^{(1)}$ is allowed to take more than two values, *e.g.* $S_1^{(1)} \in \{a, b, c\}$ according to each scenario? In this case the system (1.15) would be rewritten as

$$\begin{cases} a\mathbb{P}^*(\{\omega^-\}) + b\mathbb{P}^*(\{\omega^o\}) + c\mathbb{P}^*(\{\omega^+\}) = (1+r)S_0^{(1)} \\ \\ \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^o\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \end{cases}$$

and this system of two equations with three unknowns does not admit a unique solution, hence the market can be without arbitrage but it cannot be complete, cf. Exercise 1.5.

Market completeness can be reached by adding a second risky asset to the portfolio, *i.e.* by taking $d := 2$, in which case we will get three equations and three unknowns. More generally, when $\Omega$ contains $n \geqslant 2$ market scenarios, completeness of the market can be reached provided that we consider $d$ risky assets with $d + 1 \geqslant n$. This is related to the Meta-Theorem 8.3.1 of Björk (2004a), in which the number $d$ of traded risky underlying assets is linked to the number of random sources through arbitrage and market completeness.

## Exercises

Exercise 1.1 Consider a risky asset valued $S_0 = \$3$ at time $t = 0$ and taking only two possible values $S_1 \in \{\$1, \$5\}$ at time $t = 1$, and a financial claim given at time $t = 1$ by

$$C := \begin{cases} \$0 & \text{if } S_1 = \$5 \\ \\ \$2 & \text{if } S_1 = \$1. \end{cases}$$

Is $C$ the payoff of a call option or of a put option? Give the strike price of the option.

Exercise 1.2 Consider a risky asset valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values $S_1 \in \{\$2, \$5\}$ at time $t = 1$. Find the portfolio allocation $(\xi, \eta)$ hedging the claim payoff

$$C = \begin{cases} \$0 & \text{if } S_1 = \$5 \\ \\ \$6 & \text{if } S_1 = \$2 \end{cases}$$

at time $t = 1$, compute its price $V_0 = \xi S_0 + \$\eta$ at time $t = 0$, and determine the corresponding risk-neutral probability measure $\mathbb{P}^*$.

Exercise 1.3 Consider a risky asset valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values $S_1 \in \{\$5, \$2\}$ at time $t = 1$, and the claim payoff

$$C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases} \qquad \text{at time } t = 1.$$

We assume that the issuer charges \$1 for the option contract at time $t = 0$.

a) Compute the portfolio allocation $(\xi, \eta)$ made of $\xi$ stocks and $\$\eta$ in cash, so that:

  i) the full \$1 option price is invested into the portfolio at time $t = 0$,

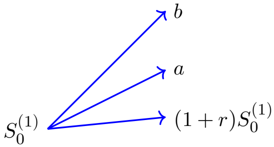
  *and*

  ii) the portfolio reaches the $C = \$3$ target if $S_1 = \$5$ at time $t = 1$.

b) Compute the loss incurred by the option issuer if $S_1 = \$2$ at time $t = 1$.

Exercise 1.4

a) Consider the following market model:



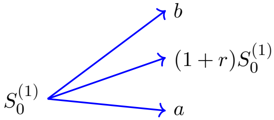

  i) Does this model allow for arbitrage?

  | Yes | | No | |

  ii) If this model allows for arbitrage opportunities, how can they be realized?

  | By shortselling | | By borrowing on savings | | N.A. | |

b) Consider the following market model:



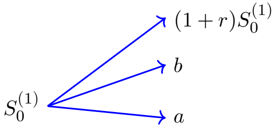

i) Does this model allow for arbitrage?

| Yes | | No | |

ii) If this model allows for arbitrage opportunities, how can they be realized?

| By shortselling | | By borrowing on savings | | N.A. | |

c) Consider the following market model:




i) Does this model allow for arbitrage?

| Yes | | No | |

ii) If this model allows for arbitrage opportunities, how can they be realized?

| By shortselling | | By borrowing on savings | | N.A. | |

Exercise 1.5 In a market model with two time instants $t = 0$ and $t = 1$ and risk-free interest rate $r$, consider

- a riskless asset valued $S_0^{(0)}$ at time $t = 0$, and value $S_1^{(0)} = (1+r)S_0^{(0)}$ at time $t = 1$.

- a risky asset with price $S_0^{(1)}$ at time $t = 0$, and three possible values at time $t = 1$, with $a < b < c$, *i.e.*:

$$S_1^{(1)} = \begin{cases} S_0^{(1)}(1+a), \\ \\ S_0^{(1)}(1+b), \\ \\ S_0^{(1)}(1+c). \end{cases}$$

a) Show that this market is without arbitrage but not complete.
b) In general, is it possible to hedge (or replicate) a claim with three distinct claim payoff values $C_a, C_b, C_c$ in this market?

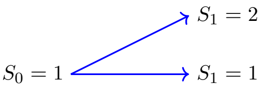
Exercise 1.6 We consider a riskless asset valued $S_1^{(0)} = S_0^{(0)}$, at times $k = 0, 1$, with risk-free interest rate is $r = 0$, and a risky asset $S^{(1)}$ whose return $R_1 := \big(S_1^{(1)} - S_0^{(1)}\big)/S_0^{(1)}$ can take three values $(-b, 0, b)$ at each time step, with $b > 0$ and

$$p^* := \mathbb{P}^*(R_1 = b) > 0, \quad \theta^* := \mathbb{P}^*(R_1 = 0) > 0, \quad q^* := \mathbb{P}^*(R_1 = -b) > 0,$$

a) Determine all possible risk-neutral probability measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in the sense of Definition 1.5 in terms of the parameter $\theta^* \in (0, 1)$, from the condition $\mathbb{E}^*[R_1] = 0$.
b) We assume that the variance $\mathrm{Var}^*\left[\frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}}\right] = \sigma^2 > 0$ of the asset return is known to be equal to $\sigma^2$. Show that this condition provides a way to select a risk-neutral probability measure $\mathbb{P}^*_\sigma$ under a certain condition on $b$ and $\sigma$.
c) Help clear the doubts expressed by Confused Quant in this post by providing an answer to his/her "what I am doing wrong?" question.

Exercise 1.7

a) Consider the following binary one-step model $(S_t)_{t=0,1,2}$ with interest rate $r = 0$ and $\mathbb{P}(S_1 = 2) = 1/3$.



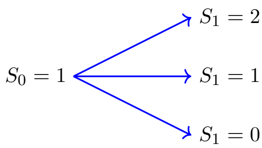

i) Is the model without arbitrage?

| Yes \| | No \| |
|---|---|

ii) Does there exist a risk-neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in the sense of Definition 1.5?

| Yes \| | No \| |
|---|---|

b) Consider the following ternary one-step model with $r = 0$, $\mathbb{P}(S_1 = 2) = 1/4$ and $\mathbb{P}(S_1 = 1) = 1/9$.




i) Does there exist a risk-neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in the sense of Definition 1.5?

Yes |     No |

ii) Is the model without arbitrage?

Yes |     No |

iii) Is the market complete?

Yes |     No |

iv) Does there exist a *unique* risk-neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in the sense of Definition 1.5?

Yes |     No |

Exercise 1.8 The Ross (2015) Recovery Theorem allows for estimates of the real-world transition probabilities of an underlying asset from option prices in a Markovian state model. We consider a one-step asset price model with two possible states $\{s_1, s_2\}$, and let $X_t \in \{s_1, s_2\}$ denote the state of the market at times $t = 0, 1$. Let $\mathcal{B}_1$, $\mathcal{B}_2$ denote two binary options maturing at $t = 1$, with respective payoffs

$$\mathbb{1}_{\{X_1 = s_1\}} \quad \text{and} \quad \mathbb{1}_{\{X_1 = s_2\}},$$

*i.e.* $\mathcal{B}_l$ pays \$1 at $t = 1$ if and only if $X_1 = s_l$, $l = 1, 2$. For $k, l = 1, 2$ we denote by $b_{k,l} > 0$ the **known** market price of the binary option $\mathcal{B}_l$ given that $X_0 = s_k$.

a) In this question, we aim at recovering the *risk-neutral* probabilities

$$\begin{aligned} P^* &= \begin{array}{c} \\ s_1 \\ s_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} p^*_{1,1} & p^*_{1,2} \\ p^*_{2,1} & p^*_{2,2} \end{bmatrix} \end{array} \\ &= \begin{array}{c} \\ s_1 \\ s_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad s_1 \quad & \quad s_2 \quad \end{array} \\ \begin{bmatrix} \mathbb{P}^*(X_1 = s_1 \mid X_0 = s_1) & \mathbb{P}^*(X_1 = s_2 \mid X_0 = s_1) \\ \mathbb{P}^*(X_1 = s_1 \mid X_0 = s_2) & \mathbb{P}^*(X_1 = s_2 \mid X_0 = s_2) \end{bmatrix} \end{array} \end{aligned}$$

using risk-neutral pricing.



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1) From Proposition 1.16, express $b_{k,l}$ using:
   i) the price $d_k$ of the bond paying \$1 at $t = 1$ when the market starts from state $s_k$ at $t = 0$, and
   ii) the risk-neutral probability $p^*_{k,l}$ that the market switches from state $s_k$ at $t = 0$ to state $s_l$ at $t = 1$.
2) Write the price $d_k$ of the bond paying \$1 at $t = 1$ in terms of the prices $b_{k,1} > 0$ and $b_{k,2} > 0$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ when the market starts from state $s_k$ at $t = 0$, $k = 1, 2$.
3) For $k = 1, 2$, show that the risk-neutral probabilities $p^*_{k,1}$, $p^*_{k,2}$ can be recovered in terms of the **known** prices $b_{k,1} > 0$, $b_{k,2} > 0$, and provide their expressions.

b) In this question, we aim at recovering the *real-world* probabilities

$$P = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} \mathbb{P}(X_1 = s_1 \mid X_0 = s_1) & \mathbb{P}(X_2 = s_2 \mid X_0 = s_1) \\ \mathbb{P}(X_1 = s_1 \mid X_0 = s_2) & \mathbb{P}(X_1 = s_2 \mid X_0 = s_2) \end{bmatrix} \end{array}$$

using *marginal utility pricing*. For this, we price the binary option $\mathcal{B}_l$ by the relation

$$u_l b_{k,l} = \delta u_k p_{k,l}, \qquad k, l = 1, 2, \tag{1.23}$$

where $\delta > 0$ is an **unknown** time discount factor and $u_1 > 0$, $u_2 > 0$ represent **unknown** marginal utilities in states $s_1, s_2$.

1) How many unknowns do we have? How many equations do we have?
2) Show that the following matrix equation holds:

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \delta \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

3) Prove that there is a unique set of strictly positive values $\delta$, $u_1$, $u_2$ that satisfy the pricing equation (1.23), provide their expressions in terms of $b_{k,l}$, $k, l = 1, 2$, and justify your choice.

   *Hint.* Diagonalize the matrix $B$.
4) Show that the real-world transition probabilities $p_{k,l}$ and the time discount factor $\delta > 0$ can be recovered from the **known** binary option prices $b_{k,l} > 0$ for $k, l = 1, 2$, and provide their expressions.

Exercise 1.9 Consider a one-step market model with two time instants $t = 0$ and $t = 1$ and two assets:

- a riskless asset $\pi$ with price $\pi_0$ at time $t = 0$ and value $\pi_1 = \pi_0(1 + r)$ at time $t = 1$,

- a risky asset $S$ with price $S_0$ at time $t = 0$ and random value $S_1$ at time $t = 1$.

We assume that $S_1$ can take only the values $S_0(1+a)$ and $S_0(1+b)$, where $-1 < a < r < b$. The *return* of the risky asset is defined as

$$R = \frac{S_1 - S_0}{S_0}.$$

a) What are the possible values of $R$?
b) Show that under the probability measure $\mathbb{P}^*$ defined by

$$\mathbb{P}^*(R = a) = \frac{b-r}{b-a}, \qquad \mathbb{P}^*(R = b) = \frac{r-a}{b-a},$$

the expected return $\mathbb{E}^*[R]$ of $S$ is equal to the return $r$ of the riskless asset.
c) Does there exist arbitrage opportunities in this model? Explain why.
d) Is this market model complete? Explain why.
e) Consider a contingent claim with payoff $C$ given by

$$C = \begin{cases} \xi & \text{if } R = a, \\ \\ \eta & \text{if } R = b. \end{cases}$$

Show that the portfolio allocation $(\eta, \xi)$ defined* by

$$\eta = \frac{\xi(1+b) - \eta(1+a)}{\pi_0(1+r)(b-a)} \quad \text{and} \quad \xi = \frac{\eta - \xi}{S_0(b-a)},$$

hedges the contingent claim with payoff $C$, *i.e.* show that at time $t = 1$, we have

$$\eta\pi_1 + \xi S_1 = C.$$

*Hint*: Distinguish two cases $R = a$ and $R = b$.
f) Compute the arbitrage-free price $\pi_0(C)$ of the contingent claim payoff $C$ using $\eta$, $\pi_0$, $\xi$, and $S_0$.
g) Compute $\mathbb{E}^*[C]$ in terms of $a, b, r, \xi, \eta$.
h) Show that the arbitrage-free price $\pi_0(C)$ of the contingent claim with payoff $C$ satisfies

$$\pi_0(C) = \frac{1}{1+r}\mathbb{E}^*[C]. \tag{1.24}$$

i) What is the interpretation of Relation (1.24) above?

* Here, $\eta$ is the (possibly fractional) quantity of asset $\pi$ and $\xi$ is the quantity held of asset $S$.

j) Let $C$ denote the payoff at time $t = 1$ of a put option with strike price K=\$11 on the risky asset. Give the expression of $C$ as a function of $S_1$ and $K$.
k) Letting $\pi_0 = S_0 = 1$, $r = 5\%$ and $a = 8$, $b = 11$, $\xi = 2$, $\eta = 0$, compute the portfolio allocation $(\xi, \eta)$ hedging the contingent claim with payoff $C$.
l) Compute the arbitrage-free price $\pi_0(C)$ of the claim payoff $C$.

Exercise 1.10 We consider the Constant Product Automated Market Maker (CPAMM) model, in which the quantities $X_t > 0$, $Y_t > 0$ of the assets $X$, $Y$ present in a liquidity pool are linked at all times $t \geqslant 0$ by the relation

$$C = X_t Y_t, \qquad t \geqslant 0,$$

where $C > 0$ is a constant. In this model, the exchange rate $S_t$ from $X$ to $Y$ at time $t \geqslant 0$ is given by $S_t = Y_t / X_t$.

a) Write down the liquidity pool value $\mathrm{LP}_t$ at any time $t \geqslant 0$ in terms of $C$ and $S_t$ only, quoted in units of $Y$, and show that at any time $t \geqslant 0$, the dollar value of the quantity $Y_t$ of asset $Y$ in the pool equals the dollar value of the quantity $X_t$ of asset $X$ in the pool.
b) An investor builds a long portfolio with $Y_0 > 0$ units of $Y$ and $X_0 > 0$ of $X$ at time $t = 0$, and holds those assets at all times. Write down the value $V_t$ of this long portfolio at any time $t \geqslant 0$ in terms of $C$, $S_0$ and $S_t$ only, quoted in units of $Y$.
c) Show that the difference in value $V_t - \mathrm{LP}_t$ between the long portfolio and the liquidity pool is always nonnegative.
d) We model the exchange rate $(S_t)_{t=0,1}$ in a one-step model on the probability space $\Omega = \{\omega^-, \omega^+\}$, with

$$S_1(\omega^-) = a, \qquad \text{and} \qquad S_1(\omega^+) = b,$$

and $ab = S_0^2$, $a < S_0 < b$. Show that the potential loss $V_1 - \mathrm{LP}_1$ of the liquidity provider at time $t = 1$ can be exactly hedged by holding a quantity $\xi$ of an option with payoff $|S_1 - S_0|$, and compute $\xi$ in terms of $C$, $S_0$ and/or $a, b$.

*Hints*: Here the payoff is $C = V_1 - \mathrm{LP}_1$ and the underlying asset value at time $t = 1$ is $|S_1 - S_0|$. The portfolio allocation is $(\xi, 0)$, as no riskless asset is used here.

Exercise 1.11 A company issues share rights, so that ten rights allow an investor to purchase three shares at the price of €6.35. Knowing that the stock is currently valued at €8, estimate the price of the right by absence of arbitrage.

Exercise 1.12 Consider a stock valued $S_0 = \$180$ at the beginning of the year. At the end of the year, its value $S_1$ can be either \$152 or \$203, and the risk-free interest rate is $r = 3\%$ per year. Given a put option with strike price $K$ on this underlying asset, find the value of $K$ for which the price of the option at the beginning of the year is equal to the intrinsic option payoff. This value of $K$ is called the break-even strike price. In other words, the break-even price is the value of $K$ for which immediate exercise of the option is equivalent to holding the option until maturity.

How would a decrease in the interest rate $r$ affect this break-even strike price?



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